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A new variable shape parameter strategy for Gaussian radial basis function approximation methods

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Abstract. In this article, we introduce a new variable shape parameter is called symmetric variable shape parameter (SVSP) for Gaussian radial basis functions (GRBFs). The GRBF has the shape parameter c, which plays an important role in the accuracy of the approximation. In this work, we will use it to interpolate functions and solve linear boundary value problems (LBVP). Some numerical experiments are presented to show accuracy and robustness of the GRBF with SVSP strategy. These results have the best accuracy for the one- and two-dimensional interpolations and LBVP. Besides, the numerical results show that the SVSP for GRBF often outperforms constant shape parameter strategy.

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Key words and phrases. Gaussian radial basis function, variable shape parameter, interpolation, boundary value problem.

1. Introduction

During the last two decades, radial basis functions (RBFs) have become a well established tool for interpolating multivariate data and solving linear boundary value problems. Approximating the solution of PDEs with RBF methods has drawn many researchers in science and engieering [1, 2, 3, 4, 5, 6, 7, 8]. In genral form, RBFs can be sorted into two types: the globally supported RBF (GSRBF) and the compactly supported RBF (CSRBF). The GSRBFs mainly include multi-quadric (MQ), inverse multi-quadric (IMQ), thin-plate spline and Gaussian. It is clear that some of the functions (GA, MQ, IMQ, IQ), are globally supported, infinity differentiable and depend on a free parameter c. Some of the most general fundamental functions that generate is RBFs listed in Table 1. In all numerical examples, we have used the GA which is representative of this class and popular application.

These functions have shape parameter that plays an important role in the accuracy of the approximation. However, it is still a challenge to find the best shape parameter of GSRBF. The choice of the optimal shape parameters has been studied by Carlson and Foley [9], Rippa [10], Wang and Liu [11], Wang [12], Cheng et al. [13], and Ferreira et al. [14]. Kansa et al. [15] noted the shape parameter must be adjusted with the number of centers in order to produce equation systems that are sufficiently well conditioned to be solved with standard finite precision arithmetic.

The RBFs is an efficient tool in multivariate approximation, but it usually suffers from an ill-conditioned interpolation matrix when interpolation points are very dense or irregularly spaced. To guarantee the robust of the interpolation many researchers have sought for the theoretical results about the convergence and stability of the RBF interpolation [16]-[20].

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So far, many methods have been proposed, such as compactly supported RBF, multilevel method, precondition method, domain decomposition method, truncated RBF method, RBF with variable shape parameter and knot optimization method [21]. The advantage of using variable shape parameter lies in the fact that the RBFs with variable shape parameters can usually improve the interpolation matrix condition number. The concept of variable shape parameters in the RBF interpolation has been proposed by many researchers, e.g. Kansa and Carlson [22].

The main idea is to determine the shape parameter of a RBF in terms of the local density of its corresponding interpolation point. Thus the columns of the interpolation matrix elements are more distinct and the condition number becomes smaller. However, new problems may be caused by the shape parameter variation such as a singular interpolation matrix, lower convergence rate and difficulties to choose the variation schemes [23].

In many cases for multi-quadric RBF and inverse multi-quadric RBF with variable shape parameter produced more accurate results than if a CSP had been used [24, 25]. Towards the singular interpolation matrix that may appear in the new interpolation method, Bozzini et al.[26] proposed some criterions for the variable shape parameter, which guarantee the unique solvability of the interpolation. Sarra and Sturgill [24] have investigated a random variation scheme and Li et al. [27] have implemented a linear scheme successfully. Xiang et al. [25] for generalized MQ-RBF presented a trigonometric variable shape parameter and exponent strategy. The paper is organized as follows : first, in section 2 we briefly review the RBF method for interpolation problem. In section 3, we describe the idea of variable shape parameters and propose the SVSP. In section 4, we study the validity and effectiveness of the proposed strategy which , is called symmetric variable shape parameter (SVSP). It is applied to four examples in one and two dimensional problems. The paper ends with conclusions and discussions on future work is section 5.

Name of RBF	Abbreviations	Definition
Gaussian	(GA)	$\varphi(r) = e^{-c^2r^2}$
Multi-Quadric	(MQ)	$\varphi(r) = \sqrt{1 + c^2r^2}$
Inverse Multi-Quadric	(IMQ)	$\varphi(r) = 1/\sqrt{1 + c^2r^2}$
Inverse Quadric	(IQ)	$\varphi(r) = 1/(1 + c^2r^2)$

Table 1. Globally Support RBFs

2. Gaussian RBF interpolation

RBFs were introduced by Hardy in 1971 [28]. The RBF known as a truly mesh free computational method, does not require the mesh generation of a regular grid as in the traditional finite difference or a mesh as in the finite element and boundary element methods. In fact, the mesh generation of high dimensional problems costs a great deal in terms of computer resources. Accordingly the recent motivation is to cut down modeling costs by avoiding the mesh generation. Overall, RBFs method aims to eliminate the structure of the mesh and approximate the solution using a set of quasi random points rather than points from a grid discretization.

In this section, the RBFs method is defined as a technique for interpolation of the scattered data. Let r be the Euclidean distance between two points i.e., $r =$

$\|\mathbf{x}-\mathbf{x}^*\|_2 = \sqrt{(x_1-x_1^*)^2+(x_2-x_2^*)^2+...+(x_d-x_d^*)^2}$ for $\mathbf{x} \in \mathbb{R}^d$ and fixed point $\mathbf{x}^* \in \mathbb{R}^d$. The RBF interpolation method uses linear combinations of translates of one function $\varphi(r)$ of a single real variable. Given a set ofcenters $\mathbf{x}_1^*, ..., \mathbf{x}_N^*$ in $\mathbb{R}^d$, the RBF interpolate takes the form

$$\mathcal{P}(x) = \sum_{i=1}^{N} \lambda_i \varphi_i(r) = \sum_{i=1}^{N} \lambda_i \varphi(\|\mathbf{x}-\mathbf{x}_i^*\|_2). \tag{1}$$

Many different basis functions $\varphi(r)$ have been used, but we concentrate on the Gaussian RBF

$$\varphi_i(r) = e^{-c^2 r^2}, \tag{2}$$

where $c > 0$ is a free shape parameter. The coefficients λ, are chosen by enforcing the interpolation condition

$$\mathcal{P}(\mathbf{x}_j^*) = f(\mathbf{x}_j^*), \quad j = 1, 2, ...N. \tag{3}$$

at a set of nodes that typically coincide with the centers. Collocation with the interpolation conditions at the N nodes leads to a $N \times N$ linear system

$$\mathbf{A}\lambda = \mathbf{f}. \tag{4}$$

Solving this linear system, the solution of the interpolation problem is obtained.

$$\lambda = \mathbf{A}^{-1}\mathbf{f}. \tag{5}$$

The matrix A with entries

$$a_{ji} = \varphi(\|\mathbf{x}_j^* - \mathbf{x}_i^*\|_2), \quad i, j = 1, ..., N. \tag{6}$$

is an interpolation matrix . For distinct center locations, the system matrix for the GARBF is known to be nonsingular [29] if a CSP is used. For given a square matrix A , the condition number $\kappa_s(A)$ is defined as:

$$\kappa_s(A) = \|A\|_s \|A^{-1}\|_s, \qquad s = 1, 2, \infty, \tag{7}$$

if the inverse of exists. If the inverse does not exist, then we say that the condition number is infinite. The condition number of A depends directly on the shape parameter c. Theoretically, RBF methods are most accurate when the shape parameter is small. However, the use of small shape parameters results in system matrices that are very poorly conditioned.

3. Variable shape strategies

One of the key issues when applying RBF to interpolation or to the numerical solution of PDEs is the choice of a suitable value for shape parameters of RBF. A large shape parameter results in a well conditioned system matrix; however, the approximation using the RBF is poor. If one chooses to use a small shape parameter this results in a very accurate RBF approximation, but now the system matrix is ill-conditioned. So, chooses the shape parameter of RBF is an important factor that affects the interpolation error and stability. A variable shape parameter strategy refers to use a possibly different value of the shape parameter at each center. This results in shape parameters that are the same in each column of the interpolation matrix or the evaluation matrix.

One positive aspect of a variable shape parameter is that it creates distinct entries in the RBF matrices which lead to lower condition numbers [30]. To the authors' knowledge, there is not yet any paper that implements GARBF methods with variable shape parameter for improving the accuracy of the interpolation and boundary value problems. In most papers, variable shape parameter methods have been successfully used in MQ-RBF or IMQ-RBF approximation methods [23, 24, 25].

In order to solve the scattered data interpolation problem, one can consider the GARBF interpolation in the following form

$$\mathcal{P}(x) = \sum_{j=1}^{N} \lambda_j \exp^{-c_j^2(x-x_j^*)^2}, \tag{8}$$

where N is the total number of nodes, λ_j is unknown coefficients, c_j is the variable shape parameters. The coefficients λ are chosen by Enforcing the interpolation condition (3) at N centers results in the equations

$$\begin{bmatrix} \exp^{-c_1^2(x_1^*-x_1^*)^2} & \exp^{-c_2^2(x_1^*-x_2^*)^2} & \cdots & \exp^{-c_N^2(x_1^*-x_N^*)^2} \\ \exp^{-c_1^2(x_2^*-x_1^*)^2} & \exp^{-c_2^2(x_2^*-x_2^*)^2} & \cdots & \exp^{-c_N^2(x_2^*-x_N^*)^2} \\ & \vdots & & \\ \exp^{-c_1^2(x_N^*-x_1^*)^2} & \exp^{-c_2^2(x_N^*-x_2^*)^2} & \cdots & \exp^{-c_N^2(x_N^*-x_N^*)^2} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_N \end{bmatrix} = \begin{bmatrix} f(x_1^*) \\ f(x_2^*) \\ \vdots \\ f(x_N^*) \end{bmatrix}. \tag{9}$$

where $f(x)$ is used to generate the data to be interpolated. The matrix form Eq.(9) can be written as

$$A\lambda = f. \tag{10}$$

The system (10) contain N equation in N unknowns λ_j which can be easily solved by

$$\lambda = A^{-1} f, \tag{11}$$

where A is the interpolation matrix and nonsingular matrix. Steady PDE problems are discretized by the RBF method in the following manner. Consider the linear boundary value problem

$$\begin{cases} \mathcal{L}u(x) = f(x), & x \in \Omega, \\ \mathcal{B}u(x) = g(x), & x \in \partial\Omega. \end{cases} \tag{12}$$

Where $\partial\Omega$ is the boundary of the domain Ω, $\mathcal{L}$ is a linear elliptic partial differential operator, $\mathcal{B}$ is a linear boundary operator,f(X) and g(X) are the known functions. The solution of Eq.(12)can be approximated in the following form

$$u(x) = \sum_{j=1}^{N} \lambda_j \varphi_j(r) = \sum_{j=1}^{N} \lambda_j \exp^{-c_j^2(x-x_j^*)^2} \tag{13}$$

The set of N distinct center centers are divided into two part. Assume that there are N_I centers in the interior of the domain Ω and N_B centers on the boundary $\partial\Omega X$. For the interior centers, we have

$$\mathcal{L}u(x_i^*) = \sum_{j=1}^{N} \lambda_j \mathcal{L}\varphi(\|x_i^* - x_j^*\|) = f(x_i^*), \quad i = 1, 2, \ldots N_I. \tag{14}$$

For the boundary centers, we apply the operator $\mathcal{B}$ to the RBF interpolation as

$$\mathcal{B}u(x_i^*) = \sum_{j=1}^{N} \lambda_j \mathcal{B}\varphi(\|x_i^* - x_j^*\|) = g(x_i^*) \quad i = N_I + 1, 2, \ldots N. \tag{15}$$

In matrix form, Eqs. (14) and (15) can be expressed as

$$\begin{bmatrix} \mathcal{L}\varphi \\ \mathcal{B}\varphi \end{bmatrix} \begin{bmatrix} \lambda \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} \tag{16}$$

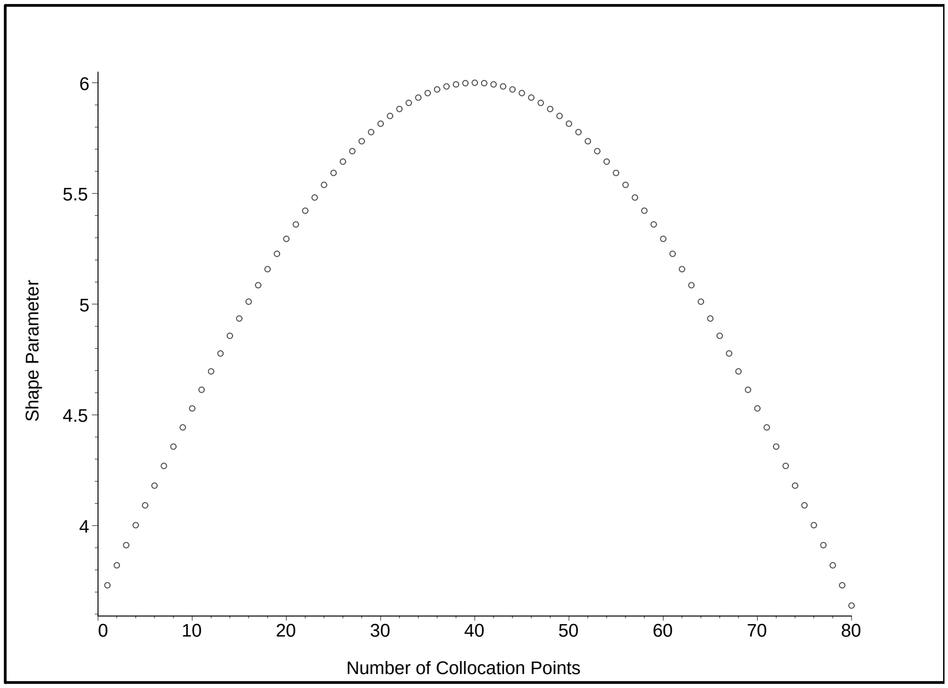

FIGURE 1. Plot of SVSP (with $N = 40$ and $c^* = 6$) for one dimensional problems.

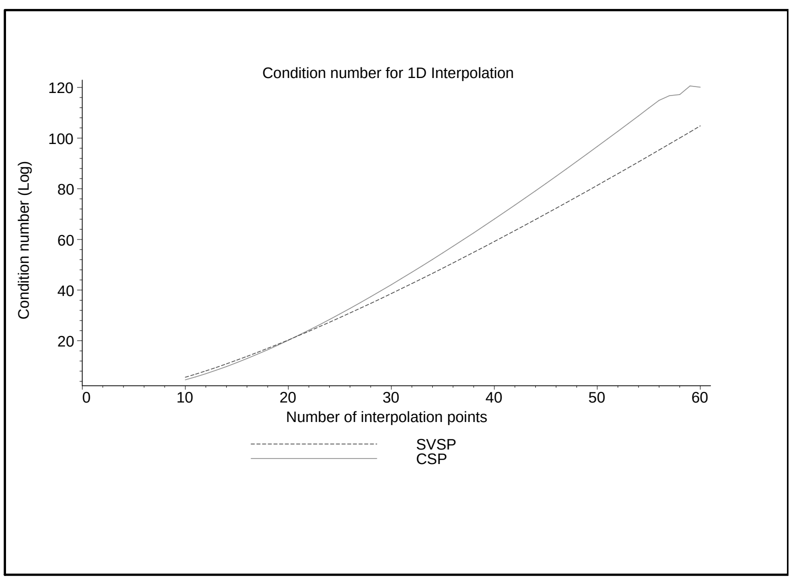

FIGURE 2. Condition number versus the density of the 1D interpolation points for the GA.

Then

$$\left[\begin{array}{c} \lambda \end{array}\right] = \left[\begin{array}{c} \mathcal{L}\varphi \\ \mathcal{B}\varphi \end{array}\right]^{-1} \left[\begin{array}{c} f \\ g \end{array}\right] \tag{17}$$

Number of Centers	Condition Number CSP ($c^* = 6$)	Condition Number SVSP
N=20	5.756e+8	6.527e+8
N=30	1.983e+18	5.923e+16
N=40	3.462e+29	4.966e+25
N=50	9.653e+41	2.133e+35

TABLE 2. Compare Condition Number 1 D Interpolation with CSP ($c^* = 6$) SVSP with $d = 50$ (the number of floating point arithmetics)

As can be seen from Eq.(9), the shape parameter c_j is different at each column of the coefficient matrix.

In this work, we propose a new variable shape parameter c strategy for GARBF, that it is called symmetric variable shape parameter (SVSP), as follows

$$c_j = c^* \exp(\frac{1}{2}(\frac{j-\mu}{\sigma})^2), \tag{18}$$

where c^* is a arbitrary shape parameter such that $c^* \in [c_{min}, c_{max}]$. We define values μ and σ as follows

$$\mu = \begin{cases} 0.5N, & 1D, \\ 0.5N^2, & 2D, \end{cases} \tag{19}$$

and

$$\sigma = \begin{cases} 0.5N, & 1D, \\ 0.25N^2, & 2D, \end{cases} \tag{20}$$

Note that, the shape parameter values c_j are controlled around c^* by μ and σ. The symmetric variable shape parameter c_j is shown in Figure 1. In the one dimensional problems N is the total number of centers and in the two dimensional N is the centers number of each coordinate axis.

In Table. 2, some values of condition number of 1D interpolation matrix, the corresponding number of centers, for CSP and SVSP are listed.

4. Numerical results and comparison with explicit solutions

In this section, we apply numerical comparisons of the SVSP to that of the CSP. The formula for the root-mean-square(RMS) error is given by

$$RMSE = \sqrt{\frac{1}{N}\sum_{j=0}^{N}(u_e(x_j) - u_a(x_j))^2}, \tag{21}$$

The L_∞ and L_2 error norms of the solution are defined by

$$L_\infty = \|u_e - u_a\|_\infty = \max_{j=0,\dots N} |u_e(x_j) - u_a(x_j)|, \tag{22}$$

and

$$L_2 = \|u_e - u_a\|_2 = \sqrt{\int_0^1 (u_e(x) - u_a(x))^2 dx}. \tag{23}$$

Where u_e and u_a are the exact and approximate solutions of the problems, respectively. In each case, we calculate errors of the RMS, the L_∞ and the L_2. We use L_∞ and L_2 error norms to measure the difference between the approximation and exact solutions that they are applied to one dimensional problems. Also, the error norm L_∞ and RMS error are used for two dimensional problems. The numerical result obtained, are in excellent agreement with the exact solution, as shown in figures and tables. Here we try to reduce the error by using SVSP strategy. Now we present the results for four test examples. We have taken these problems from the literature [25, 24].

4.1. One-dimensional interpolation. Three one-dimensional functions are used to examine the numerical accuracy of the present symmetric variable shape parameter strategy in the one-dimensional interpolation. The domain is discretized by the uniformly spaced centers. The function f_1 is an exponential and trigonometric combination:

$$f_1 = exp(-x) + sin(2x); \qquad x \in [0, 1]$$

The function f_2 is a third degree polynomial:

$$f_2 = x^3 + x^2 + x; \qquad x \in [0, 1]$$

The function f_3 is a rational function:

$$f_3 = \frac{1}{1 + 25x^2}; \qquad x \in [0, 1]$$

In Fig 3. and Fig 4. the point-wise error functins $f_i(x) - \tilde{f}_i(x)$ for $i = 1, 2, 3$ are plotted using GRBF with CSP and SVSP respectively. Table 3, lists the L_2 and L_∞ errors of GARBFs interpolating functions f_1, f_2 and f_3 by CSP (c=6) and SVSP strategies. According to the Table 3, GARBFs with SVSP results in the best accuracy for interpolating functions f_1, f_2 and f_3.

4.2. Two-dimensional interpolation. Our secant numerical experiment of the present SVSP strategy, involves interpolating two functions in two dimensional. The functions f_4 and f_5 are

$$f_4 = e^{(x+y)}; \qquad (x, y) \in [0, 1] \times [0, 1],$$

$$f_5 = x^2 + 2y^3; \qquad (x, y) \in [0, 1] \times [0, 1].$$

In Fig 5. the point-wise error functins $f_i(x) - \tilde{f}_i(x)$ for $i = 4, 5$ are plotted using GRBF with CSP and SVSP. The L_∞ and RMS errors of GARBFs interpolating functions f_4 and f_5 using CSP (c=0.95) and SVSP strategies are listed in Table 4. It is found from Table 4 that SVSP strategy produces the best accuracy for interpolating functions f_4 and f_5.

4.3. One-dimensional boundary value problem. Consider the one-dimensional linear boundary value problem

$$-u_{xx} + \pi^2 u = 2\pi^2 sin(\pi x); \qquad (x, y) \in [0, 1],$$

$$u(0) = u(1) = 0.$$

with the exact solution $u(x) = sin(\pi x)$. The set of collocation centers (uniformly spaced) have been used to solve the one-dimensional boundary value problem for $N = 40, 60, 80$. The L_∞ and L_2 errors of GARBFs solving one dimensional linear elliptic boundary problem above by CSP (c=0.95) and SVSP strategies are listed in Table 5. According to the Table 5, the SVSP strategy produces the best accuracy for solving the one-dimensional boundary value problem.

Number of Centers	Function	Error	CSP ($c^* = 6$)	SVSP
N=10	f_1	L_2	1.189398e-2	2.756890e-3
		L_∞	3.280115e-2	8.938321e-3
	f_2	L_2	3.429439e-2	7.552350e-3
		L_∞	1.253790e-1	2.910077e-2
	f_3	L_2	2.938253e-2	2.893205e-3
		L_∞	9.003832e-3	1.128364e-2
N=20	f_1	L_2	6.080262e-5	1.901896e-6
		L_∞	3.133009e-4	1.366943e-5
	f_2	L_2	1.988121e-4	5.567550e-6
		L_∞	1.398668e-3	4.697162e-5
	f_3	L_2	4.246478e-5	3.036971e-5
		L_∞	2.957292e-4	2.027269e-4
N=30	f_1	L_2	6.751047e-9	2.870895e-11
		L_∞	7.786065e-7	4.985295e-9
	f_2	L_2	2.643284e-8	8.600925e-11
		L_∞	4.332845e-6	1.713434e-8
	f_3	L_2	2.994808e-8	3.433226e-8
		L_∞	4.059901e-6	6.969993e-6
N=40	f_1	L_2	1.151938e-11	7.427225e-15
		L_∞	6.500154e-10	6.363788e-13
	f_2	L_2	5.500126e-11	2.162779e-14
		L_∞	4.156552e-9	2.113284e-12
	f_3	L_2	2.994808e-8	4.654414e-9
		L_∞	4.059901e-6	5.376785e-7
N=50	f_1	L_2	6.339441e-10	1.057222e-16
		L_∞	1.360138e-9	2.006284e-16
	f_2	L_2	5.619426e-10	1.104614e-16
		L_∞	1.165221e-9	1.980058e-16
	f_3	L_2	2.624138e-11	5.591093e-11
		L_∞	2.573009e-8	1.644512e-8

TABLE 3. 1 D Interpolation Error Results with ($d = 50$)

4.4. Two-dimensional boundary value problem. We use the two dimensional linear elliptic boundary problem

$$u_{xx} + u_{yy} = -2\pi^2 sin(\pi x) sin(\pi y); \qquad (x, y) \in [0, 1] \times [0, 1]$$

$$u(x, y) = 0; \qquad (x, y) \in \partial([0, 1] \times [0, 1])$$

as a steady PDE test problem for the shape parameter strategies. The exact solution of above boundary value problem is $u(x, y) = sin(\pi x)sin(\pi y)$. The L_∞ and RMS errors of GARBFs solving two dimensional linear elliptic boundary problem above with CSP ($c = 2.5$) and SVSP strategies are listed in Table 6. It is found from Table 6. that the SVSP strategy produces the best accuracy for solving the two-dimensional boundary value problem.

These test problems are chosen such that their exact solutions are known. But the GARBF method with SVSP strategy developed in this research can be applied to more complicated problems. Obtained Tables, show that under a new shape parameter

Number of Centers	Function	Error	CSP ($c^* = 6$)	SVSP
$N^2 = 16$	f_4	RMSE	7.864511e-3	1.118120e-2
		L_∞	9.152483e-3	2.136528e-2
	f_5	RMSE	2.758493e-3	2.868528e-3
		L_∞	1.102749e-2	6.130106e-3
$N^2 = 64$	f_4	RMSE	9.961191e-7	1.860040e-6
		L_∞	1.879069e-6	3.681840e-6
	f_5	RMSE	7.808936e-7	3.727625e-10
		L_∞	1.512060e-6	1.684241e-10
$N^2 = 100$	f_4	RMSE	1.033037e-8	2.201717e-9
		L_∞	2.306630e-8	5.379347e-9
	f_5	RMSE	6.397018e-9	3.549075e-15
		L_∞	1.519658e-8	8.793413e-15
$N^2 = 144$	f_4	RMSE	9.230352e-11	4.112013e-12
		L_∞	2.591107e-10	1.071311e-11
	f_5	RMSE	4.519395e-11	8.128081e-21
		L_∞	1.339111e-10	2.118603e-20
$N^2 = 225$	f_4	RMSE	6.006413e-14	1.159306e-15
		L_∞	2.121976e-13	3.663498e-15
	f_5	RMSE	8.613981e-14	8.314440e-28
		L_∞	2.720643e-13	2.694277e-26

TABLE 4. 2 D Interpolation Error Results with ($d = 50$)

Number of Centers	Error	CSP ($c^* = 0.95$)	SVSP
N=40	L_2	1.076395e-23	1.4037386-28
	L_∞	1.186638e-23	2.496621e-28
	RMSE	3.446426e-23	1.559586e-28
N=60	L_2	2.489795e-24	5.742747e-31
	L_∞	5.111960e-24	1.156160e-30
	RMSE	2.377938e-24	3.293193e-31
N=80	L_2	2.295270e-24	4.960491e-31
	L_∞	4.176405e-24	7.284569e-31
	RMSE	1.725887e-24	2.606880e-31

TABLE 5. Error Results 1 D Boundary Value Problem with ($d = 50$)

variation scheme the same accuracy level, the interpolation matrix condition number by our scheme grows much slower than that of the constant shaped RBF interpolation matrix with increase in the number of interpolation points.

5. Conclusion

In this study, we present a symmetric variable shape parameter strategy for GARBF, and apply it to interpolations and linear elliptic boundary value problems for verifying the numerical accuracy of present method. Comparison studies showed that the symmetric variable shaped Gaussian with the proposed scheme out performed the

Number of Centers	Error	CSP ($c^* = 2.5$)	SVSP
$N^2 = 25$	L_∞	3.783330e-2	2.206133e-2
	RMSE	2.195807e-2	1.109765e-2
$N^2 = 64$	L_∞	3.910419e-3	6.203643e-4
	RMSE	5.744870e-3	2.693574e-4
$N^2 = 100$	L_∞	9.6399669e-4	3.226619e-5
	RMSE	7.4621224e-4	1.302579e-5
$N^2 = 144$	L_∞	1.356141e-4	2.066263e-6
	RMSE	1.085606e-4	9.3973580e-7
$N^2 = 225$	L_∞	2.160298e-6	4.015188e-9
	RMSE	1.768520e-6	1.709078e-9

TABLE 6. Error Results 2 D Boundary Value Problem with ($d = 50$)

constant shaped Gaussian on the accuracy. Several numerical experiments with the uniformly spaced centers show that the GARBF with the symmetric variable shape parameter c strategy produces the best accuracy for the one and two-dimensional interpolations and linear boundary value problems.

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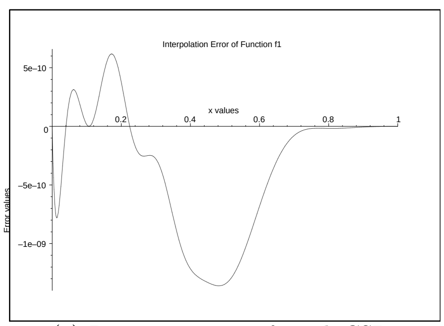
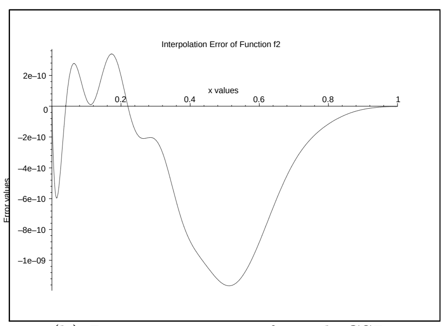
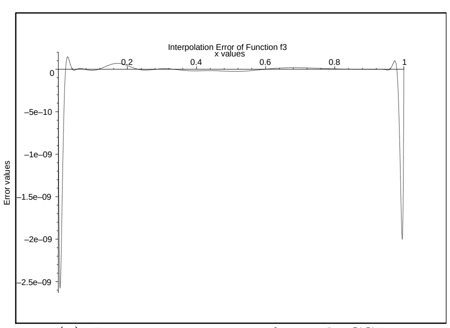


(a) Point-wise errors f_1 with CSP.

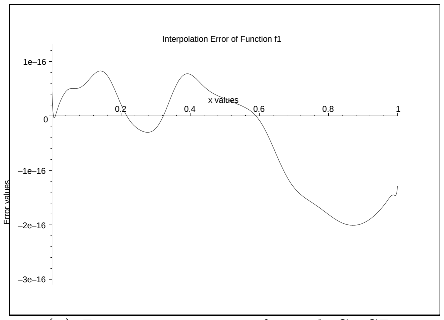


(b) Point-wise errors f_2 with CSP.

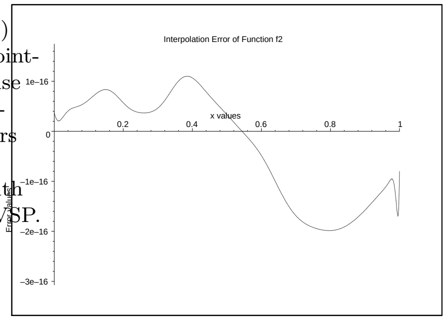


(c) Point-wise errors f_3 with CSP.

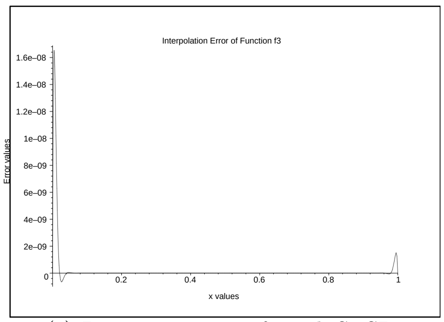
FIGURE 3. Plot of point-wise errors for the one dimensional interpolation of $f_i(x)$ for $i = 1, 2, 3$ using GRBF with CSP with $N = 40$ and $d = 50$.



(a) Point-wise errors f_1 with SVSP.



(b) Point-wise errors f_2 with SVSP.



(c) Point-wise errors f_3 with SVSP.

FIGURE 4. Plot of point-wise errors for the one dimensional interpolation of $f_i(x)$ for $i = 1, 2, 3$ using GRBF with SVSP with $N = 40$ and $d = 50$.

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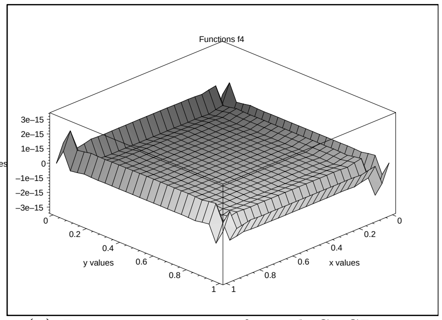
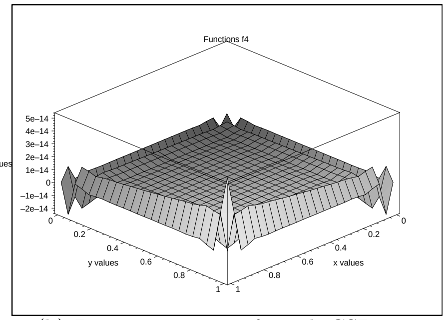
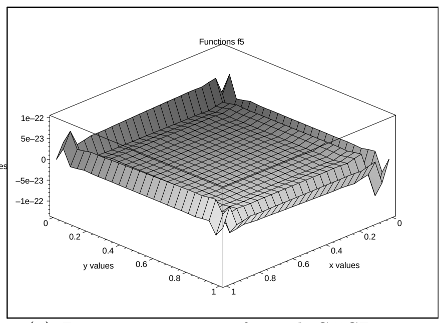


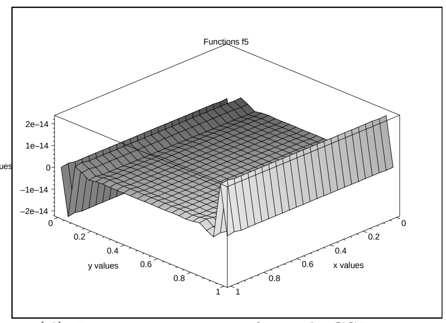
(a) Point-wise errors f_4 with SVSP.



(b) Point-wise errors f_4 with CSP.

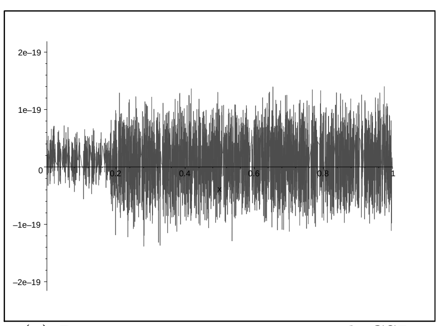


(c) Point-wise errors f_5 with SVSP.

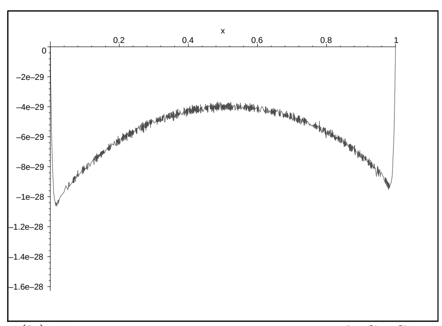


(d) Point-wise errors f_5 with CSP.

FIGURE 5. Plot of point-wise errors for the two dimensional interpolation of $f_i(x)$ for $i = 4, 5$ with $N = 40$ and $d = 50$.



(a) Point-wise errors $u_e - u_a$ with CSP.



(b) Point-wise errors $u_e - u_a$ with SVSP.

FIGURE 6. Plot of point-wise errors $u_e - u_a$ one dimensional linear boundary value problems with CSP ($c = 0.95$) and SVSP, $N = 40$ and $d = 40$.

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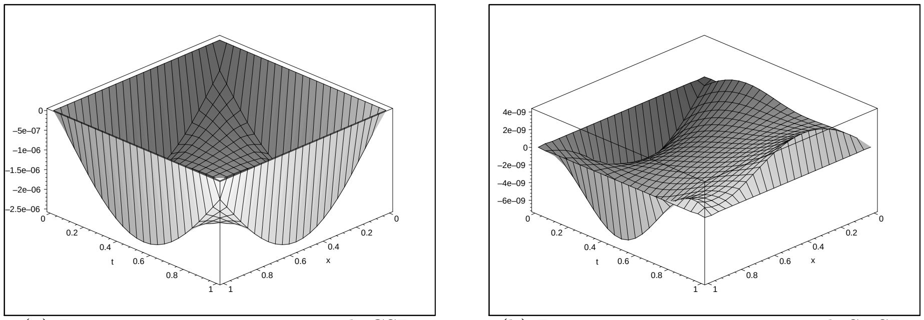

(a) Point-wise errors $u_e - u_a$ with CSP. (b) Point-wise errors $u_e - u_a$ with SVSP.

FIGURE 7. Plot of point-wise errors $u_e - u_a$ two dimensional linear boundary value problems with CSP ($c = 2.5$) and SVSP, $N = 15$ and $d = 50$.

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