Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 42(2), 2015, Pages 450–461
ISSN: 1223-6934

Spherical functions on transitive groupoids

Ibrahima Toure and Kinvi Kangni

Abstract. Let G be a topological locally compact, Hausdorff and second countable groupoid with a Haar system and K a compact subgroupoid of G with a Haar system too. (G, K) is a Gelfand pair if the algebra of K-biinvariant functions is commutative under convolution. In this paper, we study spherical functions on groupoids and generalize some well-known results on groups.

2010 Mathematics Subject Classification. 22A22, 46J10.
Key words and phrases. Transitive groupoids, groupoid representation, spherical functions, positive definite functions.

1. Introduction

Since the works of E. Cartan and H. Weyl around 1930, the theory of spherical functions plays an important part in non commutative harmonic analysis. They showed that spherical harmonics arise in a natural way from the study of functions on $n-$ dimensional sphere. The first generalization of the theory were given by I.M. Gelfand in 1950. In [6], Gelfand considers a Lie group G (not necessarily compact) and a compact subgroup K of G such that the bi-K-invariant integrable functions on G, $L^1(G\backslash\backslash K)$, forms a commutative Banach algebra for the convolution product. Such a pair (G, K) is called Gelfand pair. Under this condition, a function φ from G to the complex field is spherical if it is continuous, bi-K-invariant and the linear form $f \longmapsto \varphi(f) = \int f(x)\varphi(x)dx$ is a homomorphism of $L^1(G\backslash\backslash K)$ onto the complex field. The spherical functions play the role of the exponential function for the Gelfand pairs. The theory was extended to locally compact group and several other generalizations exist in the literature. For more details on Gelfand pairs and spherical functions the reader can consult [2, 3, 5, 6, 7]. Groupoids are generalizations of groups. Thus in [15, 16], we have extending the notion of Gelfand pair from groups to groupoids. In this paper, our main goal is to study spherical functions associated to Gelfand pairs on groupoids and establish some classical theorems of harmonic analysis including Bochner theorem. As in [15] we are interested in transitive groupoids. More precisely, this paper is a continuation of [15]. The outline of this work is as follows. In the following section we give notations and setup useful for the remainder of this paper. In section 3, we establish the basic properties of spherical functions on groupoids including functional equation characterizing them. Also the closed connection between positive definite spherical functions and irreducible representations with $K-$ fixed vector is proved in groupoids context where $K-$ fixed vector is replaced by $K-$ fixed square integrable section. The notion of positive definite function has been extended to groupoids by Ramsay and Walter [12]. In section 4, thanks to the main result of [15] which asserts that for transitive groupoids G, (G, K) is a Gelfand pair if

Received November 06, 2013. Revised October 14, 2014.

and only if for each $m \in G^{(0)}$, (G_m^m, K_m^m) is a Gelfand pair, we establish a correspondence between the positive definite spherical functions on G and the positive definite spherical functions on G_m^m. Thanks to this correspondence, we have generalized to groupoids some classical theorems of harmonic analysis on groups.

2. Preliminaries

We use the notations and setup of this section in the rest of the paper without mentioning. We shall use definition of a locally compact groupoid and the definition of a Haar system on groupoid giving by J. Renault in [14]. Let G be a locally compact, Hausdorff, second countable groupoid. $G^{(0)}$ denotes the unit space of G and $G^{(2)}$ the set of composable pairs. For $x \in G$, $r(x) = xx^{-1}$ and $d(x) = x^{-1}x$ are respectively the range and the domain of x. For $u, v \in G^{(0)}$, let us put $G^u = r^{-1}(u)$, $G_v = d^{-1}(v)$, $G_v^u = G^u \cap G_v$ and for each unit element u, $G_u^u = \{x \in G : r(x) = d(x) = u\}$ is the isotropy group at u. The group bundle $G' = \{x \in G : r(x) = d(x)\}$ is called the isotropy group bundle of G. There exists an equivalent relation on $G^{(0)}$ defined as follows: $u, v \in G^{(0)}$, $u \sim v$ iff $G_v^u \neq \emptyset$. The equivalence class of u is denoted by $[u]$ and is called the orbit of u. As a subset of $G^{(0)} \times G^{(0)}$, the graph $R = \{(r(x), d(x)) : x \in G\}$ of this equivalent relation is a groupoid on $G^{(0)}$. The anchor map $\theta = (r, d)$ is a continuous homomorphism of G into $G^{(0)} \times G^{(0)}$ with image R. A groupoid is transitive if θ is onto i.e. the range of θ is equal to $G^{(0)} \times G^{(0)}$. Otherwise, the groupoid is transitive if it has a single orbit. Let $\{\lambda^u,\ u \in G^{(0)}\}$ be a left Haar system on G. For $u \in G^{(0)}$, λ_u will denote the image of λ^u by the inverse map and $\{\lambda_u,\ u \in G^{(0)}\}$ is a right Haar system on G. Let μ be a quasi-invariant measure on $G^{(0)}$ for the Haar system $\{\lambda^u, u \in G^{(0)}\}$, $\nu = \int \lambda^u d\mu(u)$ be the induced measure by μ on G, $\nu^{-1} = \int \lambda_u d\mu(u)$ be the inverse of ν, $\nu^2 = \int \lambda^u \times \lambda_u d\mu(u)$ be the induced measure by μ on $G^{(2)}$ and Δ the modular function of μ. In [9], it was proved that Δ is a homomorphism ν^2 a.e from G to $\mathbb{R}_+^*$, the group of multiplicative positive real numbers. There is a decomposition of the left Haar system $\{\lambda^u, u \in G^{(0)}\}$ for G over R. Firstly, there is a measure β_v^u concentrated on G_v^u for all $(u, v) \in R$ such that:

- β_u^u is a left Haar measure on G_u^u,
- β_v^u is a translate of β_v^v i.e. $\beta_v^u = x\beta_v^v$ if $x \in G_v^u$.

Notice that β_v^u is independent of the choice of $x \in G_v^u$. Then, there is a unique Borel Haar system $\alpha = \{\alpha^u : u \in G^{(0)}\}$ for R with the property that for every $u \in G^{(0)}$, we have $\lambda^u = \int \beta_v^\omega d\alpha^u(\omega, v)$. Renault [13] proves that there exists a continuous homomorphism δ of G to $\mathbb{R}_+$ such that for any quasi-invariant measure μ on $G^{(0)}$, the modular functions Δ of G, defined by μ and λ, and $\widetilde{\Delta}$ of R, defined by μ and α, satisfy $\Delta = \delta\widetilde{\Delta} \circ \theta$. We can also notice that for all $u \in G^{(0)}$, $\delta \mid G_u^u$ is the modular function of G_u^u relatively to the left Haar measure β_u^u. It is proved in [12, 13] that there is a transitive quasi-invariant measure $\widetilde{\mu}$ (i.e. a quasi-invariant measure concentrated on an orbit) such that $\widetilde{\Delta} = 1$ and so $\Delta = \delta$. In particular, for a transitive groupoid there is a unique quasi-invariant measure on $G^{(0)}$ with full support such that the modular function Δ is a continuous homomorphism of G. $\mathcal{C}_c(G)$ will denote the space of complex-valued continuous functions on G with compact support, endowed with the inductive limit topology and $L^1(G, \nu)$ the space of $\nu-$ integrable functions on G. In [8], P. Hahn defines the following norm on $L^1(G, \nu)$: $\| f \|_I = \max(\| f \|_{I,r}; \| f \|_{I,d})$ where $\| f \|_{I,r} = \sup\{\int_{G^u} f(x) d\lambda^u(x), u \in G^{(0)}\}$, $\| f \|_{I,d} = \sup\{\int_{G_u} f(x) d\lambda_u(x), u \in G^{(0)}\}$

and introduces the following groupoid algebra,

$$I(G,\lambda,\mu) = \{f \in L^1(G,\nu) : \| f \|_I < \infty\}$$

$I(G,\lambda,\mu)$ is a Banach $*$-algebra under the following convolution product: for all $f, g \in I(G,\lambda,\mu)$,

$$f * g(x) = \int_{G^{r(x)}} f(y)g(y^{-1}x)d\lambda^{r(x)}(y).$$

The involution is defined by: for $f \in I(G,\lambda,\mu)$,

$$f^*(x) = \Delta(x^{-1})\overline{f(x^{-1})} = \Delta(x^{-1})\overline{\check{f}(x)}.$$

Let K be a compact subgroupoid of G with unit space $G^{(0)}$. We shall assume that K is equipped with a normalized Haar system $\gamma = \{\gamma^u,\ u \in G^{(0)}\}$ that means $\gamma_u(K_u) = \gamma^u(K^u) = 1$, for each $u \in G^{(0)}$. As it is explain above, $\{\gamma^u,\ u \in G^{(0)}\}$ has a decomposition $\{(\gamma_v^u)_{(u,v)\in R_K}, (\rho^u)_{u\in G^{(0)}}\}$, where R_K is the graph of the equivalence relation on $G^{(0)}$ seen as unit space of K, such that $\gamma^u = \int \gamma_v^\omega d\rho^u(\omega, v)$. We put

$$I(G\backslash\backslash K) = \{f \in I(G,\lambda,\mu) : f(kxk') = f(x) \forall x \in G, \forall k \in K_{r(x)}, \forall k' \in K^{d(x)}\};$$

the space of bi-$K-$ invariant integrable functions which is a Banach $*$-subalgebra of $I(G,\lambda,\mu)$. For any $f \in I(G,\lambda,\mu)$, let us denote by $f^\natural$ the bi-K-invariant function defined by: for all $x \in G$,

$$f^\natural(x) = \int\int f(kxk')d\gamma_{r(x)}(k)d\gamma^{d(x)}(k').$$

If $I(G\backslash\backslash K)$ is commutative for convolution product, we say that (G,K) is a Gelfand pair. This notion in groupoids case has been studied by authors in [15, 16]. Let $\mathcal{H} = (H_u)_{u\in G^{(0)}}$ be a Hilbert bundle over $G^{(0)}$ and $\mathcal{U}(\mathcal{H})$ the unitary groupoid of the bundle $\mathcal{H}$. $(\pi,\mathcal{H})$ is a unitary continuous representation of G if π is a groupoid morphism of G into $\mathcal{U}(\mathcal{H})$ such that for all square integrable sections ξ and η of $\mathcal{H}$, the map $x \mapsto < \pi(x)\xi(d(x)), \eta(r(x)) >$ is continuous. A closed nonzero subbundle $\mathcal{M}$ of $\mathcal{H}$ (i.e. M_u is a nonzero closed subspace of H_u for each $u \in G^{(0)}$) is invariant under π if $\pi(x)M_{d(x)} \subset M_{r(x)}$, for each $x \in G$. If π admits a non trivial closed invariant subbundle $\mathcal{M}$, it is called reducible. Otherwise it is called irreducible. If ξ is a section of $\mathcal{H}$, the subbundle $\mathcal{M}_\xi$ whose leaf at $u \in G^{(0)}$ is the closed linear span of the set $\{\pi(x)\xi(d(x)) : x \in G^u\}$ is called the cyclic subbundle generated by ξ. We say that ξ is cyclic if $(\mathcal{M}_\xi)_u$ is dense in H_u, for each $u \in G^{(0)}$. We denote by $\Gamma_\mu(\mathcal{H})$, the Hilbert space of square integrable section of $\mathcal{H}$. In [14], J. Renault associates to any unitary representation $(\pi,\mathcal{H})$ a representation L of $\mathcal{C}_c(G)$ on $\Gamma_\mu(\mathcal{H})$ defined by:

$$(L(f)\xi,\eta) = \int f(x) < \pi(x)\xi(d(x)), \eta(r(x)) > d\nu_0(x),$$

for all $f \in \mathcal{C}_c(G)$, $\xi,\eta \in \Gamma_\mu(\mathcal{H})$, where $\nu_0 = \Delta^{\frac{-1}{2}}\nu$. L is a bounded non-degenerate $*$-representation of $\mathcal{C}_c(G)$ where $\mathcal{C}_c(G)$ is equipped with the norm $\|\cdot\|_I$. We may also define L by: $L(f)\xi(u) = \int_{G^u} f(x)\pi(x)\xi(d(x))\Delta^{\frac{-1}{2}}(x)d\lambda^u(x)$. In [12], the authors extend the notion of positive definite function to groupoids. In fact, a bounded continuous function $p : G \to \mathbb{C}$ is positive definite if for each $u \in G^{(0)}$ and for each $f \in \mathcal{C}_c(G)$ we have

$$\int\int f(x)\bar{f}(y)p(y^{-1}x)d\lambda^u(x)d\lambda^u(y) \geq 0.$$

Ramsay and Walter establish for groupoids the well-known correspondence between positive definite functions and representations. In fact, for any bounded continuous positive definite function $p : G \to \mathbb{C}$, there exists a unitary representation π of G on a Hilbert bundle $\mathcal{H}$, and a bounded continuous cyclic section ξ of $\mathcal{H}$ such that for each $x \in G$, $p(x) =< \pi(x)\xi(d(x)), \xi(r(x)) >$.

3. Spherical functions

Throughout this section and the followings G is Hausdorff, second countable, transitive locally compact groupoid with a left Haar system $\lambda = \{\lambda^u,\ u \in G^{(0)}\}$. K is a compact subgroupoid of G containing $G^{(0)}$. It follows that $G^{(0)}$ is the unit space of K. Let μ be the quasi-invariant measure on $G^{(0)}$ such that $supp(\mu) = G^{(0)}$ and the modular function Δ associated to (λ, μ) is a continuous homomorphism. Since K contains $G^{(0)}$, it follows that $G^{(0)}$ is compact. So we assume that μ is normalized. For each $u \in G^{(0)}$, the measure α^u is concentrated on $u \times [u]$, and $\alpha^u = \varepsilon_u \times \mu$, where ε_u is the unit point mass at u. So $\lambda^u = \int \beta^u_\omega d\mu(\omega)$. K is equipped with a normalized Haar system $\gamma = \{\gamma^u,\ u \in G^{(0)}\}$. In this section, (G, K) is a Gelfand pair. It follows that μ is invariant (see [15]) i.e $\nu = \nu^{-1}$. Let $\varphi : G \to \mathbb{C}$ be a bounded continuous bi-K-invariant function.

Definition 3.1. φ is a *spherical function* if the map $\chi_\varphi : f \mapsto \int f(y)\varphi(y)d\nu(y)$, from $I(G\backslash\backslash K)$ to $\mathbb{C}$, is a non-trivial character of $I(G\backslash\backslash K)$.

We will study some properties of these spherical functions, in particular, extend to groupoids case some well-known results on groups. The following result extends to groupoids the classical functional equation of spherical functions.

Theorem 3.1. *φ is a spherical function if and only if for all $x, y \in G$*

$$\int_{K^{d(x)}_{r(y)}} \varphi(xky)d\gamma^{d(x)}_{r(y)}(k) = \varphi(x)\varphi(y).$$

Proof. Let f, g be in $I(G, \lambda, \mu)$. We will first compute $\chi_\varphi(f^\natural * g^\natural)$.

$$\begin{aligned}
\chi_\varphi(f^\natural * g^\natural) &= \int f^\natural(x)g^\natural(x^{-1}y)\varphi(y)d\lambda^{r(y)}(x)d\nu(y) \\
&= \int f^\natural(x)g^\natural(y)\varphi(xy)d\lambda^u(x)d\lambda^{d(x)}(y)d\mu(u) \\
&= \int f(xk_2)g(k_3yk_4)\varphi(xy)d\gamma^{d(x)}_{\omega_2}(k_2)d\gamma^{\omega_3}_u(k_3)d\gamma^{d(y)}_{\omega_4}(k_4)d\tilde{H}^{u,d(k_3)}_{\lambda,\mu}
\end{aligned}$$

$$\begin{aligned}
&= \int f(xk_2)g(yk_4)\varphi(xk_3^{-1}y)d\gamma^{d(x)}_{\omega_2}(k_2)d\gamma^{\omega_3}_{d(x)}(k_3)d\gamma^{d(y)}_{\omega_4}(k_4)d\tilde{H}^{u,r(k_3)}_{\lambda,\mu} \\
&= \int f(xk_2)g(yk_4)\varphi(xk_3^{-1}y)d\gamma^{u}_{\omega_2}(k_2)d\gamma^{\omega_3}_{u}(k_3)d\gamma^{d(y)}_{\omega_4}(k_4)d\bar{H}^{r(k_2),r(k_3)}_{\lambda,\mu} \\
&= \int f(x)g(yk_4)\varphi(xk_2^{-1}k_3^{-1}y)d\gamma^{u}_{\omega_2}(k_2)d\gamma^{\omega_3}_{u}(k_3)d\gamma^{d(y)}_{\omega_4}(k_4)d\bar{H}^{r(d_2),r(k_3)}_{\lambda,\mu} \\
&= \int f(x)g(yk_4)\varphi(xk_3^{-1}y)d\gamma^{r(y)}_{u}(k_3)d\gamma^{\omega_3}_{\omega_4}(k_4)d\ddot{H}^{u,r(k_4)}_{\lambda,\mu} \\
&= \int f(x)g(y)\varphi(xk_3^{-1}y)d\gamma^{r(y)}_{u}(k_3)d\gamma^{\omega_3}_{\omega_4}(k_4)d\ddot{H}^{u,d(k_4)}_{\lambda,\mu} \\
&= \int f(x)g(y)\varphi(xky)d\gamma^{d(x)}_{r(y)}(k)d\nu(x)d\nu(y)
\end{aligned}$$

where $d\tilde{H}^{v,t}_{\lambda,\mu} = d\lambda^v(x)d\lambda^t(y)d\mu(\omega_2)d\mu(\omega_3)d\mu(\omega_4)d\mu(u)$,
$d\bar{H}^{r(k_2),r(k_3)}_{\lambda,\mu} = d\lambda_v(x)d\lambda^t(y)d\mu(\omega_2)d\mu(\omega_3)d\mu(\omega_4)d\mu(u)$
and $d\ddot{H}^{v,t}_{\lambda,\mu} = d\lambda_v(x)d\lambda_t(y)d\mu(\omega_3)d\mu(\omega_4)d\mu(u)$.
Then we compute $\chi_\varphi(f^\natural)\ \chi_\varphi(g^\natural)$.

$$\begin{aligned}\chi_\varphi(f^\natural)\chi_\varphi(g^\natural) = &\int f(xk_2)\varphi(x)d\gamma^u(k_2)d\lambda_{r(k_2)}(x)d\mu(u)\times\\ &\int g(yk_4)\varphi(y)d\gamma^v(k_4)d\lambda_{r(k_4)}(y)d\mu(v)\\ =&\int f(x)\varphi(x)d\gamma^u(k_2)d\lambda_{d(k_2)}(x)d\mu(u)\times\int g(y)\varphi(y)d\gamma^v(k_4)d\lambda_{d(k_4)}(y)d\mu(v)\\ =&\int f(x)\varphi(x)d\gamma_u(k_2)d\lambda_{d(k_2)}(x)d\mu(u)\times\int g(y)\varphi(y)d\gamma_v(k_4)d\lambda_{d(k_4)}(y)d\mu(v)\\ =&\int f(x)\varphi(x)d\gamma_{d(x)}(k_2)d\lambda_u(x)d\mu(u)\times\int g(y)\varphi(y)d\gamma_{d(y)}(k_4)d\lambda_v(y)d\mu(v)\\ =&\int f(x)g(y)\varphi(x)\varphi(y)d\nu(x)d\nu(y)\end{aligned}$$

Thus if φ is spherical then $\chi_\varphi(f^\natural * g^\natural) = \chi_\varphi(f^\natural)\chi_\varphi(g^\natural)$. So according above calculations, we have $\int\varphi(xky)d\gamma^{d(x)}_{r(y)}(k) = \varphi(x)\varphi(y)$ for all $x, y \in G$. Conversely if the equality of theorem is verified then φ is spherical always according above calculations. □

The next result shows that spherical functions are eigenfunctions of certain convolution operator.

Theorem 3.2. *Let φ be a bi-K-invariant, bounded continuous function on G. We suppose φ is not identically zero. Then the function φ is spherical if and only if*
(i) $\varphi(u) = 1$ for all $u \in G^{(0)}$,
(ii) for every $f \in I(G\backslash\backslash K)$ there is a complex number $\chi(f)$ such that $f * \varphi = \chi(f)\varphi$.

Proof. Let φ be a spherical function. Then by the previous theorem, we have for all $x, y \in G$, $\int\varphi(xky)d\gamma^{d(x)}_{r(y)}(k) = \varphi(x)\varphi(y)$. In particular for $y = u \in G^{(0)}$, we have $\varphi(x) = \varphi(x)\varphi(u)$ for all $x \in G$ and since $\varphi \neq 0$ we conclude $\varphi(u) = 1$. For $f \in I(G\backslash\backslash K)$ and $x \in G$,

$$\begin{aligned} f*\varphi(x) &= \int f(ky)\varphi(y^{-1}x)d\gamma^v_{r(y)}(k)d\lambda^{r(x)}(y)d\mu(v)\\ &= \int f(y)\varphi(y^{-1}kx)d\gamma^v_{r(x)}(k)d\lambda^{r(k)}(y)d\mu(v)\\ &= \int f(y)(\int\varphi(y^{-1}kx)d\gamma^{r(y)}_{r(x)}(k))d\lambda^v(y)d\mu(v)\\ &= (\int f(y)\varphi(y^{-1})d\nu(y))\varphi(x)\end{aligned}$$

thanks to the left K-invariance of φ and the previous theorem. Conversely, suppose the conditions (i) and (ii) satisfied. Since $\varphi(u) = 1$ for all $u \in G^{(0)}$, we have
$f * \varphi(u) = \chi(f)$. Hence $\int f(y)\varphi(y^{-1})d\nu(y) = \chi(f)$, so $\chi_\varphi(\check{f}) = \chi(f)$. Let now $f, g \in I(G\backslash\backslash K)$. Thanks to (ii) and the associativity of convolution product, we have $(f * g) * \varphi = \chi(f * g)\varphi$ and $(f * g) * \varphi = \chi(f)\chi(g)\varphi$. So $\chi(f * g) = \chi(f)\chi(g)$.

Consequently, we have

$$\chi_\varphi(f * g) = \chi(\check{g} * \check{f}) = \chi(\check{g})\chi(\check{f}) = \chi_\varphi(f)\chi_\varphi(g)$$

So the map χ_φ is a non-trivial character of $I(G\backslash\backslash K)$. □

Theorem 3.3. *Let φ be a bounded spherical function. The mapping $f \mapsto \chi_\varphi(f) = \int f(x)\varphi(x)d\nu(x)$ is a character of $I(G\backslash\backslash K)$ and each non-trivial continuous character of $I(G\backslash\backslash K)$ is of this form.*

Proof. The proof is similar to group case. □

Let $(\pi, \mathcal{H})$ be a unitary representation of G where $\mathcal{H}$ is a Hilbert bundle on the unit space $G^{(0)}$. Let us notice that any continuous section ξ of $\mathcal{H}$ is bounded since $G^{(0)}$ is compact. It follows that any continuous section ξ of $\mathcal{H}$ is square integrable since μ is normalized. We set

$$\Gamma_\mu^K(\mathcal{H}) = \{\xi \in \Gamma_\mu(\mathcal{H}) : \pi(k)\xi(d(k)) = \xi(r(k))\forall k \in K\},$$

the space of K-invariant square integrable section of $\mathcal{H}$. If $\xi \in \Gamma_\mu(\mathcal{H})$, then the section $P_K\xi$ defined by $P_K\xi(u) = \int \pi(k)\xi(d(k))d\gamma^u(k)$ is K-invariant and square integrable. Also $\Gamma_\mu^K(\mathcal{H})$ is a closed subspace of $\Gamma_\mu(\mathcal{H})$.

Theorem 3.4. *Let $\mathcal{H}$ be a Hilbert bundle on $G^{(0)}$. Let $(\pi, \mathcal{H})$ be an irreducible unitary representation of G on $\mathcal{H}$ and let ξ be a K-invariant continuous section of $\mathcal{H}$ such that $\| \xi(u) \|_u = 1$ for all $u \in G^{(0)}$. Then the map $\varphi : x \mapsto < \pi(x)\xi(d(x)), \xi(r(x)) >$ on G is a positive definite spherical function.*

Proof. First, for all $u \in G^{(0)}$, $\varphi(u) = < \pi(u)\xi(u), \xi(u) > = \| \xi(u) \|_u^2 = 1$. Now, let $x \in G$, $k \in K_{r(x)}$ and $k' \in K^{d(x)}$,

$$\begin{aligned}\varphi(kxk') &= < \pi(kxk')\xi(d(k')), \xi(r(k)) > \\ &= < \pi(x)\xi(r(k')), \xi(d(k)) > \\ &= < \pi(x)\xi(d(x)), \xi(r(x)) > = \varphi(x).\end{aligned}$$

Hence φ is bi-K-invariant. φ is bounded since π is unitary and $\| \xi(u) \|_u = 1$ for all $u \in G^{(0)}$. Since ξ is continuous then φ is also continuous (see [12]). In [12] it is proved that φ is positive definite. So it remains to prove that φ is spherical. Let $\tilde{L}^\natural$ be the corresponding representation to π on $I(G\backslash\backslash K)$ in $\Gamma_\mu^K(\mathcal{H})$. Note that $\xi \in \Gamma_\mu^K(\mathcal{H})$ by hypothesis. So according to theorem 3.3 in [15] $dim\Gamma_\mu^K(\mathcal{H}) = 1$. Let $f \in I(G\backslash\backslash K)$ and $x \in G$, we have

$$\begin{aligned}f * \varphi(x) &= \int_{G^{r(x)}} f(y) < \pi(y^{-1}x)\xi(d(x)), \xi(d(y)) > d\lambda^{r(x)}(y) \\ &= < \pi(x)\xi(d(x)), \int_{G^{r(x)}} \overline{f(y)}\pi(y)\xi(d(y))d\lambda^{r(x)}(y) > \\ &= < \pi(x)\xi(d(x)), \tilde{L}^\natural(\bar{f})\xi(r(x)) > \\ &= < \pi(x)\xi(d(x)), c_{\bar{f}}\xi(r(x)) > \\ &= \bar{c}_{\bar{f}}\varphi(x),\end{aligned}$$

where $\bar{c}_{\bar{f}} \in \mathbb{C}$. From Theorem 3.2, it follows that φ is a spherical function. □

As we have recall in preliminaries, for any bounded continuous positive definite function $\varphi : G \to \mathbb{C}$, there exists a unitary representation of G on a Hilbert bundle $\mathcal{H}$, and a bounded continuous cyclic section ξ of $\mathcal{H}$ such that: $\varphi(x) = < \pi(x)\xi(d(x)), \xi(r(x)) >$ for all $x \in G$.

Lemma 3.5. *ξ belong to $\Gamma_\mu^K(\mathcal{H})$ if and only if φ is bi-K-invariant.*

Proof. Since ξ is bounded and the quasi-invariant measure μ is normalized then ξ is square integrable. If ξ is K-invariant then clearly φ is bi-K-invariant. Assume now that φ is bi-K-invariant. Then for all $k \in K$ and $x \in G^{r(k)}$, we have

$$\begin{aligned} < \pi(x)\xi(d(x)), \xi(r(k)) > &= \varphi(x) = \varphi(k^{-1}x) \\ &= < \pi(k^{-1}x)\xi(d(x)), \xi(d(k)) > \\ &= < \pi(x)\xi(d(x)), \pi(k)\xi(d(k)) > . \end{aligned}$$

Since ξ is cyclic, then $\pi(k)\xi(d(k)) = \xi(r(k))$ for all $k \in K$. Hence $\xi \in \Gamma_\mu^K(\mathcal{H})$. □

Lemma 3.6. *Let π be a unitary representation of G on a Hilbert bundle $\mathcal{H}$ admitting a continuous K-invariant cyclic section ξ. If $dim\Gamma_\mu^K(\mathcal{H}) = 1$, then the representation π is irreducible.*

Proof. Let $T = (T_u)_{u \in G^{(0)}}$ be an intertwining unitary operator bundle for π i.e. for all $x \in G$, $\pi(x)T_{d(x)} = T_{r(x)}\pi(x)$. In particular for all $k \in K$, we have $T_{r(k)}\pi(k)\xi(d(k)) = \pi(k)T_{d(k)}\xi(d(k))$. Hence $T_{r(k)}\xi(r(k)) = \pi(k)[T_{d(k)}\xi(d(k))]$. So the section $T\xi$ defined by $T\xi(u) = T_u\xi(u)$ for all $u \in G^{(0)}$ is K-invariant. It is easy to show that $< \pi(f)\xi, \xi >=< \pi(f)T\xi, T\xi >$ for all $f \in L^1(G, \nu)$. Since $T\xi$ is continuous on $G^{(0)}$, which is compact, then $T\xi \in \Gamma_\mu(\mathcal{H})$. So, finally $T\xi \in \Gamma_\mu^K(\mathcal{H})$. It follows that $T\xi = \lambda\xi$, for some complex number λ. For an arbitrary $x \in G$, we obtain:

$$T_{r(x)}\pi(x)\xi(d(x)) = \pi(x)T_{d(x)}\xi(d(x)) = \lambda\pi(x)\xi(d(x)).$$

So, because ξ is a cyclic section we have $T = \lambda I$, where I is the identity operator. Consequently, π is irreducible thanks to Schur Lemma. □

A positive definite function φ is said to be elementary if the unitary representation associated with φ is irreducible.

Theorem 3.7. *Let φ be a bi-K-invariant, continuous, positive definite function such that $\varphi(u) = 1$ for all $u \in G^{(0)}$. Then φ is spherical if and only if φ is elementary.*

Proof. Let φ be a spherical function and π a unitary representation of G associated with φ. Then $\varphi(x) =< \pi(x)\xi(d(x)), \xi(r(x)) >$, where ξ is a continuous cyclic section on $G^{(0)}$. Notice that by the Lemma 3.5, $\xi \in \Gamma_\mu^K(\mathcal{H})$. For any $f \in I(G\backslash\backslash K)$, we have $f * \varphi = \chi(f)\varphi$, where $\chi(f) \in \mathbb{C}$. So, we obtain for $x \in G$;

$$\begin{aligned} < \pi(x)\xi(d(x)), \tilde{L}^\natural(\bar{f})\xi(r(x)) > &= \int f(y) \times < \pi(y^{-1}x)\xi(d(x)), \xi(d(y)) > d\lambda^{r(x)}(y) \\ &= \int f(y) < \pi(y^{-1}x)\xi(d(y^{-1}x)), \xi(r(y^{-1}x)) > d\lambda^{r(x)}(y) \\ &= \int_{G^{r(x)}} f(y)\varphi(y^{-1}x)d\lambda^{r(x)}(y) \\ &= f * \varphi(x) =< \pi(x)\xi(d(x)), \overline{\chi(f)}\xi(r(x)) >, \end{aligned}$$

and since ξ is cyclic then $\tilde{L}^\natural(\bar{f})\xi = \overline{\chi(f)}\xi$. Now ξ is also a cyclic section for $\tilde{L}^\natural$ in $\Gamma_\mu^K(\mathcal{H})$, so we have $\dim\Gamma_\mu^K(\mathcal{H})$=1. By Lemma 3.6, π is irreducible and hence φ is elementary.

Conversely suppose that φ is elementary, that is π is irreducible. Since $\xi \in \Gamma_\mu^K(\mathcal{H})$, then $\dim\Gamma_\mu^K(\mathcal{H}) > 0$. Hence, according to theorem 3.3 of [15], $\dim\Gamma_\mu^K(\mathcal{H}) = 1$. For any $f \in I(G\backslash\backslash K)$ and for all $x \in G$, we now have

$$\begin{aligned} f * \varphi(x) &= < \pi(x)\xi(d(x)), \tilde{L}^\natural(f)\xi(r(x)) > \\ &= < \pi(x)\xi(d(x)), \chi(f)\xi(r(x)) > \\ &= \overline{\chi(f)}\varphi(x), \end{aligned}$$

where $\chi(f)$ is a constant depending on f. From Theorem 3.2, it now follows that φ is a spherical function. □

4. Harmonic analysis on pair (G,K)

In [15], the authors prove that if K is transitive then (G, K) is a Gelfand pair if and only if (G_m^m, K_m^m) is a Gelfand pair for any $m \in G^{(0)}$. So let $m \in G^{(0)}$ be a fixed unit. In the following result, we establish a connection between the positive definite spherical function on G and those of G_m^m.

Theorem 4.1. *(a) If φ is a positive definite spherical function on G then the restriction φ_m of φ to G_m^m is a positive definite spherical function on G_m^m.*
(b) Given a positive definite spherical function φ_m on G_m^m, there is a positive definite spherical function φ on G such that $\varphi \mid G_m^m = \varphi_m$.
(c) If φ and φ' are two positive definite spherical functions on G then $\varphi = \varphi'$ if and only if $\varphi \mid G_m^m = \varphi' \mid G_m^m$.

Proof. (a) Let φ be a positive definite spherical function on G and $\varphi_m = \varphi \mid G_m^m$ its restriction to G_m^m. If π denotes the irreducible unitary representation on G associated to φ then $\varphi(x) =< \pi(x)\xi(d(x)), \xi(r(x)) >$ for all $x \in G$; where ξ is a cyclic continuous section on G. In particular, for all $x \in G_m^m$, $\varphi_m(x) =< \pi_m(x)\xi(m), \xi(m) >$, where π_m is the restriction of π to G_m^m and $\xi(m)$ is a cyclic vector for π_m. We know that π_m is a unitary representation [1, 17] and it is irreducible (see [15]). So according to classical theory of spherical functions [3, 5], φ_m is a positive definite spherical function.
(b) Let (π_m, H) and h be respectively the irreducible unitary representation of G and the cyclic vector associated to φ_m such that $\varphi_m(x) =< \pi_m(x)h, h >$ for all $x \in G_m^m$. Let π be the unitary representation of G on the Hilbert bundle $\mathcal{H} = G^{(0)} \times H$ associated to π_m such that $\pi \mid G_m^m = \pi_m$ (see [1, 17]). It is irreducible (see [15]). Put $\xi(u) = (u, h)$ for any $u \in G^{(0)}$. ξ is a section of $G^{(0)} \times H$ which is square integrable. Since h is cyclic, then ξ is cyclic . Then the function φ defined by $\varphi(x) =< \pi(x)\xi(d(x)), \xi(r(x)) >$ for all $x \in G$ is a positive definite spherical function on G thanks to Theorem 3.4. Also, for $x \in G_m^m$, we have $\varphi(x) =< \pi_m(x)\xi(m), \xi(m) >=< \pi_m(x)h, h >= \varphi_m(x)$.
(c) Let φ and φ' be two positive definite spherical functions on G such that $\varphi \mid G_m^m = \varphi' \mid G_m^m$. For $x \in G$, $k \in K_{r(x)}^m$ and $k' \in K_m^{d(x)}$ we have $kxk' \in G_m^m$ and since φ and φ' are bi-$K-$invariant then

$$\varphi(x) = \varphi(kxk') = \varphi_m(kxk') = \varphi'_m(kxk') = \varphi'(kxk') = \varphi'(x),$$

where we have set $\varphi_m = \varphi \mid G_m^m$ and $\varphi'_m = \varphi' \mid G_m^m$. The part "If" is trivial. □

Denote by Z (respectively by Z_m) the set of positive definite spherical functions on G (respectively on G_m^m). We know by classical theory (see [3, 5]) that Z_m equipped with the topology $\sigma(L^\infty, L^1)$ is locally compact. Let us consider the map $\psi : \varphi \mapsto \varphi_m$ from Z to Z_m. It is bijective thanks to Theorem 4.1. If we equip Z with the coarsest topology making continuous the map ψ, then it is locally compact.

Definition 4.1. The *Fourier transform* of a function $f \in I(G\backslash\backslash K)$ will be the function $\hat{f}$, defined on Z by $\hat{f}(\varphi) = \int_G f(x)\varphi(x^{-1})d\nu(x)$ for all $\varphi \in Z$.

We have the following properties

Theorem 4.2. (1) *For all* $f \in I(G\backslash\backslash K)$, $\hat{f}(\varphi) = \hat{f_m}(\varphi_m)$ *for all* $\varphi \in Z$*;*

(2) *For all* $f, g \in I(G\backslash\backslash K)$, $\widehat{f * g}(\varphi) = \hat{f}(\varphi)\hat{g}(\varphi)$ *for all* $\varphi \in Z$

(3) $\hat{f}$ *is a continuous function on* Z*, vanishing at "infinity" ; moreover* $\| \hat{f} \|_\infty \leq \| f \|_I$ *where* $\| \hat{f} \|_\infty = sup \mid \hat{f}(\varphi) \mid$*;*

(4) *The function* $f \mapsto \hat{f}$ *is a linear transformation;*

(5) $\bar{\hat{f}} = (\hat{f^*})$

Proof. (1)

$$\begin{aligned}
\hat{f}(\varphi) &= \int f(x)\varphi(x^{-1})d\nu(x) \\
&= \int f(x^{-1})\varphi(kxk^{'})d\gamma^m_{r(x)}(k)d\gamma^{d(x)}_m(k^{'})d\lambda^u(x)d\mu(u) \\
&= \int f(x^{-1})\varphi(kxk^{'})d\gamma^m_{r(x)}(k)d\gamma^{d(x)}_m(k^{'})d\beta^u_v(x)d\mu(v)d\mu(u) \\
&= \int f(k^{'}x^{-1}k)\varphi(x)d\gamma^m_u(k)d\gamma^v_m(k^{'})d\beta^{r(k)}_{d(k^{'})}(x)d\mu(v)d\mu(u) \\
&= \int f_m(x^{-1})\varphi_m(x)d\beta^m_m(x) = \hat{f_m}(\varphi_m)
\end{aligned}$$

(2)

$$\begin{aligned}
\widehat{f * g}(\varphi) &= \int (f * g)(x)\varphi(x^{-1})d\nu(x) \\
&= \int f(y)g(x)\varphi(x^{-1}y^{-1})d\lambda^{d(y)}(x)d\lambda^u(y)d\mu(u) \\
&= \int f(y)g(kx)\varphi(x^{-1}y^{-1})d\gamma^v_{d(y)}(k)d\lambda^{d(k)}(x)d\lambda^u(y)d\mu(u)d\mu(v) \\
&= \int f(y)g(x^{-1})\varphi(xky^{-1})d\gamma^{d(x)}_{d(y)}(k)d\lambda_v(x)d\lambda^u(y)d\mu(u)d\mu(v) \\
&= \int f(y)g(x^{-1})(\int \varphi(xky^{-1})d\gamma^{d(x)}_{d(y)}(k))d\nu(x)d\nu(y) \\
&= \int f(y)g(x^{-1})\varphi(x)\varphi(y^{-1})d\nu(x)d\nu(y) = \hat{f}(\varphi)\hat{g}(\varphi)
\end{aligned}$$

For (3), (4) and (5) we can use (1) since these properties are true on groups. □

Denote by $M^1(Z)$ (respectively by $M^1(Z_m)$) the space of bounded complex measure on Z (respectively on Z_m). Put $M_0(Z) = \{\mu \in M^1(Z) : \mu \text{ positive}, \| \mu \| \leq 1\}$ and $M_0(Z_m) = \{\mu \in M^1(Z_m) : \mu \text{ positive}, \| \mu \| \leq 1\}$ where $\| . \|$ denotes the norm of μ. The following result extends the theorem of Bochner to groupoids.

Theorem 4.3. *Let* φ *be a bi-*K*-invariant continuous positive definite function on* G *such that* $\varphi(u) \leq 1$ *for all* $u \in G^{(0)}$*. Then there exists a unique* $\mu_Z \in M_0(Z)$ *such that for all* $x \in G$*,*

$$\varphi(x) = \int_Z \omega(x)d\mu_Z(\omega).$$

Proof. For $m \in G^{(0)}$, the restriction φ_m of φ to G_m^m is also bi-K_m^m-invariant continuous positive definite function on G_m^m such that $\varphi_m(m) \leq 1$. Then, according to the classical Bochner theorem on groups, there exists a unique measure $\mu_m \in M_0(Z_m)$ such that for all $x \in G_m^m$, $\varphi_m(x) = \int_{Z_m} \omega_m(x) d\mu_m(\omega_m)$. The choice above of the topology of Z makes ψ a continuous open bijection and therefore an homeomorphism. So let us put $\mu_Z = \psi^{-1}(\mu_m)$ the image measure of μ_m by ψ^{-1}. For all $x \in G$, $k \in K_{r(x)}^m$ and $k' \in K_m^{d(x)}$ we have,

$$\begin{aligned} \varphi(x) &= \varphi(kxk') = \varphi_m(kxk') \\ &= \int_{Z_m} \omega_m(kxk') d\mu_m(\omega_m) \\ &= \int_Z \psi(\omega)(kxk') d\mu_Z(\omega) \\ &= \int_Z \omega(kxk') d\mu_Z(\omega) \\ &= \int_Z \omega(x) d\mu_Z(\omega) \end{aligned}$$

□

Let f be a bi-K-invariant continuous positive definite function on G and belong to $I(G\backslash\backslash K)$. By previous theorem, there exists a unique $\mu_Z^f \in M_0(Z)$ such that for all $x \in G$, $f(x) = \int_Z \omega(x) d\mu_Z^f(\omega)$. $\mu_Z^f = \psi^{-1}(\mu_m^{f_m})$ where $\mu_m^{f_m}$ corresponds to the restriction f_m of f to G_m^m. According to Fourier's inversion formula on groups there exists a unique positive measure σ_m on Z_m such that $d\mu_m^{f_m} = \hat{f_m} d\sigma_m$ and $\hat{f_m} \in L^1(Z_m, \sigma_m)$. Let us put $\sigma = \psi^{-1}(\sigma_m)$. So thanks to Theorem 4.2 (1) $\hat{f} \in L^1(Z, \sigma)$ and for all $x \in G$, $k \in K_{r(x)}^m$ and $k' \in K_m^{d(x)}$ we have,

$$\begin{aligned} f(x) &= f(kxk') = f_m(kxk') \\ &= \int_{Z_m} \omega_m(kxk') \hat{f_m}(\omega_m) d\sigma_m(\omega_m) \\ &= \int_Z \psi(\omega)(kxk') \hat{f}(\omega) d\sigma(\omega) \\ &= \int_Z \omega(kxk') \hat{f}(\omega) d\sigma(\omega) \\ &= \int_Z \omega(x) \hat{f}(\omega) d\sigma(\omega). \end{aligned}$$

Let us denote by V^1 the set of bi-K-invariant continuous positive definite function on G which belong to $I(G\backslash\backslash K)$. Thus we have proved the following result.

Theorem 4.4. *There exists a unique positive measure σ on Z such that for all $f \in V^1$, $d\mu_Z^f = \hat{f} d\sigma$ and $\hat{f} \in L^1(Z, \sigma)$.*

Let σ be the positive measure on Z, obtained in Theorem 4.4, we have the analogue of Plancherel formula for $f \in I(G\backslash\backslash K)$.

Theorem 4.5. *For every $f \in I(G\backslash\backslash K) \cap L^2(G, \nu)$ one has:*
(i)$\hat{f} \in L^2(Z, \sigma)$
(ii) $\int_G \mid f(x) \mid^2 d\nu(x) = \int_Z \mid \hat{f}(\omega) \mid^2 d\sigma(\omega)$

Proof. Let us put $g = f * \tilde{f}$ for $f \in C_c(G)$, where $\tilde{f} = \overline{\check{f}}$. We know that $g \in C_c(G)$ and is definite positive (see proposition 3.3 of [11]). So by density, for $f \in I(G\backslash\backslash K)$, $g \in V^1$ and hence $\hat{g} \in L^1(Z, \sigma)$. Also for all $x \in G$, $g(x) = \int_Z \omega(x)\hat{g}(\omega)d\sigma(\omega)$ thanks to Theorem 4.4. We have thanks to Theorem 4.2 (1) and (5), $\hat{g}(\omega) =\mid \hat{f}(\omega) \mid^2$, so $\hat{f} \in L^2(Z, \sigma)$. And since $g(u) = \int \mid f(x) \mid^2 d\nu(x)$, we have $\int \mid f(x) \mid^2 d\nu(x) = \int_Z \hat{g}(\omega)d\sigma(\omega) = \int_Z \mid \hat{f}(\omega) \mid^2 d\sigma(\omega)$. □

5. Examples

Example 5.1. Let G be a locally compact group and K a compact subgroup of G such that (G, K) is a Gelfand pair. Let M be a topological compact space. Let us consider the transitive trivial Lie groupoid $M \times G \times M$ with groupoid structure which is defined in the following way: $d(m, g, n)=(n, e, n)$,
$r(m, g, n)=(m, e, m)$, $(m, g, n)(n, h, p) = (m, gh, p)$ and $(m, g, n)^{-1} = (n, g^{-1}, m)$ where $m, n, p \in M$, $g, h \in G$ and e the identity element of G. The set $M \times K \times M$ equipped with the above groupoid structure is a compact subgroupoid of G. $(M \times G \times M, M \times K \times M)$ is a Gelfand pair. If ω is a spherical function for (G, K) then the function φ defined by : $\varphi(m, g, n) = \omega(g)$ is spherical for $(M \times G \times M; M \times K \times M)$.

This method of construction can be generalized.

Theorem 5.1. *If ω_m is a spherical function on G_m^m for $m \in G^{(0)}$ then ω defined for $x \in G$ by $\omega(x) = \int \omega_m(kxk')d\gamma_{r(x)}^m(k)d\gamma_m^{d(x)}(k')$ is a spherical function on G.*

Proof. It suffices to prove the functional equation of Theorem 3.1. For $x, y \in G$, we have

$$\begin{aligned}
\int \omega(xky)d\gamma_{r(y)}^{d(x)}(k) &= \int \omega_m(k_1xkyk_1')d\gamma_{r(x)}^m(k_1)d\gamma_m^{d(y)}(k_1')d\gamma_{r(y)}^{d(x)}(k) \\
&= \int \omega_m((k_1xk_2)(k_2^{-1}kk_3^{-1})(k_3yk_1'))d\gamma_{r(x)}^m(k_1)d\gamma_m^{d(y)}(k_1')\times \\
&\quad d\gamma_{d(k_3)}^{r(k_2)}(k)d\gamma_m^{d(x)}(k_2)d\gamma_{r(y)}^m(k_3) \\
&= \int \omega_m(k_1xk_2)\omega_m(k_3yk_1')d\gamma_{r(x)}^m(k_1)d\gamma_m^{d(y)}(k_1')\times \\
&\quad d\gamma_m^{d(x)}(k_2)d\gamma_{r(y)}^m(k_3) \\
&= \int \omega_m(k_1xk_2)d\gamma_{r(x)}^m(k_1)d\gamma_m^{d(x)}(k_2)\times \\
&\quad \int \omega_m(k_3yk_1')d\gamma_m^{d(y)}(k_1')d\gamma_{r(y)}^m(k_3) = \omega(x)\omega(y).
\end{aligned}$$

□

References

[1] M. Buneci, Groupoid Algebras For Transitive Groupoids, *Math. Rep.* **5(55)** (2003), 9–26.
[2] J. Dieudonné, Gelfand Pairs And Spherical Functions, *Int. J. Math. Math. Sci.* **2**(2) (1979), 153–162.
[3] V. Dijk, *Introduction to Harmonic Analysis and Generalized Gelfand Pairs*, De Gruyter Studies in Mathematics **36**, Walter de Gruyter and Co, Berlin, 2009.
[4] J. Dixmier, *Les C^*-Algèbres et Leurs Représentations*, Gauthier-Villars, Paris, 1964.
[5] J. Faraut, Analyse Harmonique sur les Paires de Gelfand et les Espaces Hyperboliques, Les cours du CIMPA, Nancy(1980), 315–446.

[6] I.F. Gelfand, Spherical Functions On Symmetrics Spaces, *Dokl. Akad. Nauk.* **70** (1960), 5–8.
[7] R. Godement, A Theory Of Spherical Functions I, *Trans. Amer. Math. Soc.* **73** (1952), 496–556.
[8] P. Hahn, The regular representations of measure groupoids, *Trans. Amer. Math. Soc.* **242** (1978), 34–72.
[9] P. Hahn, Haar measure for Groupoids, *Trans. Amer. Math. Soc.* **242** (1978), 1-33.
[10] K. Mackenzie, General Theory of Lie Groupoids and Lie Algebroids, *London Math. Soc. Lecture Note Ser.* **213**, Cambridge University Press, 2005.
[11] K.J. Oty, Fourier-Stieltjes Algebras of r-Discrete Groupoids, *J. Operator Theory,* **4** (1999), 175–197.
[12] A. Ramsay and M.E. Walter, Fourier-Stieltjes Algebras of Locally Compact Groupoid, *J. Funct. Anal,* **148**(2) (1997), 314–367.
[13] J. Renault, The Ideal Structure of Grupoid Crossed Product C^*-Algebras, *J. Operator Theory* **25** (1991), 3–36.
[14] J. Renault, *A Groupoid Approach to C^*-Algebras*, *Lecture Notes in Math.* **793**, Springer-Verlag, 1980.
[15] I. Toure and K. Kangni, On Gelfand Pairs associated to transitive groupoids *Opuscula Math.,* 2013, **33**,no. 4, 751–762.
[16] I. Toure and K. Kangni, On Groupoid Algebras of Biinvariant Functions *Int. J. Math. Anal.* 2012, **6**(43) (2012). 2101–2108.
[17] J.J. Westman, Harmonic Analysis on groupoids *Pacific J. Math.* **27**(3) (1968), 621–632.

(Ibrahima Toure, Kinvi Kangni) UFR de Mathématiques et Informatique, Université Félix-Houphouet Boigny, Cocody-Abidjan, 22 BP 582 Abidjan 22 Côte-d'Ivoire
E-mail address: `toureibt@yahoo.fr, kangnikinvi@yahoo.fr`