Control of singularities for the Laplace equation

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Abstract. In this paper, we prove that by acting on an arbitrarily small part of the domain or on a small part of the boundary, we obtain a regular solution of the Laplace equation. For this purpose a density result and a bi-orthogonality property of the Laplacian are used.

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1. Introduction and statement

We consider a bounded polygonal domain $\Omega$ of $\mathbb{R}^2$ with cracks whose boundary $\Gamma$ is a union of the edges $\Gamma_j$ for $0 \leq j \leq n$. We assume that $\Gamma = \Gamma_D \cup \Gamma_N$ where $\Gamma_D$ and $\Gamma_N$ are two open connected parts of $\Gamma$ such that $\Gamma_N \neq \emptyset$. We denote by $S_j$ the vertex between $\Gamma_{j-1}$ and $\Gamma_j$ for $1 \leq j \leq n$ and $S_0$ the vertex between $\Gamma_n$ and $\Gamma_0$. We denote by $\omega_j$ (resp. $\omega_0$) the measure of the internal angle between $\Gamma_j$ and $\Gamma_{j-1}$ (resp. $\Gamma_n$ and $\Gamma_0$) at the point $S_j$ (resp. $S_0$).

On this domain $\Omega$, we consider the following problem

$$
\begin{cases}
-\Delta y = f & \text{in } \Omega \\
y = 0 & \text{on } \Gamma_D \\
\frac{\partial y}{\partial \nu} = 0 & \text{on } \Gamma_N
\end{cases}
$$

where $f \in L^2(\Omega)$. In this case, the problem (1.1) has a unique variational solution in $H^1(\Omega)$. In fact this is a poor regularity of $y$.

According to [2], usually, $y$ has not the optimal regularity $H^2(\Omega)$. It is well known, that the solution of the Laplace equation in a plane non convex polygonal domain contains in general a singular part even for a very smooth data. In many cases these singularities produced undesirable phenomena. To avoid these phenomena, we consider the following problem

$$
\begin{cases}
-\Delta y = f + u & \text{in } \Omega \\
y = v & \text{on } \Gamma_D \\
\frac{\partial y}{\partial \nu} = w & \text{in } \Gamma_N
\end{cases}
$$

where $u$, $v$ and $w$ are smooth functions whose support do not meet any cracks and acting on an arbitrarily small part of the domain or on a small part of the boundary. For suitable choice of these functions, we establish that the solution of the problem (1.2) has the regularity $H^2(\Omega)$ if $v = w = 0$ (internal control). We also prove that the solution $y$ of the problem (1.2) is in $H^2(\Omega)$ in case $u = 0$, $v = 0$ (boundary control on Neumann condition) and $u = 0$, $w = 0$ (boundary control on Dirichlet condition).
proof uses a density result, bi-orthogonality property of the dual singular solutions and the Theorem of Holmgren and Cauchy-Kowalevska.

Such a problem has already been studied in the literature for the Laplacian operator but with Dirichlet condition [5]. Extensions concerning the Neumann condition and mixed Dirichlet-Neumann boundary conditions. Similar problem was studied by the authors for the heat equation [1] restricted to a finite sandwich.

The paper is organized as follows. In Section 2, we propose the calculation of the singularities coefficients. In the section 3, we state bi-orthogonality properties of harmonic functions. Finally, in section 4 we show that by acting on an arbitrarily small part of the domain or on a small part of the boundary, we obtain a regular solution of the Laplace equation with Neumann and Dirichlet-Neumann conditions.

2. Calculation of the singularities coefficients

For the sake of simplicity, we consider without loss of generality, a particular case where, we suppose that \( \Omega \) has one crack or a nonconvex angle at \( S \) and \( S \) is the origin in the local polar coordinates. The local polar coordinates at \( S \) are denoted by \((r, \theta)\) and \( \omega \) is the measure of the angle at \( S \) (see Figure 1). Consider now the unbounded operator defined on \( L^2([0, \omega]) \) as follows: \( \land \phi = -\phi'' \).

\[
\land = \begin{cases} \phi \in D(\land) = \{ \phi \in L^2([0, \omega]); \land \phi \in L^2([0, \omega]) \}.
\end{cases}
\]

\( \land \) is nonnegative, self-adjoint and has a discrete spectrum. We denote by \( \phi_m, m \geq 1 \), the normalized eigenfunctions and by \( \lambda_m, m \geq 1 \), the corresponding eigenvalues. We thus have \( -\phi''_m = -\lambda_m^2 \phi_m \), where \( \phi_m \in D(\land) \) for every \( m \).

Given \( f \in L^2(\Omega) \), the unique variational solution \( y \in H^1(\Omega) \) of the following Laplace equation

\[
\begin{align*}
\Delta y &= f \quad \text{on } \Omega, \\
\gamma y &= 0 \quad \text{in } \Gamma_D, \\
\frac{\partial y}{\partial \nu} &= 0 \quad \text{in } \Gamma_N
\end{align*}
\]

admits according to [2], the following decomposition

\[
y = y_r + \sum_{0<\lambda_m<1} \eta c_m r^{\lambda_m} \phi_m(\theta)
\]

where \( y_r \in H^2(\Omega) \) is the regular part of \( y \), \( c_m \) some real constants called singularities coefficients depending on the geometry of the domain, \( S_m = r^{\lambda_m} \phi_m(\theta) \) are the singular solutions and \( \eta \) the cut off function such that \( \eta = 1 \) near \( S \) and 0 otherwise. The dual singular solutions of the problem (2.1), \( \omega^*_m = r^{-\lambda_m} \phi_m(\theta) + \psi_m \) where \( \psi_m \in H^1_0(\Omega) \) verifies the following problem

\[
\begin{align*}
\Delta \omega^*_m &= 0 \quad \text{on } \Omega, \\
\gamma \omega^*_m &= 0 \quad \text{in } \Gamma_D, \\
\frac{\partial \omega^*_m}{\partial \nu} &= 0 \quad \text{in } \Gamma_N
\end{align*}
\]

The set of the dual singular solutions of problem (2.3) associated to a vertex \( S \) contains only one element in the following cases: if \( \omega_j \) is greater than \( \pi \) in the case of Neumann conditions.
condition or in Dirichlet condition or if $\omega$ is greater than $\frac{\pi}{2}$ and less than $\frac{3\pi}{2}$ in the case of mixed boundary conditions. In the case of mixed boundary conditions if the angle denoted $\omega$ at $S$ is greater than $\frac{3\pi}{2}$, we have two dual singular solutions. Using the dual singular solutions, the coefficient of the singularity $c_1$ at the vertex $S$, associated to the solution $y$ of the problem is given by

$$
c_1 = \int_{\Omega} f^{*}_{\omega_1} dx.
$$

In the case of mixed boundary conditions, if the angle $w$ associated to $S$ is greater than $\frac{3\pi}{2}$ then the dual singular solutions set contains two elements. In this case the calculation of the coefficients $c_2$, is quite different from $c_1$. For this purpose, we suppose that $u^{(1)} = u - S_1$ and $f^{(1)} = \Delta u^{(1)}$. Using the same technique we calculate

$$
c_2 = \int_{\Omega} f^{(1)} \omega_2^* dx.
$$
3. Bi-orthogonality property of harmonic functions

We prove here the bi-orthogonality property of harmonic functions with Neumann condition and mixed Dirichlet-Neumann boundary conditions. We recall the following fundamental result which is proved in [5].

**Theorem 3.1.** (Density property) Let $H$ be a Hilbert space, $D$ a dense subspace of $H$ and \{\(e_1, e_2, \ldots, e_m\)\} a linearly independent subset of $H$. Then, there exist \(\{d_1, d_2, \ldots, d_m\}\) a subset of $D$ such that $\forall i, j \in \{1, 2, \ldots, m\}$, \((e_i, d_j)_H = \delta_{ij}\).

Now, we can prove the bi-orthogonality property of harmonic functions.

**Theorem 3.2.** Let $\Omega$ be a nonempty domain of $\mathbb{R}^2$, and $\mathcal{W}$ a nonempty open subset of $\Omega$. If \((\omega_i^*)_{1 \leq i \leq m+1}\) are linearly independent harmonic functions of $L^2(\Omega)$ verifying $\forall i \in \{1, \cdots, m\}$ \(\int_{\Omega} \omega_i^* \, dx = 0\), then, there exist $C^\infty$ functions \((g_i)_{1 \leq i \leq m+1}\) with compact supports in $\mathcal{W}$ verifying the following two conditions

(1) $\forall i \in \{1, \cdots, m\}$ \(\int_{\Omega} g_i \, dx = 0\),

(2) $\forall i, j \in \{1, \cdots, m\}$ \(\int_{\Omega} \omega_i^* g_j \, dx = \delta_{ij}\).

**Proof.** We let $H = L^2(\mathcal{W})$, $\omega_{m+1}^* = 1$ in $\Omega$, and prove that \((\omega_i^*)_{1 \leq i \leq m+1}\) are linearly independent. Suppose that there exist real numbers $\alpha_1, \cdots, \alpha_{m+1}$ such that \(m+1 \sum_{i=1}^{m+1} \alpha_i \omega_i^* = 0\).

We have \(\int_{\Omega} \sum_{i=1}^{m+1} \alpha_i \omega_i^* \, dx = 0\), then $\alpha_{m+1} = 0$ and \(\int_{\Omega} \sum_{i=1}^{m} \alpha_i \omega_i^* \, dx = 0\). Since \((\omega_i^*)_{1 \leq i \leq m}\) are linearly independent, we deduce that $\alpha_1 = \cdots = \alpha_{m+1} = 0$.

Prove now that \((\omega_i^*|_{\mathcal{W}})_{1 \leq i \leq m+1}\) are linearly independent. Suppose that there exist reals numbers $\alpha_1, \cdots, \alpha_{m+1}$ such that \(\sum_{i=1}^{m+1} \alpha_i \omega_i^*|_{\mathcal{W}} = 0\).

Since \(\sum_{i=1}^{m+1} \alpha_i (\omega_i^*|_{\Omega})\) is harmonic, we have \(\sum_{i=1}^{m+1} \alpha_i \omega_i^* = 0\) in $\Omega$ and then $\alpha_1 = \cdots = \alpha_{m+1} = 0$.

$D(\mathcal{W})$ is dense in $L^2(\mathcal{W})$, then by Theorem 3.1 there exist $C^\infty$ functions \((g_i)_{1 \leq i \leq m+1}\) with compact supports on $\mathcal{W}$ such that

$\forall i, j \in \{1, \cdots, m+1\}$, we have: \(\int_{\Omega} \omega_i^* g_j \, dx = \delta_{ij}\).

We conclude that

$\forall i, j \in \{1, \cdots, m\}$, we have: \(\int_{\Omega} \omega_i^* g_j \, dx = \delta_{ij}\).

with the conditions $\forall i \in \{1, \cdots, m\}$, \(\int_{\Omega} g_i \, dx = 0\). \(\square\)
Theorem 3.3. Let $\Omega$ be a nonempty domain of $\mathbb{R}^2$, $\Gamma_c$ be a nonempty open subset of $\Gamma$. If $(\omega_i^*)_{1 \leq i \leq m}$ are linearly independent harmonic functions of $L^2(\Omega)$ verifying
\[ \forall i \in \{1, \cdots, m\} \int_{\Omega} \omega_i^* \, dx = 0 \] such that:
\[ \forall i \in \{1, \cdots, m\}, \frac{\partial \omega_i^*}{\partial \nu}|_{r_c} = 0 \text{ on } \Gamma_c \text{ and } \omega_i^*|_{r_c} \in L^2(\Gamma_c), \] \[ \text{(3.1)} \]
there exist $C^\infty$ functions $(h_i)_{1 \leq i \leq m}$ with compact supports in $\Gamma_c$ verifying the following two conditions
\begin{enumerate}
  \item[(1)] $\forall i \in \{1, \cdots, m\} \int_{\Gamma} h_i \, dx = 0$
  \item[(2)] $\forall i, j \in \{1, \cdots, m\} \int_{\Gamma} \omega_i^* h_j \, dx = \delta_{ij}$.
\end{enumerate}

Proof. Set $H = L^2(\Gamma_c)$ and $\omega_{m+1}^* = 1$ in $\Omega$. We have that $(\omega_i^*)_{1 \leq i \leq m+1}$ are linearly independent. Prove that $(\omega_i^*|_{r_c})_{1 \leq i \leq m}$ are linearly independent.

Suppose that there exist real numbers $\beta_1, \cdots, \beta_{m+1}$ such that $\sum_{i=1}^{m+1} \beta_i \omega_i^*|_{r_c} = 0$ on $\Gamma_c$.

Set $W = \sum_{i=1}^{m+1} \beta_i \omega_i^*$. $W$ is harmonic and verifying:
\[ \left\{ \begin{array}{l}
-\Delta W = 0 \quad \text{in } \Omega \\
\gamma W = 0 \quad \text{on } \Gamma_c \\
\gamma \left( \frac{\partial W}{\partial \nu} \right) = 0 \quad \text{on } \Gamma_c.
\end{array} \right. \]

By Cauchy-Kowalevski’s theorem, there exist a nonempty neighbourhood $O$ of $\Gamma_c$ such that $W = 0$ in $O \cap \Omega$.

We conclude by using Holmgren’s Theorem, that $W = 0$ in $\Omega$ and $\beta_1 = \cdots = \beta_{m+1} = 0$.

As $D(\Gamma_c)$ is dense in $L^2(\Gamma_c)$, then by Theorem 3.1 there exist $C^\infty$ functions $(h_i)_{1 \leq i \leq m}$ with compact supports in $\Gamma_c$ such that
\[ \forall i, j \in \{1, \cdots, m\}, \text{ we have: } \int_{\Gamma} \omega_i^* h_j \, d\sigma = \delta_{ij}. \]

Moreover, we have $\forall i \in \{1, \cdots, m\} \int_{\Gamma} h_i \, dx = 0$. \hfill \Box

Theorem 3.4. Let $\Omega$ be a nonempty domain of $\mathbb{R}^2$, $\Gamma_c$ a nonempty open subset of $\Gamma$. If $(\omega_i^*)_{1 \leq i \leq m}$ are linearly independent harmonic functions of $L^2(\Omega)$ verifying
\[ \forall i \in \{1, \cdots, m\}, \omega_i^*|_{r_c} = 0 \text{ on } \Gamma_c \text{ and } \frac{\partial \omega_i^*}{\partial \nu}|_{r_c} \in L^2(\Gamma_c), \] \[ \text{(3.2)} \] there exist $C^\infty$ functions $(h_i)_{1 \leq i \leq m}$ with compact supports in $\Gamma_c$ verifying
\[ \forall i, j \in \{1, \cdots, m\} \int_{\Gamma} \frac{\partial \omega_i^*}{\partial \nu} h_j \, dx = \delta_{ij}. \] \[ \text{(3.3)} \]

Proof. Set $H = L^2(\Gamma_c)$. We have that $(\omega_i^*)_{1 \leq i \leq m+1}$ are linearly independent.

First of all, we will prove that $(\frac{\partial \omega_i^*}{\partial \nu}|_{r_c})_{1 \leq i \leq m+1}$ are linearly independent. Suppose
that there exist reals numbers $\beta_1, \cdots, \beta_m$ such that $\sum_{i=1}^m \beta_i \frac{\partial \omega_i^*}{\partial \nu} |_{\Gamma_c} = 0$ sur $\Gamma_c$.

Set $W = \sum_{i=1}^m \beta_i \omega_i^*$. $W$ is harmonic and verifying:

\[
\begin{align*}
-\Delta W &= 0 \quad \text{in } \Omega \\
\gamma W &= 0 \quad \text{on } \Gamma_c \\
\gamma \left( \frac{\partial W}{\partial \nu} \right) &= 0 \quad \text{on } \Gamma_c.
\end{align*}
\]

By Cauchy-Kovalevskaya’s theorem, there exists a nonempty neighbourhood $O$ of $\Gamma_c$ such that $W = 0$ in $O \cap \Omega$.

Holmgren’s theorem allows to conclude that $W = 0$ in $\Omega$ hence $\beta_1 = \cdots = \beta_m = 0$.

As $\mathcal{D}(\Gamma_c)$ is dense in $L^2(\Gamma_c)$, then by Theorem 3.1 there exist $C^\infty$ functions $(h_i)_{1 \leq i \leq m}$ with compact supports in $\Gamma_c$ such that

\[\forall i, j \in \{1, \cdots, m\}, \text{ we have }: \int_{\Gamma} \omega_i^* h_j \, d\sigma = \delta_{ij}.\]

\[\Box\]

4. Main results

4.1. Control of singularities of the Laplace equation with Neumann conditions. Let $(S_j)_{1 \leq j \leq m}$ be the vertices of the nonconvex angles $(\omega_j)_{1 \leq j \leq m}$; we suppose that $\omega_j$ is greater than $\pi$ for each $j$.

**Theorem 4.1.** Let $\varpi$ be a nonempty open subset of $\Omega$, $f \in L^2(\Omega)$ satisfying the condition $\int_{\Omega} f \, dx = 0$ and $(C_i)_{1 \leq i \leq m}$ the coefficients of singularities of the following problem

\[
\begin{align*}
-\Delta y &= f \quad \text{in } \Omega, \\
\gamma \frac{\partial y}{\partial \nu} &= 0 \quad \text{on } \Gamma.
\end{align*}
\]

Then there exist $m$ $C^\infty$ functions with compact supports in $\varpi$, $(g_i)_{1 \leq i \leq m}$ such that the solution of the problem

\[
\begin{align*}
-\Delta u &= f - \sum_{i=1}^m C_i g_i \quad \text{in } \Omega \\
\gamma \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \Gamma,
\end{align*}
\]

is in $H^2(\Omega)$.

**Proof.** The dual singular solutions of problem (4.1) verifies the conditions of Theorem 3.2. There exist $m$, $C^\infty$ functions $(g_i)_{1 \leq i \leq m}$ with compact support in $\varpi$ verified

$\forall i \in \{1, \cdots, m\}$ \[\int_{\Omega} g_i \, dx = 0\] and such that

$\forall i, j \in \{1, \cdots, m\}$, we have : \[\int_{\Omega} \omega_i^* g_j \, dx = \delta_{ij}.\]
Let \((\xi_i)_{1 \leq i \leq m}\) be the coefficients of singularities of Problem (4.2). We have

\[
\begin{align*}
\xi_i &= \int_{\Omega} \omega_i^* \left( f - \sum_{j=1}^{m} C_j g_j \right) dx \\
&= \int_{\Omega} w_i^* f dx - \sum_{j=1}^{m} C_j \int_{\Omega} w_i^* g_j dx \\
&= C_i - \sum_{j=1}^{m} C_j \delta_{ij} \\
&= C_i - C_i \\
&= 0.
\end{align*}
\]

This shows that \(u \in H^2(\Omega)\).

**Theorem 4.2.** Let \(\Omega\) be a nonempty domain of \(\mathbb{R}^2\), \(\Gamma_c\) a nonempty open subset of \(\Gamma\). If \(f \in L^2(\Omega)\) satisfies the condition \(\int_{\Omega} f dx = 0\) and \((C_i)_{1 \leq i \leq m}\) are the coefficients of singularities of the following problem

\[
\begin{cases}
-\Delta y = f & \text{in } \Omega \\
\gamma \frac{\partial y}{\partial \nu} = 0 & \text{on } \Gamma,
\end{cases}
\]

then, there exist \(m\), \(C^\infty\) \((h_i)_{1 \leq i \leq m}\) functions with compact support in \(\Gamma_c\), such that the solution of the problem

\[
\begin{cases}
-\Delta u = f & \text{in } \Omega \\
\gamma \frac{\partial u}{\partial \nu} = -\sum_{i=1}^{m} C_i h_i & \text{on } \Gamma,
\end{cases}
\]

is in \(H^2(\Omega)\).

**Proof.** We suppose that \(z\) is a \(C^\infty\) extension of \(\sum_{i=1}^{m} C_i h_i\) in \(\Omega\) with support in a neighbourhood of \(\Gamma_c\). Let \(v = u - z\). Suppose that the coefficients of singularities of \(v\) are denoted by \((\chi_i)_{1 \leq i \leq m}\). As the dual singular functions \(\omega_i^*\) verify the Theorem 3.3, there exists \(C^\infty\) functions \((h_i)_{1 \leq i \leq m}\) with compact support in \(\Gamma_c\) such that

\[
\forall i, j \in \{1, \cdots, m\}, \int_{\Gamma} \omega_i^* h_j d\sigma = \delta_{ij}.
\]
We have that
\[
\chi_i = \int_{\Omega} \omega_i^*(f + \Delta z) dx
= \int_{\Omega} \omega_i^* f dx + \int_{\Omega} \omega_i^* \Delta z dx
= C_i + \int_{\Gamma} \frac{\partial z}{\partial \nu} \omega_j^* d\sigma - \int_{\Gamma} \frac{\partial \omega_i^*}{\partial \nu} z d\sigma
= C_i + \sum_{j=1}^{m} C_j \int_{\Gamma} \omega_j^* h_j d\sigma
= C_i - \sum_{j=1}^{m} C_j \delta_{ij}
= 0,
\]
then \( u \in H^2(\Omega) \).

4.2. Cancellation of singularities of the Laplace equation with mixed boundary conditions. We suppose that the angles \( (\omega_i)_{1 \leq i \leq m} \) are greater than \( \frac{\pi}{2} \) and assume that there exists \( i_0 \in \{1, \cdots, m\} \) such that \( \omega_{i_0} > \frac{3\pi}{2} \). For \( i \in \{1, \cdots, m\} \), we denoted by \( N_i \) the dimension of the dual solution associated to \( \omega_i \).

**Theorem 4.3.** Assume that \( \varpi \) is a nonempty open subset of \( \Omega \). If for \( f \in L^2(\Omega) \) and \((C_{ik})_{1 \leq i \leq m, 1 \leq k \leq N_i}\) the coefficients of singularities of the problem
\[
\begin{align*}
-\Delta y &= f & \text{in } \Omega \\
\gamma y &= 0 & \text{on } \Gamma_D \\
\frac{\partial y}{\partial \nu} &= 0 & \text{on } \Gamma_N,
\end{align*}
\]
then there exist \((g_{ik})_{1 \leq i \leq m, 1 \leq k \leq N_i}\), \( C^\infty \) functions with compact support in \( \varpi \), such that the solution of the problem
\[
\begin{align*}
-\Delta u &= f - \sum_{i=1}^{m} \sum_{k=1}^{N_i} C_{ik} g_{ik} & \text{in } \Omega \\
\gamma y &= 0 & \text{on } \Gamma_D \\
\frac{\partial y}{\partial \nu} &= 0 & \text{on } \Gamma_N,
\end{align*}
\]
belongs to \( H^2(\Omega) \).

**Proof.** For \( i \in \{1, \cdots m\} \), if \( N_i = 1 \) then we apply Theorem 4.1 to prove that the corresponding coefficients of singularities \( \xi_{i1} \) are equal to zero.

If there exists \( i_0 \in \{1, \cdots m\} \), such that \( N_{i_0} = 2 \) then the corresponding coefficients
of singularities $\xi_{i_01}$ and $\xi_{i_02}$ are calculated as follows.

Calculation of $\xi_{i_01}$. We have

$$\xi_{i_01} = \int_{\Omega} \omega_{i_01}^* \left( f - \sum_{j=1}^{m} \sum_{k=1}^{N_j} C_{jk} g_{jk} \right) dx$$

$$= \int_{\Omega} \omega_{i_01}^* f dx - \sum_{j=1}^{m} \sum_{k=1}^{N_j} C_{jk} \int_{\Omega} \omega_{i_01}^* g_{jk} dx$$

$$= C_{i_01} - \sum_{j=1}^{m} \sum_{k=1}^{N_j} C_{jk} \delta_{i_0 j} \delta_{1 k}$$

$$= C_{i_01} - C_{i_01}$$

$$= 0.$$

Calculation of $\xi_{i_02}$. Let $u^{(1)} = u - S_j$ and $f^{(1)} = \triangle u^{(1)}$. Then

$$\xi_{i_02} = \int_{\Omega} \omega_{i_02}^* \left( f - \sum_{j=1}^{m} \sum_{k=1}^{N_j} C_{jk} g_{jk} \right) dx$$

$$= \int_{\Omega} \omega_{i_02}^* f dx - \sum_{j=1}^{m} \sum_{k=1}^{N_j} C_{jk} \int_{\Omega} \omega_{i_02}^* g_{jk} dx$$

$$= C_{i_02} - \sum_{j=1}^{m} \sum_{k=1}^{N_j} C_{jk} \delta_{i_0 j} \delta_{2 k}$$

$$= C_{i_02} - C_{i_02}$$

$$= 0.$$

This allows us to conclude that $u \in H^2(\Omega)$.

**Theorem 4.4.** Let $\Omega$ be a nonempty domain of $\mathbb{R}^2$, $\Gamma_c$ a nonempty open subset of $\Gamma_D$ such that $\Gamma_c \cap \Gamma_N = \emptyset$. If $f \in L^2(\Omega)$ and $(C_{ik})_{1 \leq i \leq m}$ are the coefficients of singularities of the problem

$$\begin{cases} 
-\triangle y = f & \text{in } \Omega \\
\gamma y = 0 & \text{on } \Gamma_D \\
\frac{\partial y}{\partial \nu} = 0 & \text{on } \Gamma_N,
\end{cases} \tag{4.7}$$

then there exist $(h_{ik})_{1 \leq i \leq m}$, $C^\infty$ functions with compact supports on $\Gamma_c$, such that $1 \leq k \leq N_i$, the solution of the problem

$$\begin{cases} 
-\triangle u = f & \text{in } \Omega \\
\gamma u = \sum_{i=1}^{m} \sum_{k=1}^{N_i} C_{ik} h_{ik} & \text{on } \Gamma_D \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_N,
\end{cases} \tag{4.8}$$

belongs to $H^2(\Omega)$. 

Proof. Let \( z \) be a \( C^\infty \) extension of \( \sum_{i=1}^{m} \sum_{k=1}^{N_i} C_{ik} h_{ik} \) in \( \Omega \) with support in a neighbourhood of \( \Gamma_0 \).

Let \( v = y - z \), then \( v = 0 \) on \( \Gamma_D \) and \( -\Delta v = f + \Delta z \). We denote by \( \xi_{11}, \cdots, \xi_{mN_i} \) the coefficients of singularity associated to \( v \). For \( i \in \{1, \cdots, m\} \), if \( N_i = 1 \) then we apply Theorem 4.1 to prove that the corresponding coefficients of singularities \( \xi_{i1} \) are equal to zero.

If there exists \( i_0 \in \{1, \cdots, m\} \), such that \( N_{i_0} = 2 \) then the corresponding coefficients of singularities \( \xi_{i_01} \) and \( \xi_{i_02} \) are as follows

\[
\xi_{i_01} = \int_{\Omega} \omega^*_{i_01} (f + \Delta z) \, dx
\]

\[
= \int_{\Omega} \omega^*_{i_01} f \, dx - \sum_{j=0}^{m} \sum_{k=1}^{N_j} C_{jk} \int_{\Gamma} \frac{\partial \omega^*_{i_01}}{\partial \nu} h_{ij} \, d\sigma
\]

\[
= C_{i_01} - \sum_{j=1}^{m} \sum_{k=1}^{N_j} C_{jk} \delta_{i_0j} \delta_{1k}
\]

\[
= C_{i_01} - C_{i_01}
\]

\[
= 0.
\]

and

\[
\xi_{i_02} = \int_{\Omega} \omega^*_{i_02} (f + \Delta z) \, dx
\]

\[
= \int_{\Omega} \omega^*_{i_02} f \, dx - \sum_{j=1}^{m} \sum_{k=1}^{N_j} C_{jk} \int_{\Gamma} \frac{\partial \omega^*_{i_02}}{\partial \nu} h_{ij} \, d\sigma
\]

\[
= C_{i_02} - \sum_{j=1}^{m} \sum_{k=1}^{N_j} C_{jk} \delta_{i_0j} \delta_{2k}
\]

\[
= C_{i_02} - C_{i_02}
\]

\[
= 0.
\]

Then we conclude that \( u \in H^2(\Omega) \). \( \square \)

**Theorem 4.5.** Let \( \Omega \) be a nonempty domain of \( \mathbb{R}^2 \), \( \Gamma_c \) a nonempty open subset of \( \Gamma_N \) such that \( \overline{\Gamma_c} \cap \overline{\Gamma_D} = \emptyset \). If \( f \in L^2(\Omega) \) and \( (C_{ik})_{1 \leq i \leq m} \) are the coefficients of singularities of the problem

\[
\begin{align*}
-\Delta y &= f \quad \text{in } \Omega \\
\gamma y &= 0 \quad \text{on } \Gamma_D \\
\frac{\partial y}{\partial \nu} &= 0 \quad \text{on } \Gamma_N,
\end{align*}
\]

(4.9)
Then there exist \((h_{ik})_{1 \leq i \leq m, \ 1 \leq k \leq N_i}\), \(C^\infty\) functions with compact support on \(\Gamma_c\), such that the solution \(u\) of the problem

\[
\begin{align*}
-\Delta u &= f \quad \text{in } \Omega \\
\gamma u &= 0 \quad \text{on } \Gamma_D \\
\gamma \frac{\partial u}{\partial \nu} &= \sum_{i=1}^m \sum_{k=1}^{N_i} C_i g_i^k \quad \text{on } \Gamma_N,
\end{align*}
\]

belongs to \(H^2(\Omega)\).

The proof is similar to the proof of Theorem 4.4.

References


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