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Some Hilbert-type inequalities on time scales

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Abstract. A time scale version of the Hilbert inequality is presented, which unifies and extends well-known Hilbert inequalities in the continuous and in the discrete setting.

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1. Introduction

Few years ago, M. Krnić and J. Pečarić [4], provided an unified treatment of the Hilbert and Hardy-Hilbert type inequalities in general form and extended them to cover the case when p and q are conjugate exponents. More precisely, they obtained the following two equivalent inequalities:

$$\begin{aligned}&\int_{\Omega\times\Omega} K(x,y)f(x)g(y)d\mu_1(x)d\mu_2(y) \\ &\quad\le \left[\int_\Omega \varphi^p(x)F(x)f^p(x)d\mu_1(x)\right]^{\frac{1}{p}}\left[\int_\Omega \psi^q(y)G(y)g^q(y)d\mu_2(y)\right]^{\frac{1}{q}}\end{aligned}\tag{1.1}$$

and

$$\int_\Omega G^{1-p}(y)\psi^{-p}(y)\left[\int_\Omega K(x,y)f(x)d\mu_1(x)\right]^p d\mu_2(y) \le \int_\Omega \varphi^p(x)F(x)f^p(x)d\mu_1(x),\tag{1.2}$$

where $p>1$, μ_1,μ_2 are positive σ-finite measures, $K:\Omega\times\Omega\to\mathbb{R}$, $f,g,\varphi,\psi:\Omega\to\mathbb{R}$ are measurable, non-negative functions and

$$F(x)=\int_\Omega \frac{K(x,y)}{\psi^p(y)}d\mu_2(y) \quad \text{and} \quad G(y)=\int_\Omega \frac{K(x,y)}{\varphi^q(x)}d\mu_1(x).\tag{1.3}$$

Before we present our main result, let us recall essentials about time scales. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. Let $a,b\in\mathbb{T}$. The interval $[a,b]$ in time scale $\mathbb{T}$ is defined by $[a,b]:=\{t\in\mathbb{T}: a\le t\le b\}$. We define the forward jump operator σ by $\sigma(t):=\inf\{s\in\mathbb{T}: s>t\}$, and the graininess μ of the time scale $\mathbb{T}$ by $\mu(t):=\sigma(t)-t$. A point $t\in\mathbb{T}$ is said to be right-dense, right-scattered, if $\sigma(t)=t$, $\sigma(t)>t$, respectively. We define $f^\sigma:=f\circ\sigma$. For a function $f:\mathbb{T}\to\mathbb{R}$ the delta derivative is defined by

$$f^\Delta(t):=\lim_{s\to t,\sigma(s)\ne t}\frac{f^\sigma(s)-f(t)}{\sigma(s)-t}.$$

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Here are some basic formulas involving delta derivatives: $f^\sigma = f + \mu f^\Delta$, $(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta = f^\Delta g^\sigma + f g^\Delta$, $(f/g)^\Delta = (f^\Delta g - f g^\Delta)/(g g^\sigma)$, where f, g are delta differentiable and $gg^\sigma \neq 0$ in the last formula. A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$. The classes of real rd-continuous functions on an interval I will be denoted by $C_{rd}(I, \mathbb{R})$. For a, $b \in \mathbb{T}$ and a delta differentiable function f, the Cauchy integral is defined by $\int_a^b f^\Delta(t)\Delta t = f(b) - f(a)$. For the concept of the Riemann delta integral and the Lebesgue delta integral, see [2]. Note that the definition of the Riemann delta integrability is similar to the classical one of a real variable, and that the Lebesgue delta integral is the Lebesgue integral associated with the so-called Lebesgue delta measure. Every rd-continuous function is Riemann delta integrable, and every Riemann delta integrable function is Lebesgue delta integrable. Throughout, for convenience, when we speak about a delta integrability, we mean the integrability in some of the above senses. The integration by parts formula is given by:

$$\int_a^b u(t)v^\Delta(t)\Delta t = [u(t)v(t)]_a^b - \int_a^b u^\Delta(t)v^\sigma(t)\Delta t. \tag{1.4}$$

The chain rule formula (see [1], Theorem 1.90) that we will use in this paper is

$$(u^\gamma(t))^\Delta = \gamma \left(\int_0^1 [hu^\sigma(t) + (1-h)u(t)]^{\gamma-1} dh \right) u^\Delta(t), \tag{1.5}$$

where $\gamma > 1$ and $u : \mathbb{T} \to \mathbb{R}$ is delta differentiable function.

Our results will be based on the mentioned results of Krnić and Pečarić. First step is to reformulate the inequalities (1.1) and (1.2) for time scales. Namely, rewriting inequalities (1.1) and (1.2) for Lebesgue delta measures Δx, Δy and time scale interval $[a, b]$, we have

$$\int_a^b \int_a^b K(x,y)f(x)g(y)\Delta x \Delta y \leq \left[\int_a^b \varphi^p(x)F(x)f^p(x)\Delta x \right]^{\frac{1}{p}} \left[\int_a^b \psi^q(y)G(y)g^q(y)\Delta y \right]^{\frac{1}{q}} \tag{1.6}$$

and

$$\int_a^b G^{1-p}(y)\psi^{-p}(y) \left[\int_a^b K(x,y)f(x)\Delta x \right]^p \Delta y \leq \int_a^b \varphi^p(x)F(x)f^p(x)\Delta x, \tag{1.7}$$

where $p > 1$, $K : [a,b] \times [a,b] \to \mathbb{R}$, $f, g, \varphi, \psi : [a,b] \to \mathbb{R}$ are delta measurable, non-negative functions and

$$F(x) = \int_a^b \frac{K(x,y)}{\psi^p(y)} \Delta y \quad \text{and} \quad G(y) = \int_a^b \frac{K(x,y)}{\varphi^q(x)} \Delta x. \tag{1.8}$$

In what follows, without further explanation, we assume that all integrals exist on the respective domains of their definitions.

2. Main results

By applying the inequalities (1.6) and (1.7) we obtain the following result.

Theorem 2.1. *Let $\mathbb{T}$ be a time scale with $a \in \mathbb{T}$. Let $\lambda \geq 2$, $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$, and define*

$$\Lambda(x) := \int_a^\infty \frac{1}{\sigma(y)} \left(\frac{1}{(x+\sigma(y))^\lambda (x+y)} + \frac{1}{(x+y)^\lambda (x+\sigma(y))} \right) \Delta y, \qquad x \in [a, \infty).$$

Then the following inequality

$$\int_a^\infty \int_a^\infty \frac{f(x)g(y)}{(x+y)^\lambda} \Delta x \Delta y \leq \left(\int_a^\infty [x\sigma(x)]^{p-1} \left(\frac{1}{a(a+x)^\lambda} - \Lambda(x) \right) f^p(x) \Delta x \right)^{\frac{1}{p}} \times \left(\int_a^\infty [y\sigma(y)]^{q-1} \left(\frac{1}{a(a+y)^\lambda} - \Lambda(y) \right) g^q(y) \Delta y \right)^{\frac{1}{q}} \tag{2.1}$$

holds for all non-negative and delta measurable functions $f, g : \mathbb{T} \to \mathbb{R}$.

Proof. Rewrite the inequality (1.6) for the functions $K(x,y) = (x+y)^{-\lambda}$, $\lambda \geq 2$, $\varphi(x) = [x\sigma(x)]^{1/q}$, $\psi(y) = [y\sigma(y)]^{1/p}$, $x, y \in [a, \infty)$. Further, making use of (1.8), it follows that

$$F(x) = G(x) = \int_a^\infty \frac{1}{y\sigma(y)} \frac{1}{(x+y)^\lambda} \Delta y, \qquad x \in [a, \infty). \tag{2.2}$$

Using the integration by parts formula (1.4) on the term $F(x)$ with

$$u^\lambda(y) = \frac{1}{(x+y)^\lambda} \qquad \text{and} \qquad v^\Delta(y) = \frac{1}{y\sigma(y)},$$

we have

$$F(x) = u^\lambda v|_a^\infty - \int_a^\infty (u^\lambda(y))^\Delta v^\sigma(y) \Delta y, \tag{2.3}$$

where

$$v(y) = -\frac{1}{y} \qquad \text{and} \qquad v^\sigma(y) = -\frac{1}{\sigma(y)}. \tag{2.4}$$

Applying the chain rule (1.5) we obtain

$$(u^\lambda(y))^\Delta = \lambda \left(\int_0^1 [hu^\sigma + (1-h)u]^{\lambda-1} dh \right) u^\Delta(y), \tag{2.5}$$

where

$$u^\Delta(y) = -\frac{1}{(x+y)(x+\sigma(y))}. \tag{2.6}$$

Taking into account (2.5) and the well-known inequality

$$(a+b)^\gamma \geq a^\gamma + b^\gamma, \;\; a, b \geq 0, \;\; \gamma \geq 1,$$

we observe that

$$\begin{aligned}\lambda \int_0^1 \left[\frac{h}{x+\sigma(y)} + \frac{1-h}{x+y} \right]^{\lambda-1} dh &\geq \lambda \int_0^1 \left[\left(\frac{h}{x+\sigma(y)} \right)^{\lambda-1} + \left(\frac{1-h}{x+y} \right)^{\lambda-1} \right] dh \\ &= \frac{1}{(x+\sigma(y))^{\lambda-1}} + \frac{1}{(x+y)^{\lambda-1}},\end{aligned}$$

and therefore

$$\begin{aligned} F(x) &\le u^{\lambda} v|_a^{\infty} - \int_a^{\infty} \frac{1}{\sigma(y)} \left(\frac{1}{(x+\sigma(y))^{\lambda-1}} + \frac{1}{(x+y)^{\lambda-1}} \right) \frac{1}{(x+y)(x+\sigma(y))} \Delta y \\ &= \frac{1}{a(a+x)^{\lambda}} - \Lambda(x). \end{aligned} \tag{2.7}$$

Finally, using (1.6) and (2.7) we obtain (2.1). □

The Hardy-Hilbert type inequality is proved in the following theorem.

Theorem 2.2. *Let $\mathbb{T}$ be a time scale with $a \in \mathbb{T}$. Let $\lambda \ge 2$, $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$, and let Λ be defined as in the statement of Theorem 2.1. Then the following inequality*

$$\begin{aligned} &\int_a^{\infty} \frac{1}{y\sigma(y)} \left(\frac{1}{a(a+y)^{\lambda}} - \Lambda(y) \right)^{1-p} \left[\frac{f(x)}{(x+y)^{\lambda}} \Delta x \right]^p \Delta y \\ &\quad \le \int_a^{\infty} [x\sigma(x)]^{p-1} \left(\frac{1}{a(a+x)^{\lambda}} - \Lambda(x) \right) f^p(x) \Delta x \end{aligned} \tag{2.8}$$

holds for all non-negative and delta measurable functions $f : \mathbb{T} \to \mathbb{R}$.

Proof. The proof follows directly from the inequalities (1.7) and (2.7). Namely, if $p > 1$, then we have

$$\left[\frac{1}{a(a+y)^{\lambda}} - \Lambda(y) \right]^{1-p} \le G^{1-p}(y),$$

where $G(y)$ is defined by (2.2). Now, the inequality (2.8) follows easily from (1.7). □

Remark 2.1. For $\mathbb{T} = \mathbb{R}$, we have $\sigma(y) = y$, $y \in \mathbb{R}$, and the term $\Lambda(x)$ defined in Theorem 2.1 takes form

$$\Lambda(x) = 2 \int_a^{\infty} \frac{dy}{y(x+y)^{\lambda+1}}, \qquad x \in [a, \infty),\ a \in \mathbb{R}_+.$$

For example, if $a = 1$, $\lambda \ge 2$, then applying the inequality (2.1) we obtain the following result

$$\begin{aligned} &\int_1^{\infty} \int_1^{\infty} \frac{f(x)g(y)}{(x+y)^{\lambda}} dx dy \\ &\quad \le \left(\int_1^{\infty} x^{2(p-1)} \left(\frac{1}{(x+1)^{\lambda}} - 2F(1+\lambda, 1+\lambda; 2+\lambda; -x) \right) f^p(x) dx \right)^{\frac{1}{p}} \\ &\quad \times \left(\int_1^{\infty} y^{2(q-1)} \left(\frac{1}{(y+1)^{\lambda}} - 2F(1+\lambda, 1+\lambda; 2+\lambda; -y) \right) g^q(y) dy \right)^{\frac{1}{q}}, \end{aligned}$$

where $F(\alpha, \beta; \gamma; z)$ denotes the Gaussian hypergeometric function defined by

$$F(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha} dt,\ \gamma > \beta > 0, z < 1.$$

Remark 2.2. Similarly, for $\mathbb{T} = \mathbb{N}$, $a \in \mathbb{N}$, we obtain

$$\Lambda(n) = \sum_{s=a}^{\infty} \frac{1}{s+1} \left(\frac{1}{(n+s+1)^{\lambda}(n+s)} + \frac{1}{(n+s)^{\lambda}(n+s+1)} \right),\ n \in \mathbb{N},$$

and the inequalities (2.1) and (2.8) become

$$\sum_{m=a}^{\infty}\sum_{n=a}^{\infty}\frac{f(m)g(n)}{(m+n)^{\lambda}} \leq \left(\sum_{m=a}^{\infty}[m(m+1)]^{p-1}\left(\frac{1}{a(a+m)^{\lambda}}-\Lambda(m)\right)f^{p}(m)\right)^{\frac{1}{p}} \times \left(\sum_{n=a}^{\infty}[n(n+1)]^{q-1}\left(\frac{1}{a(a+n)^{\lambda}}-\Lambda(n)\right)g^{q}(n)\right)^{\frac{1}{q}}$$

and

$$\sum_{n=a}^{\infty}\frac{1}{n+1}\left(\frac{1}{a(a+n)^{\lambda}}-\Lambda(n)\right)^{1-p}\left[\sum_{m=a}^{\infty}\frac{f(m)}{(m+n)^{\lambda}}\right]^{p} \leq \sum_{m=a}^{\infty}[m(m+1)]^{p-1}\left(\frac{1}{a(a+m)^{\lambda}}-\Lambda(m)\right)f^{p}(m).$$

Now, our further step is to derive corresponding inequalities for the kernel $K(x,y) = (1+xy)^{-\lambda}$, $\lambda > 0$, and the weight functions $\varphi^{q}(x) = \psi^{p}(x) = x\sqrt{\sigma(x)} + \sigma(x)\sqrt{x}$.

Proceeding as in the proof of Theorem 2.1, we can prove the following result.

Theorem 2.3. *Let $\mathbb{T}$ be a time scale with $a \in \mathbb{T}$. Let $\lambda \geq 0$, $\frac{1}{p}+\frac{1}{q}=1$ with $p>1$, and define*

$$\Lambda(x) := \int_{a}^{\infty}\frac{x}{\sqrt{\sigma(y)}}\left(\frac{1}{(1+x\sigma(y))^{\lambda}(1+xy)}+\frac{1}{(1+xy)^{\lambda}(1+x\sigma(y))}\right)\Delta y, \quad x \in [a,\infty).$$

Then the following inequality

$$\int_{a}^{\infty}\int_{a}^{\infty}\frac{f(x)g(y)}{(1+xy)^{\lambda}}\Delta x\Delta y \leq \left(\int_{a}^{\infty}[x\sqrt{\sigma(x)}+\sigma(x)\sqrt{x}]^{p-1}\left(\frac{1}{\sqrt{a}(1+ax)^{\lambda}}-\Lambda(x)\right)f^{p}(x)\Delta x\right)^{\frac{1}{p}} \times \left(\int_{a}^{\infty}[y\sqrt{\sigma(y)}+\sigma(y)\sqrt{y}]^{q-1}\left(\frac{1}{\sqrt{a}(1+ay)^{\lambda}}-\Lambda(y)\right)g^{q}(y)\Delta y\right)^{\frac{1}{q}} \tag{2.9}$$

holds for all non-negative and delta measurable functions $f,g:\mathbb{T}\to\mathbb{R}$.

In the following, instead of the formula (1.5) we use the chain rule (see [1], Theorem 1.87):

$$(f\circ g)^{\Delta}(t) = f'(g(c))g^{\Delta}(t), \text{ for some } c \in [t,\sigma(t)], \tag{2.10}$$

where $g:\mathbb{R}\to\mathbb{R}$ is continuous, $g:\mathbb{T}\to\mathbb{R}$ is delta differentiable and $f:\mathbb{R}\to\mathbb{R}$ is continuously differentiable.

Theorem 2.4. *Let $\mathbb{T}$ be a time scale with $a \in \mathbb{T}$. Let $\lambda \geq 0$, $\frac{1}{p}+\frac{1}{q}=1$ with $p>1$, and define*

$$\Lambda(x) := \lambda\int_{a}^{\infty}\frac{1}{\sigma(y)}\frac{1}{(x+\sigma(y))^{\lambda+2}(x+\sigma(y))}\Delta y, \qquad x \in [a,\infty).$$

Then the following inequalities

$$\int_a^\infty \int_a^\infty \frac{f(x)g(y)}{(x+y)^\lambda}\Delta x\Delta y \le \left(\int_a^\infty [x\sigma(x)]^{p-1}\left(\frac{1}{a(a+x)^\lambda}+\Lambda(x)\right) f^p(x)\Delta x\right)^{\frac{1}{p}} \times \left(\int_a^\infty [y\sigma(y)]^{q-1}\left(\frac{1}{a(a+y)^\lambda}+\Lambda(y)\right) g^q(y)\Delta y\right)^{\frac{1}{q}} \tag{2.11}$$

and

$$\int_a^\infty \frac{1}{y\sigma(y)}\left(\frac{1}{a(a+y)^\lambda}+\Lambda(y)\right)^{1-p}\left[\int_a^\infty \frac{f(x)}{(x+y)^\lambda}\Delta x\right]^p \Delta y \le \int_a^\infty [x\sigma(x)]^{p-1}\left(\frac{1}{a(a+x)^\lambda}+\Lambda(x)\right) f^p(x)\Delta x \tag{2.12}$$

hold for all non-negative and delta measurable functions $f, g : \mathbb{T} \to \mathbb{R}$.

Proof. We prove (2.11) only. To show this, we follow the same procedure as in the proof of Theorem 2.1 except that we provide a new estimate for the functions $F(x)$ and $G(x)$ defined by (2.2).

More precisely, from the inequality (2.3) we get

$$F(x) = \frac{1}{a(a+x)^\lambda} + \int_a^\infty (u^\lambda(y))^\Delta \frac{1}{\sigma(y)}\Delta y, \quad x \in [a,\infty), \tag{2.13}$$

where $u(y) = 1/(x+y)$. Using (2.6) and (2.10) we have

$$(u^\lambda(y))^\Delta = \frac{\lambda}{(x+c)^{\lambda+1}(x+y)(x+\sigma(y))}, \text{ for some } c \in [y, \sigma(y)],$$

and therefore

$$(u^\lambda(y))^\Delta \le \frac{\lambda}{(x+y)^{\lambda+2}(x+\sigma(y))}. \tag{2.14}$$

Finally, making use of (1.6), (2.13) and (2.14) we obtain (2.11). □

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