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On a theorem of Ţăndăreanu and Tudor

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Abstract. For an operator domain $\mathbf{\Sigma}$, which has exactly one binary operator symbol σ, and a set M, Ţăndăreanu and Tudor have defined a homomorphism f_M from the inf-semi-lattice $\mathbf{Sub}(M)$, where $\mathrm{Sub}(M)$, the underlying set of $\mathbf{Sub}(M)$, is the set of all subsets of M, to the inf-semi-lattice $\mathbf{Sub}_{\mathbf{\Sigma}}(\mathbf{T}_{\mathbf{\Sigma}}(M))$, where $\mathrm{Sub}_{\mathbf{\Sigma}}(\mathbf{T}_{\mathbf{\Sigma}}(M))$, the underlying set of $\mathbf{Sub}_{\mathbf{\Sigma}}(\mathbf{T}_{\mathbf{\Sigma}}(M))$, is the set of all subalgebras of the free $\mathbf{\Sigma}$-algebra $\mathbf{T}_{\mathbf{\Sigma}}(M)$ on M, by assigning to each $X \subseteq M$ precisely $\mathrm{T}_{\mathbf{\Sigma}}(X)$, the underlying set of the free $\mathbf{\Sigma}$-algebra $\mathbf{T}_{\mathbf{\Sigma}}(X)$ on X, identified to $\mathrm{Sg}_{\mathbf{T}_{\mathbf{\Sigma}}(M)}(X)$, the subalgebra of $\mathbf{T}_{\mathbf{\Sigma}}(M)$ generated by X. In this note we show, on the one hand, that the aforementioned homomorphisms between inf-semi-lattices are the components of a natural transformation between two suitable contravariant functors, and, on the other hand, that when the above mentioned homomorphisms are considered as order preserving mappings, they are the components of a natural transformation between two appropriate functors.

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In memory of Professor N. Ţăndăreanu.

1. Introduction

In the article [3], the authors, Ţăndăreanu and Tudor, for a fixed operator domain $\mathbf{\Sigma} = (\Sigma, \mathrm{ar})$, which has exactly one binary operator symbol σ, and a fixed set M, define a homomorphism (i.e., a mapping which preserves the nonempty finite infs) f_M from the inf-semi-lattice $\mathbf{Sub}(M) = (\mathrm{Sub}(M), \subseteq)$, where $\mathrm{Sub}(M)$ is the set of all subsets of M, to the inf-semi-lattice $\mathbf{Sub}_{\mathbf{\Sigma}}(\mathbf{T}_{\mathbf{\Sigma}}(M)) = (\mathrm{Sub}_{\mathbf{\Sigma}}(\mathbf{T}_{\mathbf{\Sigma}}(M)), \subseteq)$, where $\mathrm{Sub}_{\mathbf{\Sigma}}(\mathbf{T}_{\mathbf{\Sigma}}(M))$ is the set of all subalgebras of the free $\mathbf{\Sigma}$-algebra $\mathbf{T}_{\mathbf{\Sigma}}(M)$ on M (which is called by the aforementioned authors the Peano $\mathbf{\Sigma}$-algebra over M and which they denote by $\overline{M}$), by sending each $X \subseteq M$ to $\mathrm{T}_{\mathbf{\Sigma}}(X)$, the underlying set of the free $\mathbf{\Sigma}$-algebra $\mathbf{T}_{\mathbf{\Sigma}}(X)$ on X, identified to the subalgebra of $\mathbf{T}_{\mathbf{\Sigma}}(M)$ generated by X.

Before explaining more fully the contents of the subsequent, and final, section of this note, we next proceed, on the one hand, to recall, mainly to keep the exposition as self-contained as possible, for an operator domain $\mathbf{\Sigma} = (\Sigma, \mathrm{ar})$ which has exactly one binary operator symbol σ, some basic facts concerning the free $\mathbf{\Sigma}$-algebras, and, on the other hand, to set up the notation and terminology that will be used in it.

Let $\mathbf{\Sigma} = (\Sigma, \mathrm{ar})$ be an operator domain which has exactly one binary operator symbol σ and let M be a set disjoint of $\{\sigma\}$, for simplicity. Then $\mathbf{W}_{\mathbf{\Sigma}}(M)$, the $\mathbf{\Sigma}$-algebra of words on M, has as underlying set $(M \cup \{\sigma\})^{\star}$, i.e., the set of all words of finite length on the alphabet $M \cup \{\sigma\}$, and as structural operation, associated to σ, the mapping F_σ from $((M \cup \{\sigma\})^{\star})^2$ to $(M \cup \{\sigma\})^{\star}$ which assigns to an ordered pair of words (P, Q) the word $(\sigma) \curlywedge P \curlywedge Q$, i.e., the concatenation of the words (σ), P, and

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Q (which, by abuse of language, is abbreviated to $\sigma(P, Q)$). Next, the free $\mathbf{\Sigma}$-algebra on M, denoted by $\mathbf{T}_{\mathbf{\Sigma}}(M)$, is $\mathbf{Sg}_{\mathbf{W}_{\mathbf{\Sigma}}(M)}(\{(x) \mid x \in M\})$, the $\mathbf{\Sigma}$-algebra canonically associated to $\mathrm{Sg}_{\mathbf{W}_{\mathbf{\Sigma}}(M)}(\{(x) \mid x \in M\})$, the subalgebra of $\mathbf{W}_{\mathbf{\Sigma}}(M)$ generated by the subset $\{(x) \mid x \in M\}$ of $(M \cup \{\sigma\})^{\star}$. We next state some relevant background material concerning $\mathrm{T}_{\mathbf{\Sigma}}(M)$, the underlying set of $\mathbf{T}_{\mathbf{\Sigma}}(M)$, which will be used afterwards.

(1) $\mathrm{T}_{\mathbf{\Sigma}}(M) = \bigcup_{n \in \mathbb{N}} M_n$, where $(M_n)_{n \in \mathbb{N}}$ is the family of subsets of $(M \cup \{\sigma\})^{\star}$ defined by recursion as follows:

$$\begin{aligned} M_0 &= \{(x) \mid x \in M\}, \\ M_{n+1} &= M_n \cup \{\sigma(P, Q) \mid P, Q \in M_n\},\ n \geq 0. \end{aligned}$$

(2) $\mathrm{T}_{\mathbf{\Sigma}}(M) = \bigcup_{n \in \mathbb{N}} B_n$, where $(B_n)_{n \in \mathbb{N}}$ is the family of subsets of $(M \cup \{\sigma\})^{\star}$ defined by recursion as follows:

$$\begin{aligned} B_0 &= \{(x) \mid x \in M\}, \\ B_{n+1} &= M_{n+1} - M_n,\ n \geq 0. \end{aligned}$$

(3) For every $P \in \mathrm{T}_{\mathbf{\Sigma}}(M)$ there exists a unique $n \in \mathbb{N}$ such that $P \in B_n$. We call this unique natural number the length of P and denote it by $\ell(P)$.

(4) For every $P \in \mathrm{T}_{\mathbf{\Sigma}}(M)$, if $\ell(P) = 0$, then there exists a unique $x \in M$ such that $P = (x)$, and if $\ell(P) \geq 1$, then there exists a unique pair $(Q, R) \in \mathrm{T}_{\mathbf{\Sigma}}(M)^2$ such that $P = \sigma(Q, R)$.

Moreover, we denote by η_M the canonical mapping from M to $\mathrm{T}_{\mathbf{\Sigma}}(M)$ which sends an x in M to (x) in $\mathrm{T}_{\mathbf{\Sigma}}(M)$, and, for a mapping h from M to the underlying set B of a $\mathbf{\Sigma}$-algebra $\mathbf{B}$, we denote by $h^{\sharp}$ the unique homomorphism from $\mathbf{T}_{\mathbf{\Sigma}}(M)$ to $\mathbf{B}$ such that $h^{\sharp} \circ \eta_M = h$ (notice that in a category of structured sets, we carefully distinguish objects from their underlying sets, while, with the customary abuse of notation, in such a type of category we identify morphisms with their underlying mappings). In addition, if f is a mapping from M to M', then we denote by $\mathrm{T}_{\mathbf{\Sigma}}(f)$ ($\equiv \mathbf{T}_{\mathbf{\Sigma}}(f)$) the canonical homomorphism $(\eta_{M'} \circ f)^{\sharp}$ from $\mathbf{T}_{\mathbf{\Sigma}}(M)$ to $\mathbf{T}_{\mathbf{\Sigma}}(M')$, with $\eta_{M'}$ the canonical mapping from M' to $\mathrm{T}_{\mathbf{\Sigma}}(M')$, the underlying set of $\mathbf{T}_{\mathbf{\Sigma}}(M')$. Furthermore, if the mapping f is injective (resp. surjective, bijective), then $\mathrm{T}_{\mathbf{\Sigma}}(f)$ is injective (resp. surjective, bijective), and, for every subset X of M, $\mathbf{T}_{\mathbf{\Sigma}}(X)$ is isomorphic to $\mathbf{Sg}_{\mathbf{T}_{\mathbf{\Sigma}}(M)}(X)$, the $\mathbf{\Sigma}$-algebra canonically associated to $\mathrm{Sg}_{\mathbf{T}_{\mathbf{\Sigma}}(M)}(X)$, the subalgebra of $\mathbf{T}_{\mathbf{\Sigma}}(M)$ generated by X, which, by abuse of language, we identify.

We will work in one of the familiar systems of set theory, as, e.g., Zermelo-Fraenkel-Skolem, but assuming, in addition, the existence of a Grothendieck universe $\boldsymbol{\mathcal{U}}$, fixed once and for all, and from now on we adopt the following conventions concerning categories and functors.

(1) **Set** denotes the category which has as objects the $\boldsymbol{\mathcal{U}}$-small sets, i.e., the elements of $\boldsymbol{\mathcal{U}}$, and as morphisms from a $\boldsymbol{\mathcal{U}}$-small set X to another $\boldsymbol{\mathcal{U}}$-small set Y the mappings from X to Y.

(2) $\mathbf{Alg}(\mathbf{\Sigma})$ denotes the category of ($\boldsymbol{\mathcal{U}}$-small) $\mathbf{\Sigma}$-algebras and homomorphisms between $\mathbf{\Sigma}$-algebras.

(3) **Pos** denotes the category of ($\boldsymbol{\mathcal{U}}$-small) partially ordered sets and order-preserving mappings.

(4) **InfSLat** denotes the category of ($\boldsymbol{\mathcal{U}}$-small) inf-semi-lattices and homomorphisms of inf-semi-lattices. i.e., mappings which preserve the nonempty finite infs.

(5) $\mathbf{T}_{\mathbf{\Sigma}}$ denotes the functor from **Set** to $\mathbf{Alg}(\mathbf{\Sigma})$ that sends a set M to $\mathbf{T}_{\mathbf{\Sigma}}(M)$, and a mapping f from M to M' to $\mathrm{T}_{\mathbf{\Sigma}}(f)$ ($\equiv \mathbf{T}_{\mathbf{\Sigma}}(f)$), the canonical homomorphisms from $\mathbf{T}_{\mathbf{\Sigma}}(M)$ to $\mathbf{T}_{\mathbf{\Sigma}}(M')$.

(6) $\mathbf{Sub}$ denotes the functor from $\mathbf{Set}$ to $\mathbf{Pos}$ that sends a set M to the partially ordered set $\mathbf{Sub}(M) = (\mathrm{Sub}(M), \subseteq)$, and a mapping f from M to M' to $f[\cdot]$, an abbreviation for $\mathbf{Sub}(f)$, the order-preserving mapping from the partially ordered set $\mathbf{Sub}(M)$ to the partially ordered set $\mathbf{Sub}(M') = (\mathrm{Sub}(M'), \subseteq)$ that sends a subset X of M to the subset $f[X]$ of M', its direct image under f.

(7) $\mathbf{Sub}^{-}$ denotes the contravariant functor from $\mathbf{Set}$ to $\mathbf{InfSLat}$ that sends a set M to the inf-semi-lattice $\mathbf{Sub}(M) = (\mathrm{Sub}(M), \cap)$, and a mapping f from M to M' to $f^{-1}[\cdot]$, an abbreviation for $\mathbf{Sub}^{-}(f)$, the meet-preserving mapping from the inf-semi-lattice $\mathbf{Sub}(M') = (\mathrm{Sub}(M'), \cap)$ to the inf-semi-lattice $\mathbf{Sub}(M) = (\mathrm{Sub}(M), \cap)$ that sends a subset Y of M' to the subset $f^{-1}[Y]$ of M, its inverse image under f.

(8) $\mathbf{Sub}_{\mathbf{\Sigma}}$ denotes the functor from $\mathbf{Alg}(\mathbf{\Sigma})$ to $\mathbf{Pos}$ that sends a $\mathbf{\Sigma}$-algebra $\mathbf{A}$ to the partially ordered set $\mathbf{Sub}_{\mathbf{\Sigma}}(\mathbf{A}) = (\mathrm{Sub}_{\mathbf{\Sigma}}(\mathbf{A}), \subseteq)$ of all subalgebras of $\mathbf{A}$, and a homomorphism h from $\mathbf{A}$ to $\mathbf{B}$ to $h[\cdot]$, an abbreviation for $\mathbf{Sub}_{\mathbf{\Sigma}}(h)$, the order-preserving mapping from the partially ordered set $\mathbf{Sub}_{\mathbf{\Sigma}}(\mathbf{A})$ to the partially ordered set $\mathbf{Sub}_{\mathbf{\Sigma}}(\mathbf{B}) = (\mathrm{Sub}_{\mathbf{\Sigma}}(\mathbf{B}), \subseteq)$ that sends a subalgebra X of $\mathbf{A}$ to the subalgebra $h[X]$ of $\mathbf{B}$, its direct image under h.

(9) $\mathbf{Sub}_{\mathbf{\Sigma}}^{-}$ denotes the contravariant functor from $\mathbf{Alg}(\mathbf{\Sigma})$ to $\mathbf{InfSLat}$ that sends a $\mathbf{\Sigma}$-algebra $\mathbf{A}$ to the inf-semi-lattice $\mathbf{Sub}_{\mathbf{\Sigma}}(\mathbf{A}) = (\mathrm{Sub}_{\mathbf{\Sigma}}(\mathbf{A}), \cap)$ of all subalgebras of $\mathbf{A}$, and a homomorphism h from $\mathbf{A}$ to $\mathbf{B}$ to $h^{-1}[\cdot]$, an abbreviation for $\mathbf{Sub}_{\mathbf{\Sigma}}^{-}(h)$, the meet-preserving mapping from the inf-semi-lattice $\mathbf{Sub}_{\mathbf{\Sigma}}(\mathbf{B}) = (\mathrm{Sub}_{\mathbf{\Sigma}}(\mathbf{B}), \cap)$ to the inf-semi-lattice $\mathbf{Sub}_{\mathbf{\Sigma}}(\mathbf{A}) = (\mathrm{Sub}_{\mathbf{\Sigma}}(\mathbf{A}), \cap)$ that sends a subalgebra Y of $\mathbf{B}$ to the subalgebra $h^{-1}[Y]$ of $\mathbf{A}$, its inverse image under h.

In this note, for a fixed operator domain $\mathbf{\Sigma} = (\Sigma, \mathrm{ar})$, which has exactly one binary operator symbol σ, we show that the family of homomorphisms of inf-semi-lattices $(f_M)_{M \in \boldsymbol{\mathcal{U}}}$, where, for every $M \in \boldsymbol{\mathcal{U}}$, we have that f_M is precisely

$$f_M \begin{cases} \mathbf{Sub}(M) \longrightarrow \mathbf{Sub}_{\mathbf{\Sigma}}(\mathbf{T}_{\mathbf{\Sigma}}(M)) \\ \quad X \quad\;\; \longmapsto \; \mathrm{T}_{\mathbf{\Sigma}}(X) \end{cases}$$

i.e., the homomorphism defined by Ţăndăreanu and Tudor in [3] for M, is a natural transformation ζ from the contravariant functor $\mathbf{Sub}^{-}$ to the contravariant functor $\mathbf{Sub}_{\mathbf{\Sigma}}^{-} \circ \mathbf{T}_{\mathbf{\Sigma}}$, both from $\mathbf{Set}$ to $\mathbf{InfSLat}$. We represent this situation by the following diagram

$$\begin{tikzcd} \mathbf{Set} \arrow[r, bend left, "\mathbf{Sub}^{-}"] \arrow[r, bend right, "\mathbf{Sub}_{\mathbf{\Sigma}}^{-} \circ \mathbf{T}_{\mathbf{\Sigma}}"'] \arrow[r, phantom, "\zeta\Downarrow"] & \mathbf{InfSLat}. \end{tikzcd}$$

Furthermore, we prove that the family of homomorphisms of inf-semi-lattices $(f_M)_{M \in \boldsymbol{\mathcal{U}}}$, when considered as a family of order-preserving mappings, is a natural transformation ξ from the functor $\mathbf{Sub}$ to the functor $\mathbf{Sub}_{\mathbf{\Sigma}} \circ \mathbf{T}_{\mathbf{\Sigma}}$, both from $\mathbf{Set}$ to $\mathbf{Pos}$. We represent this situation by the following diagram

$$\begin{tikzcd} \mathbf{Set} \arrow[r, bend left, "\mathbf{Sub}"] \arrow[r, bend right, "\mathbf{Sub}_{\mathbf{\Sigma}} \circ \mathbf{T}_{\mathbf{\Sigma}}"'] \arrow[r, phantom, "\xi\Downarrow"] & \mathbf{Pos}. \end{tikzcd}$$

For categorial and algebraic conventions, concepts, and constructions not otherwise given explicitly within the note, readers are referred to [1], [2], and [4].

2. The homomorphism of Ţăndăreanu and Tudor is doubly natural.

In this section we state the promised proofs of the facts that the family of homomorphisms of inf-semi-lattices $(f_M)_{M\in\boldsymbol{\mathcal{U}}}$, defined in Section 1, is a natural transformation ζ from the contravariant functor $\mathbf{Sub}^{-}$ to the contravariant functor $\mathbf{Sub}^{-}_{\mathbf{\Sigma}} \circ \mathbf{T}_{\mathbf{\Sigma}}$, and that the family of homomorphisms of inf-semi-lattices $(f_M)_{M\in\boldsymbol{\mathcal{U}}}$, when considered as a family of order-preserving mappings, is a natural transformation ζ from the functor $\mathbf{Sub}$ to the functor $\mathbf{Sub}_{\mathbf{\Sigma}} \circ \mathbf{T}_{\mathbf{\Sigma}}$. We warn the reader that to simplify the proofs, and without loss of generality, we identify words of length 1 with its unique component.

Proposition 2.1. *Let* $\mathbf{\Sigma} = (\Sigma, \mathrm{ar})$ *be an operator domain which has exactly one binary operator symbol* σ*. Then there exists a natural transformation* ζ *from the contravariant functor* $\mathbf{Sub}^{-}$ *to the contravariant functor* $\mathbf{Sub}^{-}_{\mathbf{\Sigma}} \circ \mathbf{T}_{\mathbf{\Sigma}}$*, both from the category* $\mathbf{Set}$ *to the category* $\mathbf{InfSLat}$*.*

Proof. Let ζ be, by definition, the mapping from $\boldsymbol{\mathcal{U}}$, the set of all objects of $\mathbf{Set}$, to $\mathrm{Mor}(\mathbf{InfSLat})$, the set of all homomorphisms of $\mathbf{InfSLat}$, which sends a set M in $\boldsymbol{\mathcal{U}}$ to the homomorphism $\zeta_M = f_M$ from $\mathbf{Sub}(M)$ to $\mathbf{Sub}_{\mathbf{\Sigma}}(\mathbf{T}_{\mathbf{\Sigma}}(M))$ in $\mathbf{InfSLat}$. We want to show that, for every mapping $f\colon M \longrightarrow M'$, the following diagram commutes

$$\begin{tikzcd}
\mathbf{Sub}(M) \arrow[r, "\zeta_M"] & \mathbf{Sub}_{\mathbf{\Sigma}}(\mathbf{T}_{\mathbf{\Sigma}}(M)) \\
\mathbf{Sub}(M') \arrow[u, "f^{-1}[\cdot]"] \arrow[r, "\zeta_{M'}"'] & \mathbf{Sub}_{\mathbf{\Sigma}}(\mathbf{T}_{\mathbf{\Sigma}}(M')) \arrow[u, "\mathrm{T}_{\mathbf{\Sigma}}(f)^{-1}[\cdot]"']
\end{tikzcd}$$

where the meaning of $f^{-1}[\cdot]$ was explained in Section 1, and $\mathrm{T}_{\mathbf{\Sigma}}(f)^{-1}[\cdot]$, an abbreviation for $\mathbf{Sub}^{-}_{\mathbf{\Sigma}}(\mathbf{T}_{\mathbf{\Sigma}}(f))$, is the homomorphism, i.e., the meet-preserving mapping, from the inf-semi-lattice $\mathbf{Sub}_{\mathbf{\Sigma}}(\mathbf{T}_{\mathbf{\Sigma}}(M')) = (\mathrm{Sub}_{\mathbf{\Sigma}}(\mathbf{T}_{\mathbf{\Sigma}}(M')), \cap)$, of all subalgebras of $\mathbf{T}_{\mathbf{\Sigma}}(M')$, to the inf-semi-lattice $\mathbf{Sub}_{\mathbf{\Sigma}}(\mathbf{T}_{\mathbf{\Sigma}}(M)) = (\mathrm{Sub}_{\mathbf{\Sigma}}(\mathbf{T}_{\mathbf{\Sigma}}(M)), \cap)$, of all subalgebras of $\mathbf{T}_{\mathbf{\Sigma}}(M)$, that sends a subalgebra Y of $\mathbf{T}_{\mathbf{\Sigma}}(M')$ to the subalgebra $\mathrm{T}_{\mathbf{\Sigma}}(f)^{-1}[Y]$ of $\mathbf{T}_{\mathbf{\Sigma}}(M)$, its inverse image under $\mathrm{T}_{\mathbf{\Sigma}}(f)[\cdot]$. Therefore, for a mapping $f\colon M \longrightarrow M'$, we should prove that, for every subset Y of M', we have that $\mathrm{T}_{\mathbf{\Sigma}}(f)^{-1}[\mathrm{T}_{\mathbf{\Sigma}}(Y)] = \mathrm{T}_{\mathbf{\Sigma}}(f^{-1}[Y])$.

Let Y be an arbitrary but fixed subset of M'. Then we have that $\mathrm{T}_{\mathbf{\Sigma}}(Y) = \mathrm{Sg}_{\mathbf{T}_{\mathbf{\Sigma}}(M')}(Y)$ and $\mathbf{T}_{\mathbf{\Sigma}}(f^{-1}[Y]) = \mathrm{Sg}_{\mathbf{T}_{\mathbf{\Sigma}}(M)}(f^{-1}[Y])$ (see Section 1). Therefore, since $\mathrm{T}_{\mathbf{\Sigma}}(f)$ $(\equiv \mathbf{T}_{\mathbf{\Sigma}}(f))$ is a homomorphism from $\mathbf{T}_{\mathbf{\Sigma}}(M)$ to $\mathbf{T}_{\mathbf{\Sigma}}(M')$, we can assert that $\mathrm{T}_{\mathbf{\Sigma}}(f)^{-1}[\mathrm{T}_{\mathbf{\Sigma}}(Y)]$ is a subalgebra of $\mathrm{T}_{\mathbf{\Sigma}}(M)$ and, because $\mathrm{T}_{\mathbf{\Sigma}}(f)^{-1}[\mathrm{T}_{\mathbf{\Sigma}}(Y)] \supseteq f^{-1}[Y]$, recall that we have agreed on identifying words of length 1 with its unique component, we conclude that $\mathrm{T}_{\mathbf{\Sigma}}(f^{-1}[Y]) \subseteq \mathrm{T}_{\mathbf{\Sigma}}(f)^{-1}[\mathrm{T}_{\mathbf{\Sigma}}(Y)]$.

To prove that $\mathrm{T}_{\mathbf{\Sigma}}(f)^{-1}[\mathrm{T}_{\mathbf{\Sigma}}(Y)] \subseteq \mathrm{T}_{\mathbf{\Sigma}}(f^{-1}[Y])$ we proceed by induction, taking into account that $\mathrm{T}_{\mathbf{\Sigma}}(Y)$ is $\bigcup_{n\in\mathbb{N}} Y_n$, where $(Y_n)_{n\in\mathbb{N}}$ is the family defined by recursion as follows:

$$\begin{aligned}
Y_0 &= \{(y) \mid y \in Y\}, \\
Y_{n+1} &= Y_n \cup \{\sigma(P', Q') \mid P', Q' \in Y_n\},\ n \geq 0,
\end{aligned}$$

and that $\mathrm{T}_{\mathbf{\Sigma}}(f)^{-1}[\mathrm{T}_{\mathbf{\Sigma}}(Y)] = \bigcup_{n\in\mathbb{N}} \mathrm{T}_{\mathbf{\Sigma}}(f)^{-1}[Y_n]$. We have that $\mathrm{T}_{\mathbf{\Sigma}}(f)^{-1}[Y_0] \subseteq \mathrm{T}_{\mathbf{\Sigma}}(f^{-1}[Y])$. In fact, let $P \in \mathrm{T}_{\mathbf{\Sigma}}(M)$ be such that $\mathrm{T}_{\mathbf{\Sigma}}(f)(P) \in Y_0$. Then there exists a unique $y \in Y$ such that $\mathrm{T}_{\mathbf{\Sigma}}(f)(P) = (y)$. Hence, necessarily, $\ell(P) = 0$ and, consequently, $P = (x)$, for a unique $x \in M$. Therefore $(f(x)) = (y)$, thus

$f(x) = y$. From this it follows that $P \in \mathrm{T}_{\mathbf{\Sigma}}(f^{-1}[Y])$. Let us suppose that, for $n \geq 0$, $\mathrm{T}_{\mathbf{\Sigma}}(f)^{-1}[Y_n] \subseteq \mathrm{T}_{\mathbf{\Sigma}}(f^{-1}[Y])$ and that $P \in \mathrm{T}_{\mathbf{\Sigma}}(f)^{-1}[Y_{n+1}]$. Then $P \in \mathrm{T}_{\mathbf{\Sigma}}(f)^{-1}[Y_n]$ or $P \in \mathrm{T}_{\mathbf{\Sigma}}(f)^{-1}[\{\sigma(Q', R') \mid Q', R' \in Y_n\}]$. If $P \in \mathrm{T}_{\mathbf{\Sigma}}(f)^{-1}[Y_n]$, then we are done. Let $P \in \mathrm{T}_{\mathbf{\Sigma}}(M)$ be such that $\mathrm{T}_{\mathbf{\Sigma}}(f)(P) \in \{\sigma(Q', R') \mid Q', R' \in Y_n\}$. Then there exists a unique pair $(Q', R') \in Y_n^2$ such that $\mathrm{T}_{\mathbf{\Sigma}}(f)(P) = \sigma(Q', R')$. Hence, necessarily, $\ell(P) \geq 1$ and, consequently, $P = \sigma(Q, R)$, for a unique pair $(Q, R) \in \mathrm{T}_{\mathbf{\Sigma}}(M)^2$. Therefore $\mathrm{T}_{\mathbf{\Sigma}}(f)(P) = \mathrm{T}_{\mathbf{\Sigma}}(f)(\sigma(Q, R)) = \sigma(\mathrm{T}_{\mathbf{\Sigma}}(f)(Q), \mathrm{T}_{\mathbf{\Sigma}}(f)(R)) = \sigma(Q', R')$, thus $\mathrm{T}_{\mathbf{\Sigma}}(f)(Q) = Q'$ and $\mathrm{T}_{\mathbf{\Sigma}}(f)(R) = R'$. From this it follows that $P \in \mathrm{T}_{\mathbf{\Sigma}}(f^{-1}[Y])$. □

Proposition 2.2. *Let* $\mathbf{\Sigma} = (\Sigma, \mathrm{ar})$ *be an operator domain which has exactly one binary operator symbol* σ*. Then there exists a natural transformation* ξ *from the functor* $\mathbf{Sub}$ *to the functor* $\mathbf{Sub}_{\mathbf{\Sigma}} \circ \mathbf{T}_{\mathbf{\Sigma}}$*, both from the category* $\mathbf{Set}$ *to the category* $\mathbf{Pos}$.

Proof. Let ξ be, by definition, the mapping from $\boldsymbol{\mathcal{U}}$, the set of all objects of $\mathbf{Set}$, to $\mathrm{Mor}(\mathbf{Pos})$, the set of all order-preserving mappings of $\mathbf{Pos}$, which sends a set M in $\boldsymbol{\mathcal{U}}$ to the order-preserving mapping $\xi_M = f_M$ from $\mathbf{Sub}(M)$ to $\mathbf{Sub}_{\mathbf{\Sigma}}(\mathbf{T}_{\mathbf{\Sigma}}(M))$ in $\mathbf{Pos}$ (this is well defined because f_M preserves the nonempty finite infs). We want to show that, for every mapping $f\colon M \longrightarrow M'$, the following diagram commutes

$$\begin{tikzcd}
\mathbf{Sub}(M) \arrow[r, "\xi_M"] \arrow[d, "f[\cdot]"'] & \mathbf{Sub}_{\mathbf{\Sigma}}(\mathbf{T}_{\mathbf{\Sigma}}(M)) \arrow[d, "\mathrm{T}_{\mathbf{\Sigma}}(f)[\cdot]"] \\
\mathbf{Sub}(M') \arrow[r, "\xi_{M'}"'] & \mathbf{Sub}_{\mathbf{\Sigma}}(\mathbf{T}_{\mathbf{\Sigma}}(M'))
\end{tikzcd}$$

where the meaning of $f[\cdot]$ was explained in Section 1, and $\mathrm{T}_{\mathbf{\Sigma}}(f)[\cdot]$, an abbreviation for $\mathbf{Sub}_{\mathbf{\Sigma}}(\mathbf{T}_{\mathbf{\Sigma}}(f))$, is the order-preserving mapping from the partially ordered set $\mathbf{Sub}_{\mathbf{\Sigma}}(\mathbf{T}_{\mathbf{\Sigma}}(M)) = (\mathrm{Sub}_{\mathbf{\Sigma}}(\mathbf{T}_{\mathbf{\Sigma}}(M)), \subseteq)$, of all subalgebras of $\mathbf{T}_{\mathbf{\Sigma}}(M)$, to the partially ordered set $\mathbf{Sub}_{\mathbf{\Sigma}}(\mathbf{T}_{\mathbf{\Sigma}}(M')) = (\mathrm{Sub}_{\mathbf{\Sigma}}(\mathbf{T}_{\mathbf{\Sigma}}(M')), \subseteq)$, of all subalgebras of $\mathbf{T}_{\mathbf{\Sigma}}(M')$, that sends a subalgebra X of $\mathbf{T}_{\mathbf{\Sigma}}(M)$ to the subalgebra $\mathrm{T}_{\mathbf{\Sigma}}(f)[X]$ of $\mathbf{T}_{\mathbf{\Sigma}}(M')$, its direct image under $\mathrm{T}_{\mathbf{\Sigma}}(f)[\cdot]$. Therefore, for a mapping $f\colon M \longrightarrow M'$, we should prove that, for every subset X of M, we have that $\mathrm{T}_{\mathbf{\Sigma}}(f)[\mathrm{T}_{\mathbf{\Sigma}}(X)] = \mathrm{T}_{\mathbf{\Sigma}}(f[X])$.

Let X be an arbitrary but fixed subset of M. Then we have that $\mathrm{T}_{\mathbf{\Sigma}}(X) = \mathrm{Sg}_{\mathbf{T}_{\mathbf{\Sigma}}(M)}(X)$ and $\mathrm{T}_{\mathbf{\Sigma}}(f[X]) = \mathrm{Sg}_{\mathbf{T}_{\mathbf{\Sigma}}(M')}(f[X])$ (see Section 1). Therefore, since $\mathrm{T}_{\mathbf{\Sigma}}(f)$ ($\equiv \mathbf{T}_{\mathbf{\Sigma}}(f)$) is a homomorphism from $\mathbf{T}_{\mathbf{\Sigma}}(M)$ to $\mathbf{T}_{\mathbf{\Sigma}}(M')$ and $\mathrm{T}_{\mathbf{\Sigma}}(f)[X] = f[X]$, because we have agreed, in particular, on identifying, for every $x \in X$, the word $(f(x))$ of length 1 with $f(x)$, we have that

$$\begin{aligned}
\mathrm{T}_{\mathbf{\Sigma}}(f)[\mathrm{T}_{\mathbf{\Sigma}}(X)] &= \mathrm{T}_{\mathbf{\Sigma}}(f)[\mathrm{Sg}_{\mathbf{T}_{\mathbf{\Sigma}}(M)}(X)] \\
&= \mathrm{Sg}_{\mathbf{T}_{\mathbf{\Sigma}}(M')}(\mathrm{T}_{\mathbf{\Sigma}}(f)[X]) \\
&= \mathrm{Sg}_{\mathbf{T}_{\mathbf{\Sigma}}(M')}(f[X]) \\
&= \mathrm{T}_{\mathbf{\Sigma}}(f[X]).
\end{aligned}$$

□

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