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Several characterizations of the 4–valued modal algebras

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Abstract. A. Monteiro, in 1978, defined the algebras he named tetravalent modal algebras, that will be called 4−*valued modal algebras* in this work. These algebras constitute a generalization of the 3−valued Lukasiewicz algebras defined by Moisil.

The theory of the 4−valued modal algebras has been widely developed by I. Loureiro in [6, 7, 8, 9, 10, 11, 12] and by A. V. Figallo in [2, 3, 4, 5].

J. Font and M. Rius indicated, in the introduction to the important work [1], a brief but detailed review on the 4−valued modal algebras.

In this work varied characterizations are presented that show the "closeness" this variety of algebras has with other well–known algebras related to the algebraic counterparts of certain logics.

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1. Introduction

In 1940 G. C. Moisil [13] introduced the notion of three–valued Łukasiewicz algebra. In 1963, A.Monteiro [14] characterized these algebras as algebras $\langle A, \wedge, \vee, \sim, \triangledown, 1\rangle$ of type $(2, 2, 1, 1, 0)$ which verify the following identities:

(A1) $x \vee 1 = 1$,
(A2) $x \wedge (x \vee y) = x$,
(A3) $x \wedge (y \vee z) = (z \wedge x) \vee (y \wedge x)$,
(A4) $\sim\sim x = x$,
(A5) $\sim (x \vee y) =\sim x \wedge \sim y$,
(A6) $\sim x \vee \triangledown x = 1$,
(A7) $x \wedge \sim x =\sim x \wedge \triangledown x$,
(A8) $\triangledown(x \wedge y) = \triangledown x \wedge \triangledown y$.

L.Monteiro [15] proved that A1 follows from A2, $\cdots$, A8, and that A2, $\cdots$, A8, are independent.

From A2, $\cdots$, A5 it follows that $\langle A, \wedge, \vee, \sim, 1\rangle$ is a De Morgan algebra with last element 1 and first element $0 =\sim 1$.

In Lemma 1.1 we will indicate other properties valid in the variety of 4–valued modal algebras necessary for the development that follows.

Lemma 1.1. *In every* 4*-valued modal algebra* $\langle A, \wedge, \vee, \sim, \nabla, 1\rangle$ *we have :* A9-A17.

(A9) $x \leq \nabla x$,
(A10) $\nabla 1 = 1$,
(A11) $\nabla x \leq \nabla\nabla x$,
(A12) $\nabla x \vee \sim \nabla x = 1$,

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(A13) $\nabla x \wedge \sim \nabla x = 0$,
(A14) $\nabla\nabla x = \nabla x$,
(A15) *If* $x \leq y$, *then* $\nabla x \leq \nabla y$,
(A16) $\nabla(\nabla x \vee \nabla y) = \nabla(x \vee y)$,
(A17) $\nabla x \vee \nabla y = \nabla(x \vee y)$,

the proof of which we will indicate in the section that follows.

In 1969 J. Varlet [16] characterized three–valued Łukasiewicz algebras by means of other operations. Let $\langle A, \wedge, \vee, *, +, 0, 1\rangle$ be an algebra of type $(2, 2, 1, 1, 0, 0)$ where $\langle A, \wedge, \vee, 0, 1\rangle$ is a bounded distributive lattice with least element 0, greatest element 1 and the following properties are satisfied:
(V1) $x \wedge x^* = 0$,
(V2) $(x \wedge y)^* = x^* \wedge y^*$,
(V3) $0^* = 1$,
(V4) $x \vee x^+ = 1$,
(V5) $(x \vee y)^+ = x^+ \wedge y^+$,
(V6) $1^+ = 0$,
(V7) If $x^* = y^*$ and $x^+ = y^+$, then $x = y$.

About these algebras he proved that it is posible to define, in the sense of [14, 15] a structure of three-valued Łukasiewicz algebra by taking $\sim x = (x \vee x^*) \wedge x^+$ and $\triangledown x = x^{**}$.

Furthermore it holds $x^* = \sim \triangledown x$ and $x^+ = \triangledown \sim x$. Therefore three–valued Łukasiewicz are double Stone lattices which satisfy the determination principle V7. Moreover V7 may be replaced by the identity

$$(x \wedge x^+) \wedge (y \vee y^*) = x \wedge x^+.$$

Later, in 1963, A. Monteiro [14] considered the 4–valued modal algebras $\langle A, \wedge, \vee, \sim, \triangledown, 1\rangle$ of type $(2, 2, 1, 1, 0)$ which satisfy A2, $\cdots$, A7 as an abstraction of three–valued Łukasiewicz algebras.

In this paper we give several characterizations of the 4–valued modal algebras. In the first one we consider the operations $\wedge, \vee, \neg, \Gamma, 0, 1$ where $\neg x = \sim \triangledown x$, $\Gamma x = \triangledown \sim x$ are called strong and weak negation respectively.

2. A characterization of the 4–valued modal algebras

Before working on Theorem 2.1 we will indicate proofs from A9 through A16. Then,

(A9) $x \leq \nabla x$:
(1) $(x \wedge \sim x) \vee \nabla x = (\nabla x \wedge \sim x) \vee \nabla x = \nabla x$, [A7]
(2) $(x \vee \nabla x) \wedge (\sim x \vee \nabla x) = \nabla x$,
(3) $(x \vee \nabla x) \wedge 1 = \nabla x$, [A6]
(4) $x \vee \nabla x = \nabla x$,
(5) $x \leq \nabla x$.

(A10) $\nabla 1 = 1$: [A9]
(A11) $\nabla x \leq \nabla\nabla x$: [A9]
(A12) $\nabla x \vee \sim \nabla x = 1$:
(1) $\nabla x \wedge \sim \nabla x = \sim \nabla x \wedge \nabla\nabla x$, [A7]
(2) $\sim \nabla x \vee \nabla x = \nabla x \vee \sim \nabla\nabla x$ [(1), A5]
$= (\nabla x \vee \sim \nabla\nabla x) \wedge 1$
$= (\nabla x \vee \sim \nabla\nabla x) \wedge (\sim x \vee \nabla x)$ [A6]

$$= ((\nabla x \vee \sim \nabla\nabla x) \wedge \sim x) \vee ((\nabla x \vee \sim \nabla\nabla x) \wedge \nabla x)$$
$$= ((\nabla x \vee \sim x) \wedge (\sim \nabla\nabla x \vee \sim x)) \vee \nabla x$$
$$= (1 \wedge (\sim \nabla\nabla x \vee \sim x)) \vee \nabla x \quad [A6]$$
$$= \sim \nabla\nabla x \vee \sim x \vee \nabla x$$
$$= 1. \quad [A6]$$

(A13) $\nabla x \wedge \sim \nabla x = 0$: [A12]

(A14) $\nabla\nabla x = \nabla x$:

$$\nabla\nabla x = \nabla\nabla x \wedge 1$$
$$= \nabla\nabla x \wedge (\nabla x \vee \sim \nabla x) \quad [A12]$$
$$= (\nabla\nabla x \wedge \nabla x) \vee (\nabla\nabla x \wedge \sim \nabla x)$$
$$= \nabla x \vee (\nabla\nabla x \wedge \sim \nabla x) \quad [A11]$$
$$= \nabla x \vee (\sim \nabla x \wedge \nabla x) \quad [A7]$$
$$= \nabla x.$$

(A15) If $x \leq y$, then $\nabla x \leq \nabla y$:

(1) $x \leq y$, [Hip.]
(2) $\sim y \leq \sim x$, [(1)]
(3) $\sim y \vee \nabla y \leq \sim x \vee \nabla y$, [(2)]
(4) $1 = \sim x \vee \nabla y$, [A6]
(5) $0 = x \wedge \sim \nabla y$, [A6]
(6) $\sim \nabla x = \sim \nabla x \vee (x \wedge \sim \nabla y)$ [(5)]
$= (\sim \nabla x \vee x) \wedge (\sim \nabla x \vee \sim \nabla y)$
$= 1 \wedge (\sim \nabla x \vee \sim \nabla y)$ [A6]
$= \sim \nabla x \vee \sim \nabla y$,
(7) $\nabla x = \nabla x \wedge \nabla y$, [(6)]
(8) $\nabla x \leq \nabla y$. [(7)]

(A16) $\nabla(\nabla x \vee \nabla y) = \nabla(x \vee y)$:

(1) $x \leq x \vee y$,
(2) $y \leq x \vee y$,
(3) $\nabla x \leq \nabla(x \vee y)$, [(1), A15]
(4) $\nabla y \leq \nabla(x \vee y)$, [(2), A15]
(5) $\nabla x \vee \nabla y \leq \nabla(x \vee y)$, [(3), (4)]
(6) $x \leq \nabla x$, [A9]
(7) $y \leq \nabla y$, [A9]
(8) $x \vee y \leq \nabla x \vee \nabla y$, [(6), (7)]
(9) $\nabla(x \vee y) \leq \nabla(\nabla x \vee \nabla y)$, [(8), A15]
(10) $\nabla(\nabla x \vee \nabla y) \leq \nabla\nabla(x \vee y)$, [(5), A15]
(11) $\nabla(\nabla x \vee \nabla y) \leq \nabla(x \vee y)$, [(10), A14]
(12) $\nabla(\nabla x \vee \nabla y) = \nabla(x \vee y)$. [(9), (11)]

(A17) $\nabla x \vee \nabla y = \nabla(x \vee y)$:

(1) $\sim (\nabla x \vee \nabla y) \wedge \nabla(\nabla x \vee \nabla y) = (\nabla x \vee \nabla y) \wedge \sim (\nabla x \vee \nabla y)$ [A7]
$= (\nabla x \vee \nabla y) \wedge \sim \nabla x \wedge \sim \nabla y$ [A5]
$= ((\nabla x \wedge \sim \nabla x) \vee (\nabla y \wedge \sim \nabla x)) \wedge \sim \nabla y$
$= \nabla y \wedge \sim \nabla x \wedge \sim \nabla y$ [A13]
$= 0$, [A13]
(2) $\sim (\nabla x \vee \nabla y) \wedge \nabla(x \vee y) = 0$, [(1), A16]
(3) $(\nabla x \vee \nabla y) \vee \sim \nabla(x \vee y) = 1$,
(4) $((\nabla x \vee \nabla y) \vee \sim \nabla(x \vee y)) \wedge \nabla(x \vee y) = \nabla(x \vee y)$, [(3)]
(5) $((\nabla x \vee \nabla y) \wedge \nabla(x \vee y)) \vee (\sim \nabla(x \vee y) \wedge \nabla(x \vee y)) = \nabla(x \vee y)$,
(6) $(\nabla x \vee \nabla y) \wedge \nabla(x \vee y) = \nabla(x \vee y)$, [(5), A13]

(7) $\nabla(x \vee y) \leq \nabla x \vee \nabla y$, [(6)]
(8) $\nabla x \vee \nabla y = \nabla(x \vee y)$. [(7), (5) of A16]

Theorem 2.1. *Let* $\langle A, \wedge, \vee, \neg, \Gamma, 0, 1\rangle$ *be an algebra of type* $(2, 2, 1, 1, 0, 0)$ *where* $\langle A, \wedge, \vee, 0, 1\rangle$ *is a bounded distributive lattice with least element* 0, *greatest element* 1 *and the operators* $\nabla, \sim$ *are defined on* A *by means of the formulas:*
(D1) $\nabla x = \neg\neg x$,
(D2) $\sim x = (x \vee \neg x) \wedge \Gamma x$.
Then (i) *and* (ii) *are equivalent:*
(i) $\langle A, \wedge, \vee, \sim, \nabla, 1\rangle$ *is a 4–valued modal algebra.*
(ii) $\langle A, \wedge, \vee, \neg, \Gamma, 0, 1\rangle$ *verifies these properties:*
(B1) $\neg\neg 1 = 1$,
(B2) $x \wedge \neg x = 0$,
(B3) $x \vee \Gamma x = 1$,
(B4) $\neg x \wedge \Gamma \neg x = 0$,
(B5) $\Gamma x \vee \neg \Gamma x = 1$,
(B6) $\Gamma(x \wedge y) = \Gamma x \vee \Gamma y$,
(B7) $\neg(x \vee y) = \neg x \wedge \neg y$,
(B8) $\neg(x \wedge \neg y) = \neg x \vee \neg\neg y$,
(B9) $\Gamma(x \vee \Gamma y) = \Gamma x \wedge \Gamma\Gamma y$,
(B10) $(x \vee y) \wedge \Gamma(x \vee y) \leq x \vee \neg x$,
(B11) $x \wedge \Gamma x \wedge y \wedge \Gamma y \leq \Gamma(x \vee y)$.
Where $a \leq b$ *if and only if* $a \wedge b = a$ *or* $a \vee b = b$. *Moreover, the operators* $\neg, \Gamma$ *are defined on* A *by means of the formulas:*
(D3) $\neg x = \sim \nabla x$,
(D4) $\Gamma x = \nabla \sim x$.

Proof. (i) $\Longrightarrow$ (ii)
(B1) $\neg\neg 1 = 1$: [D1, A10]
(B2) $x \wedge \neg x = 0$:

$$\begin{aligned} x \wedge \neg x &= x \wedge \sim \nabla x && \text{[D3]}\\ &= \sim(\sim x \vee \nabla x)\\ &= \sim 1 && \text{[A6]}\\ &= 0. \end{aligned}$$

(B3) $x \vee \Gamma x = 1$:

$$\begin{aligned} x \vee \Gamma x &= x \vee \nabla \sim x && \text{[D4]}\\ &= \sim\sim x \vee \nabla \sim x\\ &= 1. && \text{[A6]} \end{aligned}$$

(B4) $\neg x \wedge \Gamma \neg x = 0$:

$$\begin{aligned} \neg x \wedge \Gamma \neg x &= \sim \nabla x \wedge \nabla \sim\sim \nabla x && \text{[D3, D4]}\\ &= \sim \nabla x \wedge \nabla\nabla x && \text{[A5]}\\ &= \sim \nabla x \wedge \nabla x && \text{[A14]}\\ &= 0. && \text{[A13]} \end{aligned}$$

(B5) $\Gamma x \vee \neg \Gamma x = 1$:

$$\begin{aligned} \Gamma x \vee \neg \Gamma x &= \nabla \sim x \vee \sim \nabla\nabla \sim x && \text{[D3, D4]}\\ &= \nabla \sim x \vee \sim \nabla \sim x && \text{[A14]}\\ &= 1. && \text{[A12]} \end{aligned}$$

(B6) $\Gamma(x \wedge y) = \Gamma x \vee \Gamma y$:

$$\begin{aligned} \Gamma(x \wedge y) &= \nabla \sim (x \wedge y) && \text{[D4]}\\ &= \nabla(\sim x \vee \sim y)\\ &= \nabla \sim x \vee \nabla \sim y && \text{[A17]} \end{aligned}$$

$= \Gamma x \vee \Gamma y.$ [D4]

(B7) $\neg(x \vee y) = \neg x \wedge \neg y$:

$\neg(x \vee y) = \sim \nabla(x \vee y)$ [D3]

$= \sim (\nabla x \vee \nabla y)$ [A17]

$= \sim \nabla x \wedge \sim \nabla y$

$= \neg x \wedge \neg y.$ [D3]

(B8) $\neg(x \wedge \neg y) = \neg x \vee \neg\neg y$:

$\neg(x \wedge \neg y) = \sim \nabla(x \wedge \sim \nabla y)$ [D3]

$= \sim (\nabla x \wedge \nabla \sim \nabla y)$ [A8]

$= \sim \nabla x \vee \sim \nabla \sim \nabla y$

$= \neg x \vee \neg\neg y.$ [D3]

(B9) $\Gamma(x \vee \Gamma y) = \Gamma x \wedge \Gamma\Gamma y$:

$\Gamma(x \vee \Gamma y) = \nabla \sim (x \vee \nabla \sim y)$ [D4]

$= \nabla(\sim x \wedge \sim \nabla \sim y)$

$= \nabla \sim x \wedge \nabla \sim \nabla \sim y$ [A8]

$= \Gamma x \wedge \Gamma\Gamma y.$ [D4]

(B10) $(x \vee y) \wedge \Gamma(x \vee y) \leq x \vee \neg x$:

(1) $x \wedge \Gamma x = x \wedge \sim x$:

$x \wedge \Gamma x = x \wedge \nabla \sim x$ [D4]

$= \sim (\sim x) \wedge \nabla(\sim x)$ [A4]

$= \sim\sim x \wedge \sim x$ [A7]

$= x \wedge \sim x,$ [A4]

(2) $(x \vee y) \wedge \Gamma(x \vee y) \leq \sim x$:

$(x \vee y) \wedge \Gamma(x \vee y) = (x \vee y) \wedge \sim (x \vee y)$ [(1)]

$= (x \vee y) \wedge \sim x \wedge \sim y)$ [A5]

$\leq \sim x,$

(3) $\sim x \leq x \vee \neg x$:

$\sim x \wedge (x \vee \neg x) = \sim x \wedge (x \vee \sim \nabla x)$ [D3]

$= (\sim x \wedge x) \vee (\sim x \wedge \sim \nabla x)$

$= (\sim x \wedge \nabla x) \vee \sim (x \vee \nabla x)$ [A7]

$= (\sim x \wedge \nabla x) \vee \sim \nabla x$ [A9]

$= (\sim x \vee \sim \nabla x) \wedge (\nabla x \vee \sim \nabla x)$

$= \sim (x \wedge \nabla x)$ [A5, A12]

$= \sim x,$ [A9]

(4) $(x \vee y) \wedge \Gamma(x \vee y) \leq x \vee \neg x.$ [(2), (3)]

(B11) $x \wedge \Gamma x \wedge y \wedge \Gamma y \leq \Gamma(x \vee y)$:

$(x \wedge \Gamma x) \wedge (y \wedge \Gamma y) = (x \wedge \sim x) \wedge (y \wedge \sim y)$ [(1) of B10]

$= (x \wedge y) \wedge (\sim x \wedge \sim y)$

$\leq (x \vee y) \wedge \sim (x \vee y)$

$= (x \vee y) \wedge \Gamma(x \vee y)$ [(1) of B10]

$\leq \Gamma(x \vee y).$

(ii) $\Longrightarrow$ (i)

(B12) $\Gamma 0 = 1$:

(1) $x \vee \Gamma x = 1,$ [B3]

(2) $\Gamma 0 = 1.$ [(1)]

(B13) $\neg 1 = 0$:

(1) $1 \wedge \neg 1 = 0,$ [B2]

(2) $\neg 1 = 0.$ [(1)]

(B14) $\neg x \leq \Gamma x$:

$$\begin{aligned}\neg x \wedge \Gamma x &= (\neg x \wedge \Gamma x) \vee 0 \\ &= (\neg x \wedge \Gamma x) \vee (x \wedge \neg x) && [\text{B2}] \\ &= \neg x \wedge (\Gamma x \vee x) \\ &= \neg x \wedge 1 && [\text{B3}] \\ &= \neg x.\end{aligned}$$

(B15) $\neg 0 = 1$:

(1) $0 = 1 \wedge \neg 1$, [B2]

(2) $\neg 0 = \neg(1 \wedge \neg 1)$ [(1)]

$= \neg 1 \vee \neg\neg 1$ [B8]

$= 1.$ [B1]

(B16) $\Gamma 1 = 0$:

$\neg 0 \wedge \Gamma\neg 0 = 0,$ [B4]

$1 \wedge \Gamma 1 = 0,$ [B15]

$\Gamma 1 = 0.$

(B17) $\Gamma x \wedge \Gamma\Gamma x = 0$:

$$\begin{aligned}\Gamma x \wedge \Gamma\Gamma x &= \Gamma(x \vee \Gamma x) && [\text{B9}] \\ &= \Gamma 1 && [\text{B3}] \\ &= 0. && [\text{B16}]\end{aligned}$$

(B18) $\neg x \vee \neg\neg x = 1$:

$$\begin{aligned}\neg x \vee \neg\neg x &= \neg(x \wedge \neg x) && [\text{B8}] \\ &= \neg 0 && [\text{B2}] \\ &= 1. && [\text{B15}]\end{aligned}$$

(B19) $\neg\neg x = \Gamma\neg x$:

(1) $\Gamma\neg x = \Gamma\neg x \wedge 1$

$= \Gamma\neg x \wedge (\neg x \vee \neg\neg x)$ [B18]

$= (\Gamma\neg x \wedge \neg x) \vee (\Gamma\neg x \wedge \neg\neg x)$

$= 0 \vee (\Gamma\neg x \wedge \neg\neg x)$ [B4]

$= \Gamma\neg x \wedge \neg\neg x,$

(2) $\Gamma\neg x \leq \neg\neg x,$ [(1)]

(3) $\neg\neg x \leq \Gamma\neg x,$ [B14]

(4) $\neg\neg x = \Gamma\neg x.$ [(2), (3)]

(B20) $\Gamma\Gamma x = \neg\Gamma x$:

(1) $\neg\Gamma x \leq \Gamma\Gamma x,$ [B14]

(2) $\Gamma x \wedge \Gamma\Gamma x = 0,$ [B17]

(3) $\Gamma x \vee \neg\Gamma x = 1,$ [B5]

(4) $\Gamma\Gamma x = \Gamma\Gamma x \wedge 1 = \Gamma\Gamma x \wedge (\Gamma x \vee \neg\Gamma x) = (\Gamma\Gamma x \wedge \Gamma x) \vee (\Gamma\Gamma x \wedge \neg\Gamma x) = \Gamma\Gamma x \wedge \neg\Gamma x,$ [(2)]

(5) $\Gamma\Gamma x \leq \neg\Gamma x,$ [(4)]

(6) $\Gamma\Gamma x = \neg\Gamma x.$ [(1), (5)]

(B21) $\neg x \wedge \Gamma\Gamma x = 0$:

(1) $\neg x \leq \Gamma x,$ [B14]

(2) $\neg x \wedge \Gamma\Gamma x \leq \Gamma x \wedge \Gamma\Gamma x,$ [(1)]

(3) $\neg x \wedge \Gamma\Gamma x = 0.$ [(2), B17]

(B22) $x \leq \neg\neg x$:

(1) $x \wedge 1 = x \wedge (\neg x \vee \neg\neg x)$ [B18]

$= (x \wedge \neg x) \vee (x \wedge \neg\neg x)$

$= x \wedge \neg\neg x,$ [B2]

(2) $x \leq \neg\neg x.$ [(1)]

(B23) $\Gamma\Gamma x \leq x$:

(1) $x = x \vee \Gamma 1$ [B16]
$= x \vee \Gamma(x \vee \Gamma x)$ [B3]
$= x \vee (\Gamma x \wedge \Gamma\Gamma x)$ [B9]
$= (x \vee \Gamma x) \wedge (x \vee \Gamma\Gamma x)$
$= x \vee \Gamma\Gamma x,$ [B3]
(2) $\Gamma\Gamma x \leq x.$ [(1)]

(B24) $\neg\neg\neg x = \neg x$:
(1) $\neg\neg\neg x = \neg(\neg\neg x)$
$= \neg(\neg\neg x \vee x)$ [B22]
$= \neg\neg\neg x \wedge \neg x,$ [B7]
(2) $\neg\neg\neg x \leq \neg x,$ [(1)]
(3) $\neg x \leq \neg\neg\neg x,$ [B22]
(4) $\neg\neg\neg x = \neg x.$ [(2), (3)]

(B25) $\Gamma\Gamma\Gamma x = \Gamma x$:
(1) $\Gamma\Gamma\Gamma x \leq \Gamma x,$ [B23]
(2) $\Gamma\Gamma\Gamma x = \Gamma(\Gamma\Gamma x)$
$= \Gamma(\Gamma\Gamma x \wedge x)$ [B23]
$= \Gamma\Gamma\Gamma x \vee \Gamma x,$ [B6]
(3) $\Gamma x \leq \Gamma\Gamma\Gamma x,$ [(2)]
(4) $\Gamma x = \Gamma\Gamma\Gamma x.$ [(1), (3)]

(B26) $\neg\Gamma x \leq x$:
(1) $\Gamma\Gamma x = \neg\Gamma x,$ [B20]
(2) $\Gamma\Gamma x \leq x,$ [B23]
(3) $\neg\Gamma x \leq x.$ [(1), (2)]

(B27) $\Gamma\Gamma\neg x = \neg x$:
$\Gamma\Gamma\neg x = \neg\Gamma\neg x$ [B20]
$= \neg\neg\neg x$ [B19]
$= \neg x.$ [B24]

(B28) $\Gamma\Gamma\Gamma\neg x = \neg\neg x$:
$\Gamma\Gamma\Gamma\neg x = \Gamma\neg x$ [B27]
$= \neg\neg x.$ [B19]

(B29) $\Gamma((x \vee \neg x) \wedge \Gamma x) = \neg\neg x$:
$\Gamma((x \vee \neg x) \wedge \Gamma x) = \Gamma(x \vee \neg x) \vee \Gamma\Gamma x$ [B6]
$= \Gamma(x \vee \Gamma\Gamma\neg x) \vee \Gamma\Gamma x$ [B27]
$= (\Gamma x \wedge \neg\neg x) \vee \Gamma\Gamma x$ [B9, B28]
$= (\Gamma x \vee \Gamma\Gamma x) \wedge (\neg\neg x \vee \Gamma\Gamma x)$
$= \neg\neg x.$ [B3, B22, B23]

(B30) $\neg\neg\Gamma x = \Gamma x$:
$\neg\neg\Gamma x = \Gamma\neg\Gamma x$ [B19]
$= \Gamma\Gamma\Gamma x$ [B20]
$= \Gamma x.$ [B25]

(B31) $\neg\neg\neg\Gamma x = \Gamma\Gamma x$:
$\neg\neg\neg\Gamma x = \Gamma\Gamma\Gamma\neg\neg\Gamma x$ [B28]
$= \Gamma\neg\neg\Gamma x$ [B25]
$= \neg\neg\neg\Gamma x$ [B19]
$= \neg\Gamma x$ [B24]
$= \Gamma\Gamma x.$ [B20]

(B32) $\neg((x \wedge \Gamma x) \vee \neg x) = \Gamma\Gamma x$:
$\neg((x \wedge \Gamma x) \vee \neg x) = \neg(x \wedge \Gamma x) \wedge \neg\neg x$ [B7]

$= \neg(x \wedge \neg\neg\Gamma x) \wedge \neg\neg x$ [B30]
$= (\neg x \vee \Gamma\Gamma x) \wedge \neg\neg x$ [B8, B31]
$= \Gamma\Gamma x.$ [B2, B22, B23]

(B33) $\Gamma\neg\neg x = \neg x$:
$\Gamma\neg\neg x = \neg\neg\neg x$ [B19]
$= \neg x.$ [B24]

Now we are able to prove the axioms A4, A6 and A7.

Axiom A4 $\sim\sim x = x$:
First, we observe that from B14 and D2 we obtain: (D3) $\sim x = (x \wedge \Gamma x) \vee \neg x$. Then
(1) $\sim\sim x = (((x \wedge \Gamma x) \vee \neg x) \wedge \Gamma((x \wedge \Gamma x) \vee \neg x)) \vee \neg((x \wedge \Gamma x) \vee \neg x),$ [D3]
(2) $\Gamma((x \wedge \Gamma x) \vee \neg x)) = \Gamma((x \vee \neg x) \wedge (\Gamma x \vee \neg x))$
$= \Gamma((x \vee \neg x) \wedge \Gamma x)$ [B14]
$= \neg\neg x,$ [B29]
(3) $\sim\sim x = (((x \wedge \Gamma x) \vee \neg x) \wedge \neg\neg x) \vee \Gamma\Gamma x$ [(1), (2), B32]
$= (((x \wedge \Gamma x) \wedge \neg\neg x) \vee (\neg x \wedge \neg\neg x)) \vee \Gamma\Gamma x$
$= ((x \wedge \Gamma x) \wedge \neg\neg x) \vee \Gamma\Gamma x$ [B2]
$= ((x \wedge \Gamma x) \vee \Gamma\Gamma x) \wedge (\neg\neg x \vee \Gamma\Gamma x)$
$= ((x \wedge \Gamma x) \vee \Gamma\Gamma x) \wedge \neg\neg x$ [B22, B23]
$= ((x \vee \Gamma\Gamma x) \wedge (\Gamma x \vee \Gamma\Gamma x)) \wedge \neg\neg x$
$= (x \vee \Gamma\Gamma x) \wedge \neg\neg x$ [B3]
$= x \wedge \neg\neg x$ [B23]
$= x.$ [B22]

Axiom A6 $\sim x \vee \nabla x = 1$:
$\sim x \vee \nabla x = (x \wedge \Gamma x) \vee \neg x \vee \neg\neg x$ [D3, D1]
$= (x \wedge \Gamma x) \vee 1 = 1.$ [B18]

Axiom A7 $x \wedge \sim x = \sim x \wedge \nabla x$:
$\sim x \wedge \nabla x = ((x \wedge \Gamma x) \vee \neg x) \wedge \neg\neg x$ [D3, D1]
$= (x \wedge \Gamma x \wedge \neg\neg x) \vee (\neg x \wedge \neg\neg x)$
$= (x \wedge \Gamma x) \vee 0$ [B22, B2]
$= (x \wedge \Gamma x) \vee (x \wedge \neg x)$ [B2]
$= ((x \wedge \Gamma x) \vee x) \wedge ((x \wedge \Gamma x) \vee \neg x)$
$= x \wedge \sim x.$ [D3]

(B34) If $x \leq y$ then $\neg y \leq \neg x$ and $\Gamma y \leq \Gamma x$:
(1) $x \leq y,$ [Hip.]
(2) $x \vee y = y,$ [(1)]
(3) $\neg(x \vee y) = \neg y,$ [(2)]
(4) $\neg x \wedge \neg y = \neg y,$ [(3), B7]
(5) $\neg y \leq \neg x,$ [(4)]
(6) $x \wedge y = x,$ [(1)]
(7) $\Gamma(x \wedge y) = \Gamma x,$ [(6)]
(8) $\Gamma x \vee \Gamma y = \Gamma x,$ [(7), B6]
(9) $\Gamma y \leq \Gamma x.$ [(8)]

(B35) $\sim \Gamma x = \Gamma\Gamma x$:
$\sim \Gamma x = \Gamma\Gamma x \wedge (\Gamma x \vee \neg\Gamma x)$ [D2]
$= \Gamma\Gamma x \wedge (\Gamma x \vee \Gamma\Gamma x)$ [B20]
$= \Gamma\Gamma x.$ [A2]

(B36) $\sim (\neg x \wedge \Gamma y) = \neg\neg x \vee \Gamma\Gamma y$:
(1) $\sim (\neg x \wedge \Gamma y) = \Gamma(\neg x \wedge \Gamma y) \wedge ((\neg x \wedge \Gamma y) \vee \neg(\neg x \wedge \Gamma y)),$ [D2]

On the other hand

(2) $\Gamma(\neg x \wedge \Gamma y) = \Gamma\neg x \vee \Gamma\Gamma y$ [B6]

$= \neg\neg x \vee \Gamma\Gamma y,$ [B19]

and

(3) $\neg(\neg x \wedge \Gamma y) = \neg\Gamma y \vee \neg\neg x$ [B8]

$= \neg\neg x \vee \Gamma\Gamma y,$ [B20]

Then B36 follows from (1), (2) and (3).

(B37) $\Gamma\Gamma(y \vee \neg\neg x) = \Gamma\Gamma y \vee \neg\neg x$:

$\Gamma\Gamma y \vee \neg\neg x = \Gamma\Gamma y \vee \Gamma\Gamma\neg\neg x$ [B27]

$= \Gamma(\Gamma y \wedge \Gamma\neg\neg x)$ [B6]

$= \Gamma(\Gamma y \wedge \neg x)$ [B33]

$= \Gamma(\Gamma y \wedge \Gamma\Gamma\neg x)$ [B27]

$= \Gamma\Gamma(y \vee \Gamma\neg x)$ [B9]

$= \Gamma\Gamma(y \vee \neg\neg x).$ [B19]

(B38) If $x \leq y$ then $\sim y \leq \sim x$:

Let x, y be such that

(1) $x \leq y,$

Then

(2) $\sim y \vee \sim x = (y \wedge \Gamma y) \vee \neg y \vee \neg x \vee (x \wedge \Gamma x)$ [D3]

$= (y \wedge \Gamma y) \vee \neg x \vee (x \wedge \Gamma x)$ [(1), B34]

$= \neg x \vee ((y \vee x) \wedge (y \vee \Gamma x) \wedge (\Gamma y \vee x) \wedge (\Gamma y \vee \Gamma x))$

$= \neg x \vee (y \wedge (y \vee \Gamma x) \wedge (\Gamma y \vee x) \wedge \Gamma x),$ ((1), B34)

(3) $\sim y \vee \sim x = \neg x \vee (y \wedge (\Gamma y \vee x) \wedge \Gamma x)$ [(2)]

$= \neg x \vee (\Gamma x \wedge ((y \wedge \Gamma y) \vee (y \wedge x)))$

$= \Gamma x \wedge (\neg x \vee (y \wedge \Gamma y) \vee x),$ [(1), B14]

(4) $y \wedge \Gamma y \leq x \vee \neg x,$ [(1), B10]

Then

(5) $\sim y \vee \sim x = \Gamma x \wedge (\neg x \vee x) = \sim x,$ [(3), (4), D2]

(6) $\sim y \leq \sim x.$ [(5)]

(B39) $\neg x \wedge \Gamma y \leq \Gamma(x \vee y)$:

(1) $\neg x \wedge \Gamma y = \Gamma\Gamma\neg x \wedge \Gamma y$ [B27]

$= \Gamma(y \vee \Gamma\neg x),$ [B9]

(2) $x \leq \neg\neg x$ [B22]

$= \Gamma\neg x,$ [B19]

(3) $\Gamma(y \vee \Gamma\neg x) \vee \Gamma(x \vee y) = \Gamma((y \vee \Gamma\neg x) \wedge (x \vee y))$ [B6]

$= \Gamma(((y \vee \Gamma\neg x) \wedge x) \vee ((y \vee \Gamma\neg x) \wedge y))$

$= \Gamma(((y \vee \Gamma\neg x) \wedge x) \vee y)$

$= \Gamma(((y \vee \Gamma\neg x) \vee y) \wedge (x \vee y))$

$= \Gamma((y \vee \Gamma\neg x) \wedge (x \vee y))$

$= \Gamma(y \vee (\Gamma\neg x \wedge x))$

$= \Gamma(y \vee x),$ [(2)]

(4) $\Gamma(y \vee \Gamma\neg x) \leq \Gamma(x \vee y),$ [(3)]

(5) $\neg x \wedge \Gamma y \leq \Gamma(x \vee y).$ [(1), (4)]

(B40) $\neg x \wedge \sim y \leq (x \vee y) \vee \neg(x \vee y)$:

(1) $\neg x \wedge \sim y \wedge ((x \vee y) \vee \neg(x \vee y)) = \sim y \wedge ((\neg x \wedge (x \vee y)) \vee (\neg x \wedge \neg(x \vee y)))$

$= \sim y \wedge ((\neg x \wedge y) \vee (\neg x \wedge \neg y))$ [B2, B7]

$= \sim y \wedge \neg x \wedge (y \vee \neg y)$

$= \neg x \wedge \sim y,$ [D2]

(2) $\neg x \wedge \sim y \leq (x \vee y) \vee \neg(x \vee y).$ [(1)]

(B41) $x \wedge \Gamma x \wedge \sim y \leq \Gamma\,(x \vee y)$:

$$
\begin{aligned}
x \wedge \Gamma x \wedge \sim y &= x \wedge \Gamma x \wedge \Gamma y \wedge (y \vee \neg y) && \text{[D2]}\\
&= (x \wedge \Gamma x \wedge \Gamma y \wedge y) \vee (x \wedge \Gamma x \wedge \Gamma y \wedge \neg y)\\
&= (x \wedge \Gamma x \wedge \Gamma y \wedge y) \vee (x \wedge \Gamma x \wedge \neg y) && \text{[B14]}\\
&\leq \Gamma(x \vee y) \vee (x \wedge \Gamma x \wedge \neg y) && \text{[B11]}\\
&\leq \Gamma(x \vee y) \vee (x \wedge \Gamma(x \vee y)) && \text{[B39]}\\
&= \Gamma(x \vee y). && \text{[A2]}
\end{aligned}
$$

(B42) $\neg x \wedge \sim y \leq \Gamma\,(x \vee y)$:

(1) $\neg x \wedge \sim y \wedge \Gamma(x \vee y) = \neg x \wedge (y \vee \neg y) \wedge \Gamma y \wedge \Gamma(x \vee y)$ [D2]
$= \neg x \wedge \Gamma y \wedge (y \vee \neg y)$ [B39]
$= \neg x \wedge \sim y,$ [D2]

(2) $\neg x \wedge \sim y \leq \Gamma(x \vee y).$ [(1)]

Now we can prove the axiom A5.

Axiom A5 $\sim (x \vee y) = \sim x \wedge \sim y$:

(1) $\sim x \wedge \sim y = ((x \vee \neg x) \wedge \Gamma x) \wedge \sim y$ [D2]
$= (x \vee \neg x) \wedge (\Gamma x \wedge \sim y)$
$= (x \wedge \Gamma x \wedge \sim y) \vee (\sim y \wedge \neg x \wedge \Gamma x)$
$= (x \wedge \Gamma x \wedge \sim y) \vee (\sim y \wedge \neg x),$ [B14]

(2) $x \wedge \Gamma x \wedge \sim y \leq \Gamma(x \vee y),$ [B41]

(3) $\sim y \wedge \Gamma x \leq (x \vee y) \vee \sim (x \vee y),$ [B40]

(4) $\sim x \wedge \sim y \leq \Gamma(x \vee y) \wedge ((x \vee y) \vee \neg(x \vee y)),$ [(1), (2), (3)]

(5) $\sim x \wedge \sim y \leq \sim (x \vee y),$ [(4), D2]

(6) $x \leq x \vee y,$

(7) $y \leq x \vee y,$

(8) $\sim (x \vee y) \leq \sim x,$ [(6), B38]

(9) $\sim (x \vee y) \leq \sim y,$ [(7), B38]

(10) $\sim (x \vee y) \leq \sim x \wedge \sim y,$ [(8), (9)]

(11) $\sim x \wedge \sim y = \sim (x \vee y).$ [(5), (10)]

Therefore, $\langle A, \wedge, \vee, \sim, \nabla, 1\rangle$ is a 4-valued modal algebra. □

3. Other characterizations

The following characterization of 4–valued modal algebras is easier than that given in Theorem 2.1.

Theorem 3.1. *Let* $(A, \wedge, \vee, \sim, \neg, 1)$ *be an algebra of type* (2,2,1,1,0) *where* $(A, \wedge, \vee, \sim, 1)$ *is a De Morgan algebra with last element* 1 *and first element* $0 = \sim 1$. *If* ∇ *is an unary operation defined on* A *by means of the formula* $\nabla x = \sim \neg x$, *then* A *is a* 4*–valued modal algebra if and only if it verifies:*

(T1) $x \wedge \neg x = 0,$

(T2) $x \vee \neg x = x \vee \sim x.$

Furthermore $\neg x = \sim \nabla x$.

Proof. We check only sufficient condition

(A6) $\sim x \vee \nabla x = \sim x \vee \sim \neg x = \sim (x \wedge \neg x) = 1$ [T1]

(A7) $\sim x \wedge \nabla x = \sim x \wedge \sim \neg x$ [T2]
$= \sim (x \vee \neg x)$
$= \sim (x \vee \sim x)$
$= x \wedge \sim x$

□

Remark 3.1. In a 4–valued modal algebra the operation considered in Theorem 2.1, generally does not coincide with the pseudo-complement * as we can verify in the following example:

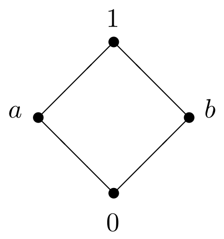

Figure 1.

x	$\sim x$	∇x
0	1	0
a	a	1
b	b	1
1	0	1

Table 1.

we have

x	$\neg x$	x^*
0	1	1
a	0	b
b	0	a
1	0	0

Table 2.

However, every finite 4–valued modal algebra is a distributive lattice pseudo complemented. We do not know whether this situation holds in the non-finite case. This suggests that we consider a particular class of De Morgan algebras.

Definition 3.1. An algebra $(A, \wedge, \vee, \sim, ^*, 1)$ of type (2,2,1,1,0) is a *modal De Morgan p–algebra* if the reduct $(A, \wedge, \vee, \sim, 1)$ is a De Morgan algebra with last element 1 and first element $0 = \sim 1$, the reduct is a pseudo–complemented meet–lattice and the following condition is verified

$$\text{H1) } x \vee \sim x \leq x \vee x^*$$

Example 3.1. The De Morgan algebra whose Hasse diagram is given in Figure 2 and the operations $\sim$ and * are defined in Table 3.

Figure 2.

x	$\sim x$	x^*
0	1	1
a	b	0
b	a	0
1	0	0

Table 3.

is not a modal De Morgan p–algebra because $b = (a \vee \sim a) \not\leq a \vee a^* = a$.

Theorem 3.2. *If we define on a modal De Morgan p–algebra* $\langle A, \wedge, \vee, \sim, ^*, 1\rangle$ *the operation* $\neg$ *by means of the formula* $\neg x = x^* \wedge \sim x$ *then the algebra* $\langle A, \wedge, \vee, \neg, 1\rangle$ *verifies the identities* T1 *and* T2.

Proof.

(T1) $x \wedge \neg x = x \wedge x^* \wedge \sim x = 0 \wedge \sim x = 0$

(T2) $x \vee \neg x = x \vee (x^* \wedge \sim x) = (x \vee x^*) \wedge (x \vee \sim x)$
$= (x \vee \sim x)$ [H1]

□

Remark 3.2. By [4] we know that every finite modal 4–valued algebra A is direct product of copies of T2, T3 and T4, where T2={0,1}, T3={0,a,1} and T4={0,a,b,1} are modal De Morgan p–algebra we conclude that A is also a modal De Morgan p–algebra.

We do not know whether this situation holds in the non-finite case.

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