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Congruence relations on pseudo BE–algebras

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Abstract. In this paper, we consider the notion of congruence relation on pseudo BE–algebras and construct quotient pseudo BE–algebra via this congruence relation. Also, we use the notion of normal pseudo filters and get a congruence relation.

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1. Introduction

Some recent researchers led to generalizations of some types of algebraic structures by pseudo structures. G Georgescu and A. Iorgulescu [3], and independently J. Rachunek [11], introduced pseudo MV–algebras which are a non-commutative generalization of MV–algebras. The notions of pseudo BL–algebras and pseudo BCK–algebras were introduced and studied by G. Georgescu and A. Iorgulescu [9, 10, 4]. A. Walendziak gave a system of axioms defining pseudo BCK– algebras [12]. Y, B. Jun and et al. introduced the concepts of pseudo-atoms, pseudo BCI–ideals and pseudo BCI–homomorphisms in pseudo BCI– algebras and characterizations of a pseudo BCI–ideal, and provide conditions for a subset to be a pseudo BCI–ideal [5]. Y. H. Kim and K. S. So [7], discuss on minimal elements in pseudo BCI–algebras.

The notion of BE–algebras was introduced by H. S. Kim and Y. H. Kim [6]. We generalized the notion of BE–algebras and introduced the notion of pseudo BE–algebras, pseudo subalgebras, pseudo filters and investigated some related properties [1]. We introduced the notion of distributive pseudo BE–algebra and normal pseudo filters and prove some basic properties. Furthermore, the notion of pseudo upper sets in pseudo BE– algebras introduced and prove that the every pseudo filter F of X is union of pseudo upper sets. We show that in distributive pseudo BE–algebras normal pseudo filters and pseudo filters are equivalent [2].

In the present paper, we apply the notion of congruence relations to pseudo BE–algebras and discuss on the quotient algebras via this congruence relations. It is a natural question which is the relationships between congruence relations on pseudo BE–algebras and (normal)pseudo filters. From here comes the main motivation for this. We show that quotient of a pseudo BE–algebra via a congruence relation is a pseudo BE–algebra and prove that, if X is a distributive pseudo BE–algebra, then it becomes to a BE–algebra.

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2. Preliminaries

In this section we review the basic definitions and some elementary aspects that are necessary for this paper.

Definition 2.1. [1] An algebra $(X; *, \diamond, 1)$ of type $(2, 2, 0)$ is called a *pseudo BE–algebra* if it satisfies in the following axioms:

$(pBE1)$ $x * x = 1$ and $x \diamond x = 1$,
$(pBE2)$ $x * 1 = 1$ and $x \diamond 1 = 1$,
$(pBE3)$ $1 * x = x$ and $1 \diamond x = x$,
$(pBE4)$ $x * (y \diamond z) = y \diamond (x * z)$,
$(pBE5)$ $x * y = 1 \Leftrightarrow x \diamond y = 1$, for all $x, y, z \in X$.

In a pseudo BE–algebra, one can introduce a binary relation " $\leq$ " by $x \leq y \Leftrightarrow x * y = 1 \Leftrightarrow x \diamond y = 1$, for all $x, y \in X$. From now on X is a pseudo BE–algebra, unless otherwise is stated and we note that if $(X; *, \diamond, 1)$ is a pseudo BE–algebra, then $(X; \diamond, *, 1)$ is a pseudo BE–algebra, too.

Remark 2.1. If X is a pseudo BE–algebra satisfying $x * y = x \diamond y$, for all $x, y \in X$, then X is a BE–algebra.

Proposition 2.1. [1, 2] *The following statements hold:*

(1) $x * (y \diamond x) = 1$, $x \diamond (y * x) = 1$,
(2) $x \diamond (y \diamond x) = 1$, $x * (y * x) = 1$,
(3) $x \diamond ((x \diamond y) * y) = 1$, $x * ((x * y) \diamond y) = 1$,
(4) $x * ((x \diamond y) * y) = 1$, $x \diamond ((x * y) \diamond y) = 1$,
(5) *If* $x \leq y * z$, *then* $y \leq x \diamond z$,
(6) *If* $x \leq y \diamond z$, *then* $y \leq x * z$,
(7) $1 \leq x$, *implies* $x = 1$.
(8) *If* $x \leq y$, *then* $x \leq z * y$ *and* $x \leq z \diamond y$,

for all $x, y, z \in X$.

Definition 2.2. [1] A non-empty subset F of X is called a *pseudo filter* of X if it satisfies in the following axioms:

$(pF1)$ $1 \in F$,
$(pF2)$ $x \in F$ and $x * y \in F$ imply $y \in F$.

Proposition 2.2. [1] *Let* $F \subseteq X$ *and* $1 \in F$. F *is a pseudo filter if and only if* $x \in F$ *and* $x \diamond y \in F$ *imply* $y \in F$, *for all* $x, y \in X$.

Theorem 2.3. [1] *Let* X *be a pseudo* BE*-algebra. Then every pseudo filter of* X *is a pseudo sub-algebra.*

Definition 2.3. [2] X is said to be *distributive* if it satisfies in the following condition:

$$x * (y \diamond z) = (x * y) \diamond (x * z), \ \textit{for all } x, y, z \in X.$$

Theorem 2.4. [2] *Let* X *be a distributive and* $x \leq y$. *Then*

(i) $z * x \leq z * y$, *and* $z * x \leq z \diamond y$,
(ii) $z \diamond x \leq z * y$, *and* $z \diamond x \leq z \diamond y$,

for all $x, y, z \in X$.

Proposition 2.5. [2] *Let* X *be a distributive. Then*

(i) $y * z \leq (x * y) * (x * z)$, *and* $y * z \leq (x * y) \diamond (x * z)$,
(ii) $y \diamond z \leq (x * y) * (x * z)$, *and* $y \diamond z \leq (x * y) \diamond (x * z)$,
(iii) $A(x * y) = A(x \diamond y)$,

for all $x, y, z \in X$.

Definition 2.4. [2] A pseudo filter F is said to be *normal*, if for all $x, y \in X$

$$x * y \in F \text{ if and only if } x \diamond y \in F.$$

Theorem 2.6. [2] *Let X be distributive. Then every pseudo filter is normal.*

Theorem 2.7. [2] *Let $(X; *, \diamond, 1)$ be a distributive pseudo BE–algebra. $(X, \diamond, *, 1)$ is a distributive pseudo BE–algebra if and only if $(X; *, 1)$ is a BE–algebra (i. e. $x * y = x \diamond y$, for all $x, y \in X$).*

3. Congruences relations on pseudo BE–algebras

Quotient algebras are a basic tool for exploring the structures of pseudo BE–algebras. There are some relations between pseudo filters, pseudo congruence and quotient pseudo BE–algebras. We define the notion of congruence relations on pseudo BE–algebras and prove that the quotient algebra $(X/\theta; *, \diamond, C_1)$ is a pseudo BE–algebra.

Definition 3.1. Let "θ" be an equivalence relation on X. "θ" is called:

(*i*) *Left congruence* relation on X if $(x, y) \in \theta$ implies $(u * x, u * y) \in \theta$ and $(u \diamond x, u \diamond y) \in \theta$, for all $u \in X$.

(*ii*) *Right congruence* relation on X if $(x, y) \in \theta$ implies $(x * v, y * v) \in \theta$ and $(x \diamond v, y \diamond v) \in \theta$, for all $v \in X$.

(*iii*) *Congruence* relation on X if has the substitution property with respect to "$*$" and "$\diamond$", that is, for any $(x, y), (u, v) \in \theta$ we have $(x*u, y*v) \in \theta$ and $(x\diamond u, y\diamond v) \in \theta$.

Example 3.1. (*i*). It is obvious that $\nabla = X \times X$ and $\triangle = \{(x, x) \mid x \in X\}$ is a congruence relation on X.
(*ii*). Let $X = \{1, a, b, c, d\}$ and operations "$*$" and "$\diamond$" defined as follows:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	a	1	1
b	1	1	1	1	1
c	1	a	a	1	1
d	1	a	b	c	1

$\diamond$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	c	1	1
b	1	1	1	1	1
c	1	a	b	1	1
d	1	a	b	c	1

Set $\theta_1 := \triangle \cup \{(d, 1), (1, d)\}$ and $\theta_2 := \triangle \cup \{(1, a), (a, 1)\}$. We can see that θ_1 is a congruence relation on X and θ_2 is a left congruence relation on X. Since $(1, a) \in \theta_2$, and $(b, a) = (1 * b, a * b) \notin \theta_2$, it follows that θ_2 is not a right congruence relation.
(*iii*). Let $X = \{1, a, b, c\}$, operations "$*$" and "$\diamond$" defined as follows:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	1	a
b	1	1	1	1
c	1	1	1	1

$\diamond$	1	a	b	c
1	1	a	b	c
a	1	1	1	b
b	1	1	1	1
c	1	1	1	1

Then, $(X; *, \diamond, 1)$ is a pseudo BE–algebra. If set $\theta_3 = \Delta \cup \{(b, c), (c, b)\}$, then θ_3 is a right congruence relation. Since $(b, c) \in \theta_3$ and $a \in X$, but $(1, a) = (a * b, a * c) \notin \theta_3$, it follows that θ_3 is not a left congruence neither a congruence relation.

For any $x \in X$, we define

$$\phi_x = \{(a, b) \in X \times X : x * a = x * b \text{ and } x \diamond a = x \diamond b\}.$$

Proposition 3.1. *ϕ_x is a left congruence relation on X, for all $x \in X$.*

Proof. It is obvious that ϕ_x is an equivalence relation on X. Let $(a, b) \in \phi_x$ and $u \in X$. Hence $x * a = x * b$. Now, we have $x * (u * a) = u * (x * a) = u * (x * b) = x * (u * b)$. Therefore, $(u * a, u * b) \in \phi_x$. By a similar way $(u \diamond a, u \diamond b) \in \phi_x$. □

The following example shows that ϕ_x is not a right congruence relation on X, in general.

Example 3.2. Let $X = \{1, a, b, c\}$ and operations " $*$ " and " $\diamond$ " defined as follows:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	a	1	c
c	1	b	1	1

$\diamond$	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	c	1	c
c	1	c	1	1

Then, $(X; *, \diamond, 1)$ is a pseudo BE–algebra. It can be seen that

$$\phi_c = \{(1, 1), (a, a), (b, b), (c, c), (1, b), (b, 1), (b, c), (c, b)\}$$

is a left congruence relation on X, but it is not right congruence relation because $(c, b) \in \phi_c$ but $(c * a, b * a) = (b, a) \notin \phi_c$.

Proposition 3.2. *Let X be distributive. Then ϕ_x is a right congruence relation on X, for all $x \in X$.*

Proof. It is sufficient to show that if $(a, b) \in \phi_x$ and $v \in X$, then $(a * v, b * v) \in \phi_x$. Let $(a, b) \in \phi_x$ and $v \in X$. Hence $x * a = x * b$ and $x \diamond a = x \diamond b$. Now, by using distributivity of X we have $x * (a \diamond v) = (x * a) \diamond (x * v) = (x * b) \diamond (x * v) = x * (b \diamond v)$. Therefore, $(a \diamond v, b \diamond v) \in \phi_x$. By a similar way $(a * v, b * v) \in \phi_x$. □

Example 3.3. In Example 3.2, consider

$$c * (c \diamond a) = c * c = 1 \neq b = 1 \diamond b = (c * c) \diamond (c * a),$$

then X is not distributive. Also we showed that ϕ_c is not a right congruence relation.

Let $pCon(X)$ be the set of all congruence relations on X and respectively $pCon_L(X)$ ($pCon_R(X)$) be the set of all the left (right) congruence relations on X. It is clear that $pCon(X) = pCon_L(X) \cap pCon_R(X)$. For $\theta \in pCon(X)$ we will denote $C_x(\theta) = \{y \in X : y \sim_\theta x\}$, abbreviated by C_x. We will call C_x the equivalence class containing x and so $X/\theta = \{C_x : x \in X\}$.

Theorem 3.3. *Let $\theta \in pCon(X)$. Then $C_1 = \{x \in X : x \sim_\theta 1\}$ is a pseudo filter of X.*

Proof. Since θ is a reflexive relation, we see that $(1, 1) \in \theta$ and so $1 \sim_\theta 1$. Thus $1 \in C_1$. Now, let $x, y \in X$. Assume that $a \in C_1$, $a * x \in C_1$. Then $a * x \sim_\theta 1$. Now, we have $x \diamond (a * x) \sim_\theta x \diamond 1$. Thus $1 \sim_\theta a \sim_\theta x$ and so $x \in C_1$. This shows that C_1 is a pseudo filter of X. □

Note. Let $\theta \in pCon(X)$. Define operations " $*$ " and " $\diamond$ " on X/θ by $C_x * C_y = C_{x*y}$ and $C_x \diamond C_y = C_{x \diamond y}$. Let $\nu : X \to X/\theta$ be such that $\nu(x) = C_x$ for all $x \in X$. Then, ν is an epimorphism. In fact $\nu(x * y) = C_{x*y} = C_x * C_y = \nu(x) * \nu(y)$ and $\nu(x \diamond y) = C_{x \diamond y} = C_x \diamond C_y = \nu(x) \diamond \nu(y)$. ν is called the *natural homomorphism* from X to X/θ.

Proposition 3.4. *The following statements hold:*
(i) *if* $\theta = X \times X$, *then* $X/\theta = \{C_1\}$,
(ii) *if* $\theta = \triangle_X$, *then* $X/\theta = \{X\}$,
(iii) *if* $x \leq y$, *then* $C_x \leq C_y$.

Proof. (i). Let $C_x \in X/\theta$, for some $x \in X$. Since $\theta = X \times X$, we have $(x, y) \in \theta$ for all $y \in X$. Hence $C_x = C_y$. Putting $y := 1$, then $C_x = C_1$. Therefore, $X/\theta = \{C_1\}$.

(ii). Let $C_x \in X/\theta$, for some $x \in X$. Since $\theta = \triangle_X$, we have $C_x = \{x\}$. Therefore, $X/\theta = \{X\}$.

(iii). Since $x \leq y$, we get that $x * y = 1$ and $x \diamond y = 1$. Hence $C_{x*y} = C_1 = C_x * C_y$ and $C_{x\diamond y} = C_1 = C_x \diamond C_y$. Therefore, $C_x \leq C_y$. □

Proposition 3.5. *Let* $\theta \in pCon(X)$. *Then* $(X/\theta; *, \diamond, C_1)$ *is a pseudo* BE*–algebra.*

Proof. If $C_x, C_y, C_z \in X/\theta$, then we have
($pBE1$) $C_x * C_x = C_1$ and $C_x \diamond C_x = C_1$,
($pBE2$) $C_x * C_1 = C_1$ and $C_x \diamond C_1 = C_1$,
($pBE3$) $C_1 * C_x = C_x$ and $C_1 \diamond C_x = C_x$,
($pBE4$) $C_x * (C_y \diamond C_z) = C_y \diamond (C_x * C_z)$,
($pBE5$) $C_x \leq C_y \Leftrightarrow C_x * C_y = C_1 \Leftrightarrow C_x \diamond C_y = C_1$.
Then, $(X/\theta; *, \diamond, C_1)$ is a pseudo BE–algebra. □

Example 3.4. Consider congruence relation θ_1 in Example 3.1(ii), then

$$X/\theta_1 = \{C_1 = C_d = \{1, d\}, C_a = \{a\}, C_b = \{b\}, C_c = \{c\}\},$$

with the operations "$*$" and "$\diamond$" defined by following table is a pseudo BE–algebra.

$*$	C_1	C_a	C_b	C_c
C_1	C_1	C_a	C_b	C_c
C_a	C_1	C_1	C_a	C_1
C_b	C_1	C_1	C_1	C_1
C_c	C_1	C_a	C_a	C_1

$\diamond$	C_1	C_a	C_b	C_c
C_1	C_1	C_a	C_b	C_c
C_a	C_1	C_1	C_c	C_1
C_b	C_1	C_1	C_1	C_1
C_c	C_1	C_a	C_b	C_1

Theorem 3.6. *Let* X *be distributive and* $\theta \in pCon(X)$. *Then* $(X/\theta; *, \diamond, C_1)$ *is too.*

Proof. Let $C_x, C_y, C_z \in X/\theta$, for any $x, y, z \in X$. Then

$$\begin{aligned} C_x * (C_y \diamond C_z) = C_x * C_{y\diamond z} &= C_{x*(y\diamond z)} \\ &= C_{(x*y)\diamond(x*z)} \\ &= C_{x*y} \diamond C_{x*z} \\ &= (C_x * C_y) \diamond (C_x * C_z). \end{aligned}$$

Therefore, X/θ is distributive. □

Proposition 3.7. *Let* $f : X \to Y$ *be a homomorphism. Then*
(i) $f(1) = 1$,
(ii) f *has the isotonic property, i. e., if* $x \leq y$, *then* $f(x) \leq f(y)$, *for all* $x, y \in X$.

Proof. (i). Let $x \in X$. Since $x * x = x \diamond x = 1$ and f is a homomorphism, we see that $f(1) = f(x * x) = f(x) * f(x) = 1$ and $f(1) = f(x \diamond x) = f(x) \diamond f(x) = 1$. Hence $f(1) = 1$.

(ii). If $x \leq y$, Then $x * y = x \diamond y = 1$. So, (i) implies

$$f(x) * f(y) = f(x * y) = f(1) = 1, \text{ and } f(x) \diamond f(y) = f(x \diamond y) = f(1) = 1.$$

Hence $f(x) \leq f(y)$. Therefore, f has the isotonic property. □

Proposition 3.8. *Let $f : X \to Y$ be a homomorphism and $\theta = \{(x, y) : f(x) = f(y)\}$. Then*
(i) θ is a congruence relation on X,
(ii) $X/\theta \cong f(X)$.

Proof. (i). It is obvious θ is an equivalence relation on X. We only show that θ satisfies the substitution property. Assume that (x, y) and $(u, v) \in \theta$. Then we have $f(x) = f(y)$ and $f(u) = f(v)$. Since f is a homomorphism and above argument yields,

$$f(x * u) = f(x) * f(u) = f(y) * f(v) = f(y * v).$$

and

$$f(x \diamond u) = f(x) \diamond f(u) = f(y) \diamond f(v) = f(y \diamond v).$$

Then $(x*u, y*v), (x\diamond u, y\diamond v) \in \theta$. In the same way we have $(u*x, v*y), (u\diamond x, v\diamond y) \in \theta$. Hence θ is a congruence relation on X.

(ii). By using the Proposition 3.5, we have $(X/\theta; *, \diamond, C_1)$ is a pseudo BE–algebra. Let $\nu : X/\theta \to f(X)$ be such that $\nu(C_x) = f(x)$, for all $C_x \in X/\theta$. Then

(i). v is well defined, because if $C_x = C_y$, for any $x, y \in X$, then $(x, y) \in \theta$. Therefore, $f(x) = f(y)$. Hence $\nu(C_x) = \nu(C_y)$.

(ii). $\ker \nu = \{C_x : \nu(C_x) = f(x) = 1\} = \{C_x : f(x) = f(1)\} = \{C_x : (x, 1) \in \theta\} = C_1$. Then v is one to one.

(iii). $\nu(C_x * C_y) = \nu(C_{x*y}) = f(x * y) = f(x) * f(y) = \nu(C_x) * \nu(C_y)$ and $\nu(C_x \diamond C_y) = \nu(C_{x\diamond y}) = f(x \diamond y) = f(x) \diamond f(y) = \nu(C_x) \diamond \nu(C_y)$. Thus ν is a homomorphism. Therefore, $X/\theta \cong f(X)$. □

4. Congruence relations induced by pseudo filters

In this section we assume that X is a distributive pseudo BE–algebra, unless otherwise is stated.

Proposition 4.1. *Let F be a pseudo filter of X. Define*

$$x \sim_F y \ \ if \ and \ only \ if \ \ x * y,\ y * x \in F.$$

Then $\sim_F \in pCon(X)$.

Proof. (i). Since $1 \in F$, we have $x * x = 1 \in F$, i.e., $x \sim_F x$. This means that " $\sim_F$ " is reflexive. Now, if $x \sim_F y$ and $y \sim_F z$, then $x * y, y * x \in F$ and $y * z, z * y \in F$. By Proposition 2.5(i), $y * z \leq (x * y) * (x * z)$. Now, since $y * z \in F$ and F is a pseudo filter, it follows that $(x * y) * (x * z) \in F$. So $x * z \in F$. By a similar way we see that $z * x \in F$. This shows that " $\sim_F$ " is transitive. The symmetry of " $\sim_F$ " is immediate from the definition. Therefore, " $\sim_F$ " is an equivalence relation on X.

(ii). Let $x \in X$ and $u \sim_F v$. Then by Proposition 2.5(i), $v * u \leq (x * v) * (x * u)$. Now, since $v * u \in F$ and F is a pseudo filter, $(x * v) * (x * u) \in F$. By a similar way, $(x * u) * (x * v) \in F$. Therefore, $x * v \sim_F x * u$. Also, by Proposition 2.5(ii), $u * v \leq (x \diamond u) * (x \diamond v)$. Now, since $u * v \in F$ and F is a pseudo filter, we see that $(x \diamond u) * (x \diamond v) \in F$. By a similar way, $(x \diamond v) * (x \diamond u) \in F$. Therefore, $x \diamond v \sim_F x \diamond u$.

By using Proposition 2.5(ii), we have $x * u \leq (y \diamond x) * (y \diamond u)$, then $(x * u) \diamond ((y \diamond x) * (y \diamond u)) = 1$ and so by $(pBE4)$ we have $(y \diamond x) * ((x * u) \diamond (y \diamond u)) = 1$, which implies that $(x * u) \diamond (y \diamond u) \in F$, because F is pseudo filter $y * x \in F$ and by Theorem 2.6, F is normal, then $y \diamond x \in F$. Hence $(x * u) * (y \diamond u) \in F$. On the other hand, we have

$x * y \leq (y \diamond u) * (x * u)$, because

$$\begin{aligned}(x * y) \diamond ((y \diamond u) * (x * u)) &= (y \diamond u) * ((x * y) \diamond (x * u)) \\ &= (y \diamond u) * (x * (y \diamond u)) = 1.\end{aligned}$$

Hence $(y \diamond u) * (x * u) \in F$, because F is pseudo filter and $x * y \in F$. Thus $x * u \sim_F y \diamond u$. Finally, since $y \diamond u \sim_F y \diamond v$ and by a similar way, $y \diamond v \sim_F y * v$. By the transitivity " $\sim_F$ " we get $x * u \sim_F y * v$. By the same manner $x \diamond u \sim_F y \diamond v$. Therefore, $\sim_F \in pCon(X)$. $\square$

Note. Now, let F be a pseudo filter of X. Denote the equivalence class of x by C_x. Then $F = C_1$. In fact, if $x \in F$, then $x * 1 = x \diamond 1 = 1 \in F$ and $1 * x = 1 \diamond x = x \in F$, i.e., $x \sim_F 1$. Hence $x \in C_1$.

Conversely, let $x \in C_1$. Then $x = 1 * x = 1 \diamond x \in F$, and so $x \in F$. Hence $F = C_1$. Denote $X/F = \{C_x : x \in X\}$ and define that $C_x * C_y = C_{x*y}$ and $C_x \diamond C_y = C_{x \diamond y}$. Since " $\sim_F$ " is a congruence relation on X, the operations " $*$ " and " $\diamond$ " are well defined.

Example 4.1. Let $X = \{1, a, b, c, d\}$ and operations " $*$ " and " $\diamond$ " defined as follows:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	c	c	1
b	1	d	1	1	d
c	1	d	1	1	d
d	1	1	c	c	1

$\diamond$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	1
b	1	d	1	1	d
c	1	d	1	1	d
d	1	1	b	c	1

Then, $(X; *, \diamond, 1)$ is a distributive pseudo BE–algebra. It can be easily seen that $F = \{1, a, d\}$ is a pseudo filter. We have

$$\sim_F = \{(1,1), (a,a), (b,b), (c,c), (d,d), (1,a), (a,1), (d,1), (1,d)(a,d), (d,a), (b,c), (c,b)\}$$

and so $\sim_F \in pCon(X)$.

Theorem 4.2. *Let $F \in pF(X)$. Then*

(i) $(X/F; *, C_1) = (X/F; \diamond, C_1)$ *is a BE–algebra (which is called quotient pseudo BE–algebra via F, and $C_1 = F$.)*

(ii) $(X/F; \diamond, *, 1)$ *is a distributive pseudo BE–algebra if and only if* $(X/F; *, 1)$ *is a BE–algebra (i. e. $C_{x*y} = C_{x \diamond y}$, for all $x, y \in X$).*

Proof. (i). By similar way of the proof of Proposition 3.5, $(X/F; *, \diamond, C_1)$ is a distributive pseudeo BE-algebra. To prove X/F is a BE-algebra it is sufficient to prove, $C_x * C_y = C_x \diamond C_y$, for all $C_x, C_y \in X/F$. By Proposition 2.5 (iii), $A(x * y) = A(x \diamond y)$. By definition of $A(x)$, it is obvious that $x * y \in A(x * y)$ and $x \diamond y \in A(x \diamond y)$. Thus $x * y \in A(x * y) = A(x \diamond y)$ and so $(x * y) * (x \diamond y) = 1 \in F$. By similar way, $x \diamond y \in A(x \diamond y) = A(x * y)$ and so $(x \diamond y) * (x * y) = 1 \in F$. Hence $x * y \sim_F x \diamond y$ and so $C_{x*y} = C_{x \diamond y}$, which means $C_x * C_y = C_x \diamond C_y$.

(ii). By (i) and Theorem 2.7, the proof is obvious. $\square$

Example 4.2. Let $X = \{1, a, b, c, d, e\}$. Define the operations " $*$ " and " $\diamond$ " on X as follows:

$*$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	c	c	d	1
b	1	a	1	1	d	e
c	1	a	1	1	d	e
d	1	a	1	1	1	e
e	1	a	c	c	d	1

$\diamond$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	b	c	d	1
b	1	a	1	1	d	e
c	1	a	1	1	d	e
d	1	a	1	1	1	e
e	1	a	c	c	d	1

Then $(X; *, \diamond, 1)$ is a distributive pseudo BE–algebra. By consider pseudo filter $F = \{1, e\}$, we have $X/F = \{C_1 = C_e = F, C_a = \{a\}, C_b = C_c = \{b, c\}, C_d = \{d\}\}$ with the operations " $*$ " and " $\diamond$ " defined by following table is a pseudo BE–algebra.

$* = \diamond$	C_1	C_a	C_b	C_d
C_1	C_1	C_a	C_b	C_d
C_a	C_1	C_1	C_b	C_d
C_b	C_1	C_a	C_1	C_d
C_d	C_1	C_a	C_1	C_1

Proposition 4.3. *Let $\theta \in pCon(X)$. Then*
(i) $F_\theta \in pF(X)$,
(ii) $F_\theta = \{x | (x, 1) \in \theta\}$.

Proof. (i). Since $(x, x) \in \theta$, we have $x * x = 1 \in F_\theta$. Suppose that $x * y, x \in F_\theta$. There are $(u, v), (p, q) \in \theta$ such that $x * y = u * v$ and $x = p * q$. Since $(u, v) \in \theta \in pCon(X)$, we have $(u * v, v * v) = (x * y, 1) \in \theta$ and by a similar way $(x, 1) \in \theta$. Now, $(x*y, 1*y) = (x*y, y) \in \theta$. Hence $(y, 1) \in \theta$. This yields that $y*1 = 1, 1*y = y \in F_\theta$. That is F_θ is a pseudo filter of X. Furthermore, we can see that, F_θ is normal pseudo filter from Theorem 2.6.

(ii). Put $F := \{x | (x, 1) \in \theta\}$. Let $x \in F_\theta$. There is $(u, v) \in \theta$ such that $x = u * v$. Since θ is a congruence, we have $(x, 1) = (u * v, 1) = (u * v, v * v) \in \theta$. Hence $F_\theta \subseteq F$.

Now, let $x \in F$. Hence $(x, 1) \in \theta$ and so $x * 1 = 1, 1 * x = x \in F_\theta$. Hence $F \subseteq F_\theta$. Therefore, $F = F_\theta$. □

In [8], M. Kondo proved that θ is a regular cogruence relation on BCI–algebra if and only if $\theta = \theta_{I_\theta}$. Now, it is natural to ask whether $\theta = \theta_{F_\theta}$ in pseudo BE–algebras, for all $\theta \in pCon(X)$. We shall investigate the relation between the congruences θ and θ_{F_θ}.

Theorem 4.4. *Let $\theta \in pCon(X)$. Then $\theta_{F_\theta} = \theta$.*

Proof. Let $(x, y) \in \theta$. Then $x * y, y * x \in F_\theta$. Since F_θ is a normal pseudo filter by Proposition 4.3(i), we have $(x, y) \in \theta_{F_\theta}$. Therefore $\theta \subseteq \theta_{F_\theta}$. Now, it is sufficient to show that $\theta_{F_\theta} \subseteq \theta$. Let $(x, y) \in \theta_{F_\theta}$. By definition, we have $x * y, y * x \in F_\theta$. Hence there are $(u, v), (p, q) \in \theta$ such that $x * y = u * v, y * x = p * q$. Since $\theta \in pCon(X)$, we have

$$(x * y, 1) = (u * v, 1) = (u * v, v * v) \in \theta.$$

By a similar way $(y * x, 1) \in \theta$. Using Proposition 4.3(ii), $x * y, y * x \in F_\theta$. Hence $(x, y) \in \theta$ and so $\theta_{F_\theta} \subseteq \theta$. Therefore, $\theta_{F_\theta} = \theta$. □

Proposition 4.5. *Let $f : X \to Y$ be a homomorphism. Then*
(i) *f is epimorphic if and only if $Im(f) = Y$,*
(ii) *f is monomorphic if and only if* $\ker(f) = \{0\}$,

(*iii*) *f is isomorphic if and only if the inverse mapping f^{-1} is isomorphic.*
(*iv*) $\ker(f)$ *is a closed pseudo filter of X,*
(*v*) *$Im(f)$ is a pseudo subalgebra of Y.*

Proof. (*iv*). By Proposition 3.7(i), $1 \in \ker(f)$. Let $x, x * y \in \ker(f)$, then $f(x) = f(x * y) = 1$, and so

$$1 = f(x * y) = f(x) * f(y) = 1 * f(y) = f(y).$$

Thus $y \in \ker(f)$. Now, let $x * y \in \ker(f)$. Then $f(x * y) = f(x) * f(y) = 1$, and so by $(pBE5)$ we have $f(x) \diamond f(y) = f(x \diamond y) = 1$. Therefore, $x \diamond y \in F$. By a similar way we can prove if $x \diamond y \in F$, then $x * y \in F$. Hence $\ker(f)$ is a closed pseudo filter of X.

(*v*). Obviously, $Im(f)$ is a non-vacuous set. If y_1, $y_2 \in Im(f)$, then there exist x_1, $x_2 \in X$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$, thus

$$y_1 * y_2 = f(x_1) * f(x_2) = f(x_1 * x_2) \in Im(f),$$

and

$$y_1 \diamond y_2 = f(x_1) \diamond f(x_2) = f(x_1 \diamond x_2) \in Im(f).$$

Consequently, $Im(f)$ is a pseudo subalgebra of Y. □

Note. In general, $Im(f)$ may not be a pseudo filter.

Example 4.3. Let $X = \{1, a, b, c\}$ and $Y = \{1, a, b, c, d\}$. Define operations "$*$" and "$\diamond$" on X and Y as follows:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	1
b	1	1	1	1
c	1	a	a	1

$\diamond$	1	a	b	c
1	1	a	b	c
a	1	1	c	1
b	1	1	1	1
c	1	a	b	1

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	a	1	a
b	1	1	1	1	a
c	1	a	a	1	a
d	1	1	1	1	1

$\diamond$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	c	1	c
b	1	1	1	1	c
c	1	a	b	1	d
d	1	1	1	1	1

Then $(X; *, \diamond, 1)$ and $(Y; *, \diamond, 1)$ are pseudo BE–algebras and $\{1, a, b, c\}$ is a pseudo filter of X. Now, if we consider $f : X \to Y$ as the identity map, then f is a homomorphism and $f(X) = X$. We can see that $X = \{1, a, b, c\}$ is a trivial pseudo filter of X, but $f(X)$ is not a pseudo filter of Y, because

$$a * d = a \in f(X), \ a \in f(X) \ but \ d \notin f(X).$$

Proposition 4.6. *Let $f : X \to Y$ be an epimorphism. If F is a pseudo filter of X, then $f(F)$ is a pseudo filter of Y.*

Proof. $f(F)$ is nonempty subset of Y because $1 \in f(F)$. Let $y \in Y$ and $a \in f(F)$ such that $a * y, \in f(F)$. Then there exist $x \in X$ and $a_1 \in F$ such that $f(x) = y$ and $f(a_1) = a$. Now, we have $a * y = f(a_1) * f(x) = f(a_1 * x) \in f(F)$. Hence $a_1 * x \in F$. Since F is a pseudo filter and $a_1 \in F$, we have $x \in F$. Therefore, $y = f(x) \in f(F)$. □

Theorem 4.7. *Let F be a closed pseudo filter of X. Then there is a canonical surjective homomorphism $\varphi : X \to X/F$ by $\varphi(x) = C_x$, and* $\ker \varphi = F$, *where* $\ker \varphi = \varphi^{-1}(C_1)$.

Proof. It is clear that φ is well-defined. Let $x, y \in X$. Then

$$\varphi(x * y) = C_{x*y} = C_x * C_y = \varphi(x) * \varphi(y)$$

and

$$\varphi(x \diamond y) = C_{x\diamond y} = C_x \diamond C_y = \varphi(x) \diamond \varphi(y).$$

Hence φ is homomorphism.

Clearly φ is onto. Also, we have

$$\begin{aligned} \ker \varphi = \{x \in X : \varphi(x) = C_1\} &= \{x \in X : C_x = C_1\} \\ &= \{x \in X : x * 1, 1 * x, x \diamond 1, 1 \diamond x \in F\} \\ &= \{x \in X : x \in F\} = F. \end{aligned}$$

□

5. Conclusion

In this paper, we consider the relation between congruence relations on pseudo BE–algebras and (normal) pseudo filters. Also, we show that the quotient of a pseudo BE–algebra via a congruence relation is a pseudo BE–algebra and prove that, if X is a distributive pseudo BE–algebra and F is a normal pseudo filter, then the quotient algebra via this filter is a BE–algebra.

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