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Control of singularities for the Laplace equation

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Abstract. In this paper, we prove that by acting on an arbitrarily small part of the domain or on a small part of the boundary,we obtain a regular solution of the Laplace equation. For this purpose a density result and a bi-orthogonality property of the Laplacian are used.

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1. Introduction and statement

We consider a bounded polygonal domain Ω of $\mathbb{R}^2$ with cracks whose boundary Γ is a union of the edges Γ_j for $0 \leq j \leq n$. We assume that $\Gamma = \bar{\Gamma}_D \cup \bar{\Gamma}_N$ where Γ_D and Γ_N are two open connected parts of Γ such that $\Gamma_N \neq \emptyset$. We denote by S_j the vertex between Γ_{j-1} and Γ_j for $1 \leq j \leq n$ and S_0 the vertex between Γ_n and Γ_0. We denote by ω_j (resp. ω_0) the measure of the internal angle between Γ_j and Γ_{j-1} (resp. Γ_n and Γ_0) at the point S_j (resp. S_0).
On this domain Ω, we consider the following problem

$$\begin{cases} -\triangle y = f & \text{in } \Omega \\ y = 0 & \text{on } \Gamma_D \\ \dfrac{\partial y}{\partial \nu} = 0 & \text{on } \Gamma_N \end{cases} \tag{1.1}$$

where $f \in L^2(\Omega$.
In this case, the problem (1.1) has a unique variational solution in $H^1(\Omega)$. In fact this is a poor regularity of y.

According to [2], usually, y has not the optimal regularity $H^2(\Omega)$. It is well known, that the solution of the Laplace equation in a plane non convex polygonal domain contains in general a singular part even for a very smooth data. In many cases these singularities produced undesirable phenomena. To avoid these phenomena, we consider the following problem

$$\begin{cases} -\triangle y = f + u & \text{in } \Omega \\ y = v & \text{on } \Gamma_D \\ \dfrac{\partial y}{\partial \nu} = w & \text{in } \Gamma_N \end{cases} \tag{1.2}$$

where u, v and w are smooth functions whose support do not meet any cracks and acting on an arbitrarily small part of the domain or on a small part of the boundary. For suitable choice of these functions, we establish that the solution of the problem (1.2) has the regularity $H^2(\Omega)$ if $v = w = 0$ (internal control). We also prove that the solution y of the problem (1.2) is in $H^2(\Omega)$ in case $u = 0, v = 0$ (boundary control on Neumann condition) and $u = 0, w = 0$ (boundary control on Dirichlet condition). The

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proof uses a density result, bi-orthogonality property of the dual singular solutions and the Theorem of Holmgren and Cauchy-Kowalevska.

Such a problem has already been studied in the literature for the Laplacian operator but with Dirichlet condition [5]. Extensions concerning the Neumann condition and mixed Dirichlet-Neumann boundary conditions. Similar problem was studied by the authors for the heat equation [1] restricted to a finite bandwich.

The paper is organized as follows. In Section 2, we propose the calculation of the singularities coefficients. In the section 3, we state bi-orthogonality properties of harmonic functions. Finally, in section 4 we show that by acting on an arbitrarily small part of the domain or on a small part of the boundary, we obtain a regular solution of the Laplace equation with Neumann and Dirichlet-Neumann conditions.

2. Calculation of the singularities coefficients

For the sake of simplicity, we consider without loss of generality, a particular case where, we suppose that Ω has one crack or a nonconvex angle at S and S is the origin in the local polar coordinates. The local polar coordinates at S are denoted by (r, θ) and ω is the measure of the angle at S, (see Figure 1). Consider now $\wedge$ the unbounded operator defined on $L^2(]0, \omega[)$ as follows: $\wedge\phi = -\phi''$.

$$\text{with } \phi \in D(\wedge) = \Big\{ \phi \in L^2(]0, \omega[);\ \wedge\phi \in L^2(]0, \omega[)\ \Big\}.$$

$\wedge$ is nonnegative, self-adjoint and has a discrete spectrum. We denote by $\phi_m, m \geq 1$, the normalized eigenfunctions and by $\lambda_m^2, m \geq 1$, the corresponding eigenvalues. We thus have $-\phi''_m = -\lambda_m^2 \phi_m$, where $\phi_m \in D(\wedge)$ for every m.

Given $f \in L^2(\Omega)$, the unique variational solution $y \in H^1(\Omega)$ of the following Laplace equation

$$\begin{cases} \triangle y = f & \text{on } \Omega, \\ \gamma y = 0 & \text{in } \Gamma_D, \\ \dfrac{\partial y}{\partial \nu} = 0 & \text{in } \Gamma_N \end{cases} \tag{2.1}$$

admits according to [2], the following decomposition

$$y = y_r + \sum_{0<\lambda_m<1} \eta c_m r^{\lambda_m} \phi_m(\theta) \tag{2.2}$$

where $y_r \in H^2(\Omega)$ is the regular part of y, c_m some real constants called singularities coefficients depending on the geometry of the domain, $S_m = r^{\lambda_m}\phi_m(\theta)$ are the singular solutions and η the cut off function such that $\eta = 1$ near S and 0 otherwise.

The dual singular solutions of the problem (2.1), $\omega_m^* = r^{-\lambda_m}\phi_m(\theta) + \psi_m$ where $\psi_m \in H_0^1(\Omega)$ verifies the following problem

$$\begin{cases} \triangle \omega_m^* = 0 & \text{on } \Omega, \\ \gamma \omega_m^* = 0 & \text{in } \Gamma_D, \\ \dfrac{\partial \omega_m^*}{\partial \nu} = 0 & \text{in } \Gamma_N \end{cases} \tag{2.3}$$

The set of the dual singular solutions of problem (2.3) associated to a vertex S contains only one element in the following cases: if ω_j is greater than π in the case of Neumann

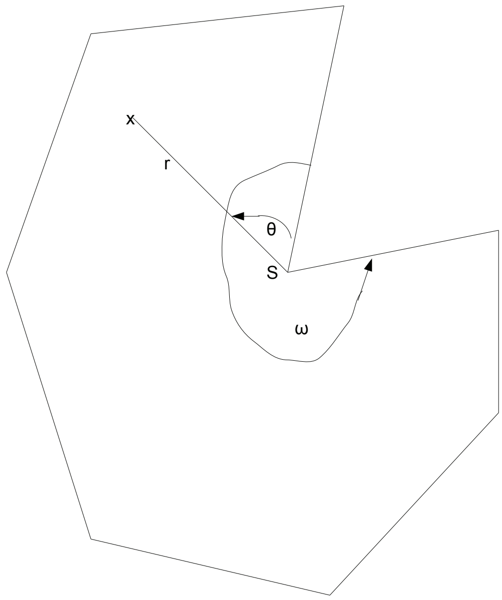

FIGURE 1. Polygonal domain.

condition or in Dirichlet condition or if ω is greater than $\frac{\pi}{2}$ and less than $\frac{3\pi}{2}$ in the case of mixed boundary conditions. In the case of mixed boundary conditions if the angle denoted ω at S is greater than $\frac{3\pi}{2}$, we have two dual singular solutions.
Using the dual singular solutions, the coefficient of the singularity c_1 at the vertex S, associated to the solution y of the problem is given by

$$c_1 = \int_\Omega f\omega_1^* \, dx. \tag{2.4}$$

In the case of mixed boundary conditions, if the angle w associated to S is greater than $\frac{3\pi}{2}$ then the dual singular solutions set contains two elements. In this case the calculation of the coefficients c_2, is quite different from c_1. For this purpose, we suppose that $u^{(1)} = u - S_1$ and $f^{(1)} = \triangle u^{(1)}$. Using the same technique we calculate

$$c_2 = \int_\Omega f^{(1)}\omega_2^* \, dx.$$

3. Bi-orthogonality property of harmonic functions

We prove here the bi-orthogonality property of harmonic functions with Neumann condition and mixed Dirichlet-Neumann boundary conditions.
We recall the following fundamental result which is proved in [5].

Theorem 3.1. (Density property) *Let H be a Hilbert space, D a dense subspace of H and $\{e_1, e_2, ..., e_m\}$ a linearly independent subset of H. Then, there exist $\{d_1, d_2, ..., d_m\}$ a subset of D such that $\forall i, j \in \{1, 2, ..., m\}$, $(e_i, d_j)_H = \delta_{ij}$.*

Now, we can prove the bi-orthogonality property of harmonic functions.

Theorem 3.2. *Let Ω be a nonempty domain of $\mathbb{R}^2$, and ϖ a nonempty open subset of Ω. If $(\omega_i^*)_{1\le i\le m}$ are linearly independent harmonic functions of $L^2(\Omega)$ verifying $\forall\, i \in \{1, \cdots, m\}$ $\displaystyle\int_\Omega \omega_i^*\, dx = 0$, then, there exist C^∞ functions $(g_i)_{1\le i\le m}$ with compact supports in ϖ verifying the following two conditions*

(1) $\forall\, i \in \{1, \cdots, m\}$ $\displaystyle\int_\Omega g_i\, dx = 0,$

(2) $\forall\, i,\, j \in \{1, \cdots, m\}$ $\displaystyle\int_\Omega \omega_i^*\, g_j\, dx = \delta_{ij}.$

Proof. We let $H = L^2(\varpi)$, $\omega_{m+1}^* = 1$ in Ω, and prove that $(\omega_i^*)_{1\le i\le m+1}$ are linearly independent. Suppose that there exist real numbers $\alpha_1, \cdots, \alpha_{m+1}$ such that $\displaystyle\sum_{i=1}^{m+1} \alpha_i \omega_i^* = 0.$

We have $\displaystyle\int_\Omega \sum_{i=1}^{m+1} \alpha_i \omega_i^*\, dx = 0$, then $\alpha_{m+1} = 0$ and $\displaystyle\int_\Omega \sum_{i=1}^{m} \alpha_i \omega_i^* dx = 0$. Since $(\omega_i^*)_{1\le i\le m}$ are linearly independent, we deduce that

$$\alpha_1 = \cdots = \alpha_{m+1} = 0.$$

Prove now that $(\omega_i^*|_\varpi)_{1\le i\le m+1}$ are linearly independent. Suppose that there exist reals numbers $\alpha_1, \cdots, \alpha_{m+1}$ such that $\displaystyle\sum_{i=1}^{m+1} \alpha_i \omega_i^*|_\varpi = 0.$

Since $\displaystyle\sum_{i=1}^{m+1} \alpha_i (\omega_i^*|_\varpi)$ is harmonic, we have $\displaystyle\sum_{i=1}^{m+1} \alpha_i \omega_i^* = 0$ in Ω and then

$$\alpha_1 = \cdots = \alpha_{m+1} = 0.$$

$\mathcal{D}(\varpi)$ is dense in $L^2(\varpi)$, then by Theorem 3.1 there exist C^∞ functions $(g_i)_{1\le i\le m+1}$ with compact supports on ϖ such that

$$\forall\, i,\, j \in \{1, \cdots, m+1\}, \text{ we have : } \int_\Omega \omega_i^*\, g_j\, dx = \delta_{ij}.$$

We conclude that

$$\forall\, i,\, j \in \{1, \cdots, m\}, \text{ we have : } \int_\Omega \omega_i^*\, g_j\, dx = \delta_{ij}.$$

with the conditions $\forall\, i \in \{1, \cdots, m\}$, $\displaystyle\int_\Omega g_i\, dx = 0.$ ☐

Theorem 3.3. *Let Ω be a nonempty domain of $\mathbb{R}^2$, Γ_c be a nonempty open subset of Γ. If $(\omega_i^*)_{1\le i\le m}$ are linearly independent harmonic functions of $L^2(\Omega)$ verifying $\forall\ i \in \{1,\cdots,m\}$ $\int_\Omega \omega_i^*\,dx = 0$ such that:*

$$\forall\ i\ \in \{1,\cdots,m\},\ \frac{\partial \omega_i^*}{\partial \nu}|_{\Gamma_c} = 0 \text{ on } \Gamma_c \text{ and } \ \omega_i^*|_{\Gamma_c} \in L^2(\Gamma_c), \tag{3.1}$$

there exist C^∞ functions $(h_i)_{1\le i\le m}$ with compact supports in Γ_c verifying the following two conditions

(1) $\forall\ i \in \{1,\cdots,m\}$ $\displaystyle\int_\Gamma h_i\,dx = 0$

(2) $\forall\, i,\, j \in \{1,\cdots,m\}$ $\displaystyle\int_\Gamma \omega_i^*\,h_j\,dx = \delta_{ij}.$

Proof. Set $H = L^2(\Gamma_c)$ and $\omega_{m+1}^* = 1$ in Ω. We have that $(\omega_i^*)_{1\le i\le m+1}$ are linearly independent. Prove that $(\omega_i^*|_{\Gamma_c})_{1\le i\le m+1}$ are linearly independent.

Suppose that there exist real numbers $\beta_1,\cdots,\beta_{m+1}$ such that $\displaystyle\sum_{i=1}^{m+1}\beta_i\omega_i^*|_{\Gamma_c} = 0$ on Γ_c.

Set $W = \displaystyle\sum_{i=1}^{m+1}\beta_i\omega_i^*$. W is harmonic and verifying:

$$\begin{cases} -\triangle W = 0 & \text{in } \Omega \\ \gamma W = 0 & \text{on } \Gamma_c \\ \gamma(\dfrac{\partial W}{\partial \nu}) = 0 & \text{on } \Gamma_c. \end{cases}$$

By Cauchy-Kowalevska's theorem, there exist a nonempty neighbourhood $\mathcal{O}$ of Γ_c such that $W = 0$ in $\mathcal{O}\cap\Omega$.

We conclude by using Holmgren's Theorem, that $W = 0$ in Ω and $\beta_1 = \cdots = \beta_{m+1} = 0$.

As $\mathcal{D}(\Gamma_c)$ is dense in $L^2(\Gamma_c)$, then by Theorem 3.1 there exist C^∞ functions $(h_i)_{1\le i\le m}$ with compact supports in Γ_c such that

$$\forall\ i,\, j \in \{1,\cdots,m\}, \text{ we have : } \int_\Gamma \omega_i^*\,h_j\,d\sigma = \delta_{ij}.$$

Moreover, we have $\forall\ i \in \{1,\cdots,m\}$ $\displaystyle\int_\Gamma h_i\,dx = 0.$ □

Theorem 3.4. *Let Ω be a nonempty domain of $\mathbb{R}^2$, Γ_c a nonempty open subset of Γ. If $(\omega_i^*)_{1\le i\le m}$ are linearly independent harmonic functions of $L^2(\Omega)$ verifying*

$$\forall\ i\ \in \{1,\cdots,m\},\ \omega_i^*|_{\Gamma_c} = 0 \text{ sur } \Gamma_c \text{ and } \frac{\partial \omega_i^*}{\partial \nu}|_{\Gamma_c} \in L^2(\Gamma_c), \tag{3.2}$$

there exist C^∞ functions $(h_i)_{1\le i\le m}$ with compact supports in Γ_c verifying

$$\forall\, i,\, j \in \{1,\cdots,m\} \int_\Gamma \frac{\partial \omega_i^*}{\partial \nu}\,h_j\,dx = \delta_{ij}. \tag{3.3}$$

Proof. Set $H = L^2(\Gamma_c)$. We have that $(\omega_i^*)_{1\le i\le m+1}$ are linearly independent.

First of all, we will prove that $\displaystyle\left(\frac{\partial \omega_i^*}{\partial \nu}|_{\Gamma_c}\right)_{1\le i\le m}$ are linearly independent. Suppose that there exist reals numbers $\beta_1, \cdots, \beta_m$ such that $\sum_{i=1}^{m} \beta_i \frac{\partial \omega_i^*}{\partial \nu}|_{\Gamma_c} = 0$ sur Γ_c.

Set $W = \sum_{i=1}^{m} \beta_i \omega_i^*$. W is harmonic and verifying:

$$\begin{cases} -\triangle W = 0 & \text{in } \Omega \\ \gamma W = 0 & \text{on } \Gamma_c \\ \gamma(\frac{\partial W}{\partial \nu}) = 0 & \text{on } \Gamma_c. \end{cases}$$

By Cauchy-Kowalevska's theorem, there exists a nonempty neighbourhood $\mathcal{O}$ of Γ_c such that $W = 0$ in $\mathcal{O} \cap \Omega$.
Holmgren's theorem allows to conclude that $W = 0$ in Ω hence $\beta_1 = \cdots = \beta_m = 0$.
As $\mathcal{D}(\Gamma_c)$ is dense in $L^2(\Gamma_c)$, then by Theorem 3.1 there exist C^∞ functions $(h_i)_{1 \leq i \leq m}$ with compact supports in Γ_c such that

$$\forall\, i,\, j \in \{1, \cdots, m\}, \text{ we have : } \int_\Gamma \omega_i^*\, h_j\, d\sigma = \delta_{ij}.$$

□

4. Main results

4.1. Control of singularities of the Laplace equation with Neumann conditions. Let $(S_j)_{1 \leq j \leq m}$ be the vertices of the nonconvex angles $(\omega_j)_{1 \leq j \leq m}$; we suppose that ω_j is greater than π for each j.

Theorem 4.1. *Let ϖ be a nonempty open subset of Ω, $f \in L^2(\Omega)$ satisfying the condition $\int_\Omega f\, dx = 0$ and $(C_i)_{1 \leq i \leq m}$ the coefficients of singularities of the following problem*

$$\begin{cases} -\triangle y = f & \textit{in } \Omega, \\ \gamma \frac{\partial y}{\partial \nu} = 0 & \textit{on } \Gamma. \end{cases} \tag{4.1}$$

Then there exist m C^∞ functions with compact supports in ϖ, $(g_i)_{1 \leq i \leq m}$ such that the solution of the problem

$$\begin{cases} -\triangle u = f - \sum_{i=1}^{m} C_i g_i & \textit{in } \Omega \\ \gamma \frac{\partial u}{\partial \nu} = 0 & \textit{on } \Gamma, \end{cases} \tag{4.2}$$

is in $H^2(\Omega)$.

Proof. The dual singular solutions of problem (4.1) verifies the conditions of Theorem 3.2. There exist m, C^∞ functions $(g_i)_{1 \leq i \leq m}$ with compact support in ϖ verified $\forall\, i \in \{1, \cdots, m\}$ $\int_\Omega g_i\, dx = 0$ and such that

$$\forall\, i,\, j \in \{1, \cdots, m\}, \text{ we have : } \int_\Omega \omega_i^*\, g_j\, dx = \delta_{ij}.$$

Let $(\xi_i)_{1\leq i\leq m}$ be the coefficients of singularities of Problem (4.2).
We have

$$\begin{aligned}\xi_i &= \int_\Omega \omega_i^* \Big(f - \sum_{j=1}^m C_j g_j\Big) dx \\ &= \int_\Omega w_i^* \, f \, dx - \sum_{j=1}^m C_j \int_\Omega w_i^* \, g_j \, dx \\ &= C_i - \sum_{j=1}^m C_j \delta_{ij} \\ &= C_i - C_i \\ &= 0.\end{aligned}$$

This shows that $u \in H^2(\Omega)$. □

Theorem 4.2. *Let Ω be a nonempty domain of $\mathbb{R}^2$, Γ_c a nonempty open subset of Γ. If $f \in L^2(\Omega)$ satisfies the condition $\int_\Omega f \, dx = 0$ and $(C_i)_{1\leq i\leq m}$ are the coefficients of singularities of the following problem*

$$\begin{cases} -\triangle y = f & in\ \Omega \\ \gamma \dfrac{\partial y}{\partial \nu} = 0 & on\ \Gamma, \end{cases} \tag{4.3}$$

then, there exist m, C^∞ $(h_i)_{1\leq i\leq m}$ functions with compact support in Γ_c, such that the solution of the problem

$$\begin{cases} -\triangle u = f & in\ \Omega \\ \gamma \dfrac{\partial u}{\partial \nu} = -\displaystyle\sum_{i=1}^m C_i h_i & on\ \Gamma, \end{cases} \tag{4.4}$$

is in $H^2(\Omega)$.

Proof. We suppose that z is a C^∞ extension of $\sum_{i=1}^m C_i h_i$ in Ω with support in a neighbourhood of Γ_C. Let $v = u - z$. Suppose that the coefficients of singularities of v are denoted by $(\chi_i)_{1\leq i\leq m}$. As the dual singular functions ω_i^* verify the Theorem 3.3, there exists C^∞ functions $(h_i)_{1\leq i\leq m}$ with compact support in Γ_c such that

$$\forall\ i,\ j \in \{1, \cdots, m\}, \int_\Gamma \omega_i^* \, h_j \, d\sigma = \delta_{ij}.$$

We have that

$$\begin{aligned} \chi_i &= \int_\Omega \omega_i^*(f+\triangle z)dx \\ &= \int_\Omega \omega_i^*\, f\, dx + \int_\Omega \omega_i^* \triangle z\, dx \\ &= C_i + \int_\Omega z\triangle\omega_i^* dx + \int_\Gamma \frac{\partial z}{\partial\nu}\omega_j^*\, d\sigma - \int_\Gamma \frac{\partial\omega_i^*}{\partial\nu}\, z\, d\sigma \\ &= C_i + \int_\Gamma \frac{\partial z}{\partial\nu}\,\omega_i^*\, d\sigma \\ &= C_i - \sum_{j=1}^m C_j \int_\Gamma \omega_i^* h_j d\sigma \\ &= C_i - \sum_{j=1}^m C_j\delta_{ij} \\ &= 0, \end{aligned}$$

then $u \in H^2(\Omega)$. □

4.2. Cancellation of singularities of the Laplace equation with mixed boundary conditions. We suppose that the angles $(\omega_i)_{1\le i\le m}$ are greater than $\frac{\pi}{2}$ and assume that there exists $i_0 \in \{1,\cdots,m\}$ such that $\omega_{i_0} > \frac{3\pi}{2}$. For $i \in \{1,\cdots,m\}$, we denoted by N_i the dimension of the dual solution associated to ω_i.

Theorem 4.3. *Assume that ϖ is a nonempty open subset of Ω. If for $f \in L^2(\Omega)$ and $(C_{ik})_{\substack{1\le i\le m\\ 1\le k\le N_i}}$ the coefficients of singularities of the problem*

$$\begin{cases} -\triangle y = f & in\ \Omega \\ \gamma y = 0 & on\ \Gamma_D \\ \gamma\dfrac{\partial y}{\partial\nu} = 0 & on\ \Gamma_N, \end{cases} \tag{4.5}$$

then there exist $(g_{ik})_{\substack{1\le i\le m\\ 1\le k\le N_i}}$, C^∞ functions with compact support in ϖ, such that the solution of the problem

$$\begin{cases} -\triangle u = f - \displaystyle\sum_{i=1}^m\sum_{k=1}^{N_i} C_{ik} g_{ik} & in\ \Omega \\ \gamma y = 0 & on\ \Gamma_D \\ \gamma\dfrac{\partial y}{\partial\nu} = 0 & on\ \Gamma_N, \end{cases} \tag{4.6}$$

belongs to $H^2(\Omega)$.

Proof. For $i \in \{1,\cdots m\}$, if $N_i = 1$ then we apply Theorem 4.1 to prove that the corresponding coefficients of singularities ξ_{i1} are equal to zero.
If there exists $i_0 \in \{1,\cdots m\}$, such that $N_{i_0} = 2$ then the corresponding coefficients

of singularities $\xi_{i_0 1}$ and $\xi_{i_0 2}$ are calculated as follows.
Calculation of $\xi_{i_0 1}$. We have

$$\begin{aligned}
\xi_{i_0 1} &= \int_\Omega \omega^*_{i_0 1}\Big(f - \sum_{j=1}^{m}\sum_{k=1}^{N_j} C_{jk} g_{jk}\Big)dx \\
&= \int_\Omega \omega^*_{i_0 1}\, f\, dx - \sum_{j=0}^{m}\sum_{k=1}^{N_j} C_{jk} \int_\Omega \omega^*_{i_0 1}\, g_{jk}\, dx \\
&= C_{i_0 1} - \sum_{j=1}^{m}\sum_{k=1}^{N_j} C_{jk}\delta_{i_0 j}\delta_{1k} \\
&= C_{i_0 1} - C_{i_0 1} \\
&= 0.
\end{aligned}$$

Calculation of $\xi_{i_0 2}$. Let $u^{(1)} = u - S_{j,1}$ and $f^{(1)} = \triangle u^{(1)}$. Then

$$\begin{aligned}
\xi_{i_0 2} &= \int_\Omega \omega^*_{i_0 2}\Big(f - \sum_{j=1}^{m}\sum_{k=1}^{N_j} C_{jk} g_{jk}\Big)dx \\
&= \int_\Omega \omega^*_{i_0 2}\, f\, dx - \sum_{j=1}^{m}\sum_{k=1}^{N_j} C_{jk} \int_\Omega \omega^*_{i_0 2}\, g_{jk}\, dx \\
&= C_{i_0 2} - \sum_{j=1}^{m}\sum_{k=1}^{N_j} C_{jk}\delta_{i_0 j}\delta_{2k} \\
&= C_{i_0 2} - C_{i_0 2} \\
&= 0
\end{aligned}$$

This allows us to conclude that $u \in H^2(\Omega)$. □

Theorem 4.4. *Let Ω be a nonempty domain of $\mathbb{R}^2$, Γ_c a nonempty open subset of Γ_D such that $\overline{\Gamma_c} \cap \overline{\Gamma_N} = \emptyset$. If $f \in L^2(\Omega)$ and $(C_{ik})_{\substack{1\le i\le m\\ 1\le k\le N_i}}$ are the coefficients of singularities of the problem*

$$\begin{cases} -\triangle y = f & \text{in } \Omega \\ \gamma y = 0 & \text{on } \Gamma_D \\ \gamma \dfrac{\partial y}{\partial \nu} = 0 & \text{on } \Gamma_N, \end{cases} \tag{4.7}$$

then there exist $(h_{ik})_{\substack{1\le i\le m\\ 1\le k\le N_i}}$, C^∞ functions with compact supports on Γ_c, such that the solution of the problem

$$\begin{cases} -\triangle u = f & \text{in } \Omega \\ \gamma u = \displaystyle\sum_{i=1}^{m}\sum_{k=1}^{N_i} C_{ik} h_{ik} & \text{on } \Gamma_D \\ \gamma \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_N, \end{cases} \tag{4.8}$$

belongs to $H^2(\Omega)$.

Proof. Let z be a C^∞ extension of $\displaystyle\sum_{i=1}^{m}\sum_{k=1}^{N_i} C_{ik}h_{ik}$ in Ω with support in a neighbourhood of Γ_0.
Let $v = y - z$, then $v = 0$ on Γ_D and $-\triangle v = f + \triangle z$. We denote by $\xi_{11}, \cdots, \xi_{mN_m}$ the coefficients of singularity associated to v. For $i \in \{1, \cdots m\}$, if $N_i = 1$ then we apply Theorem 4.1 to prove that the corresponding coefficients of singularities ξ_{i1} are equal to zero.
If there exists $i_0 \in \{1, \cdots m\}$, such that $N_{i_0} = 2$ then the corresponding coefficients of singularities ξ_{i_01} and ξ_{i_02} are as follows

$$\begin{aligned}
\xi_{i_01} &= \int_\Omega \omega^*_{i_01}\Big(f + \triangle z\Big)dx \\
&= \int_\Omega \omega^*_{i_01}\, f\, dx - \sum_{j=0}^{m}\sum_{k=1}^{N_j} C_{jk}\int_\Gamma \frac{\partial \omega^*_{i_01}}{\partial \nu}\, h_{ij}\, d\sigma \\
&= C_{i_01} - \sum_{j=1}^{m}\sum_{k=1}^{N_j} C_{jk}\delta_{i_0j}\delta_{1k} \\
&= C_{i_01} - C_{i_01} \\
&= 0.
\end{aligned}$$

and

$$\begin{aligned}
\xi_{i_02} &= \int_\Omega \omega^*_{i_02}\Big(f + \triangle z\Big)dx \\
&= \int_\Omega \omega^*_{i_02}\, f\, dx - \sum_{j=1}^{m}\sum_{k=1}^{N_j} C_{jk}\int_\Gamma \frac{\partial \omega^*_{i_02}}{\partial \nu}\, h_{ij}\, d\sigma \\
&= C_{i_02} - \sum_{j=1}^{m}\sum_{k=1}^{N_j} C_{jk}\delta_{i_0j}\delta_{2k} \\
&= C_{i_02} - C_{i_02} \\
&= 0.
\end{aligned}$$

Then we conclude that $u \in H^2(\Omega)$. □

Theorem 4.5. *Let Ω be a nonempty domain of $\mathbb{R}^2$, Γ_c a nonempty open subset of Γ_N such that $\overline{\Gamma_c} \cap \overline{\Gamma_D} = \emptyset$. If $f \in L^2(\Omega)$ and $(C_{ik})_{\substack{1\le i\le m\\ 1\le k\le N_i}}$ are the coefficients of singularities of the problem*

$$\begin{cases}
-\triangle y = f & \text{in } \Omega \\
\gamma y = 0 & \text{on } \Gamma_D \\
\gamma \dfrac{\partial y}{\partial \nu} = 0 & \text{on } \Gamma_N,
\end{cases} \tag{4.9}$$

Then there exist $(h_{ik})_{\substack{1\le i\le m\\ 1\le k\le N_i}}$, C^∞ *functions with compact support on* Γ_c, *such that the solution* u *of the problem*

$$\begin{cases} -\triangle u = f & in\ \Omega \\ \gamma u = 0 & on\ \Gamma_D \\ \gamma \dfrac{\partial u}{\partial \nu} = \displaystyle\sum_{i=1}^{m}\sum_{k=1}^{N_i} C_i^k g_i^k & on\ \Gamma_N, \end{cases} \tag{4.10}$$

belongs to $H^2(\Omega)$.

The proof is similar to the proof of Theorem 4.4.

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