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# Strongly nonlinear variational parabolic initial-boundary value problems

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Abstract. We prove an existence result for a strongly nonlinear parabolic equations with dual data in Sobolev spaces.

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## 1. Introduction

We deal with the following boundary value problems

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}\Big(a(x,t,u,\nabla u)\Big) + g(x,t,u,\nabla u) + H(x,t,\nabla u) = f & \text{in} \quad Q, \\ u(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{on} \quad \Omega, \end{cases} \tag{1}$$

where the cylinder $Q = \Omega \times [0,T]$ with a given real number $T > 0$ and $\Omega$ is a bounded domain of $\mathbb{R}^N$, $a : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function (that is, measurable with respect to $x$ in $\Omega$ for every $(t,s,\xi)$ in $[0,T] \times \mathbb{R} \times \mathbb{R}^N$, and continuous with respect to $(s,\xi)$ in $\mathbb{R} \times \mathbb{R}^N$ for almost every $x$ in $\Omega$) such that for all $\xi$, $\eta$ in $\mathbb{R}^N$, $\xi \neq \eta$,

$$|a(x,t,s,\xi)| \leq \beta\Big[k(x,t) + |s|^{p-1} + |\xi|^{p-1}\Big], \tag{2}$$

$$[a(x,t,s,\xi) - a(x,t,s,\eta)](\xi - \eta) > 0, \tag{3}$$

$$a(x,t,s,\xi)\xi \geq \alpha|\xi|^p, \tag{4}$$

where the function $k(x,t) \in L^{p'}(Q)$ and $\beta$, $\alpha$ are positives constants.

Furthermore, let $g(x,t,s,\xi) : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, and $H(x,t,\xi) : \Omega \times [0,T] \times \mathbb{R}^N \to \mathbb{R}$ be two Carathéodory functions which satisfy, for all $(x,t) \in Q$ and for all $s \in \mathbb{R}, \xi \in \mathbb{R}^N$, the following conditions

$$|g(x,t,s,\xi)| \leq L_1(|s|)\Big(L_2(x,t) + |\xi|^p\Big), \tag{5}$$

$$g(x,t,s,\xi)s \geq 0, \tag{6}$$

$$|H(x,t,\xi)| \leq h(x,t)|\xi|^{p-1}, \tag{7}$$

where $L_1 : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous increasing function in $L^1(\mathbb{R})$, while $L_2(x,t)$ belongs to $L^1(Q)$, and $h(x,t) \in L^r(Q)$ with $r > \max(N,p)$.

$$u(x,0) = u_0 \in L^1(\Omega). \tag{8}$$

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Our purpose in this paper is to prove the existence of solutions for the initial-boundary value problems (1) in the setting Sobolev space, in the case where $H \neq 0$ and $f$ belongs to $L^{p'}(0,T;W^{-1,p'}(\Omega))$, where the principal part $-\text{div}\Big(a(x,t,u,\nabla u)\Big)$, the nonlinearity $g$ and $H$ satisfying some general growth conditions. Note that, a little information is known for the parabolic case.

The study of the nonlinear partial differential equations in this type of spaces is strongly motivated by numerous phenomena of physics, namely the problems related to non-Newtonian fluids of strongly inhomogeneous behavior with a high ability of increasing their viscosity under a different stimulus, like the shear rate, magnetic or electric field.

For some classical and recent results on parabolic problems in Orlicz and Sobolev spaces see Dall'Aglio-Orsina [11] and Porretta [18] who proved the existence of solutions for the following Cauchy-Dirichlet problem (1) where $H = 0$ and the right hand side $f$ is assumed to belong to $L^1(Q)$. This result generalizes analogous one of Boccardo-Gallouët [7], J. L. Lions [17], Landes [14] with $g = 0$, and of Landes-Mustonen [15, 16] with $g = g(x,t,u)$. See also [8] and [2, 4, 9, 10, 12, 19, 20, 21, 22] for related topics. In all of these results, the function $a$ is supposed to satisfy a polynomial growth condition with respect to $u$ and $\nabla u$.

## 2. Main result

Firstly, we give the following lemma which will be used in our main result.

**Lemma 2.1.** *Given the functions $\lambda, \gamma, \varphi, \rho$ defined on $[a, +\infty[$, suppose that $a \geq 0$, $\lambda \geq 0$, $\gamma \geq 0$ and that $\lambda\gamma, \lambda\varphi$ and $\lambda\rho$ belong to $L^1(a,+\infty)$. If for a.e., $t \geq 0$ we have $\varphi(t) \leq \rho(t) + \gamma(t)\int_t^{+\infty} \lambda(\tau)\varphi(\tau)d\tau$, then for a.e., $t \geq 0$*

$$\varphi(t) \leq \rho(t) + \gamma(t)\int_t^{+\infty} \rho(\tau)\lambda(\tau)\left(\int_t^{\tau} \lambda(r)\gamma(r)dr\right)d\tau.$$

For the proof of this Lemma see [3]. Now we shall prove the following existence theorem:

**Theorem 2.2.** *Let $f \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ and assume that (2)-(8) hold. Then there exists at least one solution of the problem (1), in the following sense:*

$$\begin{cases} \displaystyle\int_\Omega S_k(u-v)(T)\,dx + \Big\langle \frac{\partial v}{\partial t}, T_k(u-v)\Big\rangle + \int_Q a(x,t,u,\nabla u)\nabla T_k(u-v)\,dx\,dt \\ \displaystyle\qquad + \int_Q g(x,t,u,\nabla u)T_k(u-v)\,dx\,dt + \int_Q H(x,t,\nabla u)T_k(u-v)\,dx\,dt \\ \displaystyle\qquad = \int_Q fT_k(u-v)\,dx\,dt + \int_\Omega S_k(u_0 - v(0))\,dx, \end{cases} \tag{9}$$

*for all $v \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$, where $S_k(s) = \int_0^s T_k(r)\,dr$.*

*Proof.* We divide the proof of this Theorem in four steps.

**Step 1: <u>Approximate problem and Energy estimate.</u>** For $n > 0$, let us define the following approximation of $u_0$, $g$ and $H$. Set

$$g_n(x,t,s,\xi) = \frac{g(x,t,s,\xi)}{1 + \frac{1}{n}|g(x,t,s,\xi)|} \quad \text{and} \quad H_n(x,t,\xi) = \frac{H(x,t,\xi)}{1 + \frac{1}{n}|H(x,t,\xi)|},$$

and $\{u_{0n}\}$ be a sequence in $L^2(\Omega)$ such that $u_{0n} \to u_0$ in $L^1(\Omega)$.

Let us now consider the following regularized problems:

$$\begin{cases} \dfrac{\partial u_n}{\partial t} - \operatorname{div}\Big(a(x,t,u_n,\nabla u_n)\Big) + g_n(x,t,u_n,\nabla u_n) + H_n(x,t,\nabla u_n) = f \text{ in } \mathcal{D}'(Q), \\ u_n(x,t=0) = 0 \qquad \text{in } \Omega, \\ u_n(x,t) = 0 \qquad \text{on } \partial\Omega \times (0,T). \end{cases} \tag{10}$$

Note that $g_n(x,t,s,\xi)$ and $H_n(x,t,\xi)$ are satisfying the following conditions

$$|g_n(x,t,s,\xi)| \le \max\Big\{|g(x,t,s,\xi)| \ ; \ n\Big\} \text{ and } |H_n(x,t,\xi)| \le \max\Big\{|H(x,t,\xi)| \ ; \ n\Big\}.$$

Moreover, since $f \in L^{p'}(0,T;W^{-1,\,p'}(\Omega))$, proving existence of a weak solution $u_n \in L^p(0,T;W_0^{1,\,p}(\Omega))$ of (10) is an easy task (see e.g. [17]).
For $\varepsilon > 0$ and $s \ge 0$, we define

$$\varphi_\varepsilon(r) = \begin{cases} \operatorname{sign}(r) & \text{if } \ |r| > s+\varepsilon, \\ \dfrac{\operatorname{sign}(r)(|r|-s)}{\varepsilon} & \text{if } \ s < |r| \le s+\varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

We choose $v = \varphi_\varepsilon(u_n)$ as test function in (10), we have

$$\begin{aligned} &\Big[\int_\Omega B^n_{\varphi_\varepsilon}(u_n)\,dx\Big]_0^T + \int_Q a(x,t,u_n,\nabla u_n)\cdot\nabla(\varphi_\varepsilon(u_n))\,dx\,dt \\ &\quad + \int_Q g_n(x,t,u_n,\nabla u_n)\varphi_\varepsilon(u_n)\,dx\,dt + \int_Q H_n(x,t,\nabla u_n)\varphi_\varepsilon(u_n)\,dx\,dt \\ &= \int_0^T \langle f;\varphi_\varepsilon(u_n)\rangle\,dt, \end{aligned}$$

where $B^n_{\varphi_\varepsilon}(r) = \displaystyle\int_0^r \varphi_\varepsilon(s)\,ds$. Using $B^n_{\varphi_\varepsilon}(r) \ge 0$, $g_n(x,t,u_n,\nabla u_n)\varphi_\varepsilon(u_n) \ge 0$, (7) and Hölder's inequality, we obtain

$$\begin{aligned} &\frac{1}{\varepsilon}\int_{\{s<|u_n|\le s+\varepsilon\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt \\ &\le \left(\int_{\{s<|u_n|\le s+\varepsilon\}} |f|^{p'}\,dx\,dt\right)^{\frac{1}{p'}} \left(\int_{\{s<|u_n|\le s+\varepsilon\}} \left(\frac{|\nabla u_n|}{\varepsilon}\right)^p dx\,dt\right)^{\frac{1}{p}} \\ &\quad + \int_{\{s<|u_n|\}} h(x,t)|\nabla u_n|^{p-1} dx\,dt. \end{aligned}$$

Observe that,

$$\begin{aligned} &\int_{\{s<|u_n|\}} h(x,t)|\nabla u_n|^{p-1} dx\,dt \\ &\le \int_s^{+\infty} \left(\frac{-d}{d\sigma}\int_{\{\sigma<|u_n|\}} h^p\,dx\,dt\right)^{\frac{1}{p}} \left(\frac{-d}{d\sigma}\int_{\{\sigma<|u_n|\}} |\nabla u_n|^p\,dx\,dt\right)^{\frac{1}{p'}} d\sigma. \end{aligned} \tag{11}$$

Because,

$$\begin{aligned}
&\int_{\{s<|u_n|\}} h(x,t)|\nabla u_n|^{p-1}dx\,dt = \int_s^{+\infty} \frac{-d}{d\sigma}\left(\int_{\{\sigma<|u_n|\}} h(x,t)|\nabla u_n|^{p-1}\,dx\,dt\right)d\sigma\\
&= \int_s^{+\infty} \lim_{\delta\to 0}\frac{1}{\delta}\left(\int_{\{\sigma<|u_n|\le\sigma+\delta\}} h(x,t)|\nabla u_n|^{p-1}\,dx\,dt\right)d\sigma\\
&\le \int_s^{+\infty} \lim_{\delta\to 0}\frac{1}{\delta}\left(\int_{\{\sigma<|u_n|\le\sigma+\delta\}} h^p\,dx\,dt\right)^{\frac{1}{p}}\left(\int_{\{\sigma<|u_n|\le\sigma+\delta\}} |\nabla u_n|^p\,dx\,dt\right)^{\frac{1}{p'}}d\sigma\\
&= \int_s^{+\infty} \left(\lim_{\delta\to 0}\frac{1}{\delta}\int_{\{\sigma<|u_n|\le\sigma+\delta\}} h^p\,dx\,dt\right)^{\frac{1}{p}}\left(\lim_{\delta\to 0}\frac{1}{\delta}\int_{\{\sigma<|u_n|\le\sigma+\delta\}} |\nabla u_n|^p\,dx\,dt\right)^{\frac{1}{p'}}d\sigma\\
&= \int_s^{+\infty} \left(\frac{-d}{d\sigma}\int_{\{\sigma<|u_n|\}} h^p\,dx\,dt\right)^{\frac{1}{p}}\left(\frac{-d}{d\sigma}\int_{\{\sigma<|u_n|\}} |\nabla u_n|^p\,dx\,dt\right)^{\frac{1}{p'}}d\sigma.
\end{aligned}$$

By (4) and (11), we deduce that

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_{\{s<|u_n|\le s+\varepsilon\}} \alpha|\nabla u_n|^p\,dx\,dt\\
&\quad\le \left(\frac{1}{\varepsilon}\int_{\{s<|u_n|\le s+\varepsilon\}} |f|^{p'}\,dx\,dt\right)^{\frac{1}{p'}}\left(\frac{1}{\varepsilon}\int_{\{s<|u_n|\le s+\varepsilon\}} |\nabla u_n|^p\,dx\,dt\right)^{\frac{1}{p}}\\
&\qquad+\int_s^{+\infty}\left(\frac{-d}{d\sigma}\int_{\{\sigma<|u_n|\}} h^p\,dx\,dt\right)^{\frac{1}{p}}\left(\frac{-d}{d\sigma}\int_{\{\sigma<|u_n|\}} |\nabla u_n|^p\,dx\,dt\right)^{\frac{1}{p'}}d\sigma.
\end{aligned}$$

Letting $\varepsilon$ go to zero, we obtain

$$\begin{aligned}
&\frac{-d}{ds}\int_{\{s<|u_n|\}} \alpha|\nabla u_n|^p\,dx\,dt\\
&\quad\le \left(\frac{-d}{ds}\int_{\{s<|u_n|\}} |f|^{p'}\,dx\,dt\right)^{\frac{1}{p'}}\left(\frac{-d}{ds}\int_{\{s<|u_n|\}} |\nabla u_n|^p\,dx\,dt\right)^{\frac{1}{p}}\\
&\qquad+\int_s^{+\infty}\left(\frac{-d}{d\sigma}\int_{\{\sigma<|u_n|\}} h^p\,dx\,dt\right)^{\frac{1}{p}}\left(\frac{-d}{d\sigma}\int_{\{\sigma<|u_n|\}} |\nabla u_n|^p\,dx\,dt\right)^{\frac{1}{p'}}d\sigma,
\end{aligned}\tag{12}$$

where $\{s<|u_n|\}$ denotes the set $\{(x,t)\in Q,\ s<|u_n(x,t)|\}$ and $\mu(s)$ stands for the distribution function of $u_n$, that is $\mu(s)=|\{(x,t)\in Q,\ |u_n(x,t)|<s\}|$ for all $s\ge 0$.

Now, we recall the following inequality (see for example [13]), we have for almost every $s>0$

$$1\le\left(NC_N^{\frac{1}{N}}\right)^{-1}(\mu(s))^{\frac{1}{N}-1}(-\mu'(s))^{\frac{1}{p'}}\left(-\frac{d}{ds}\int_{\{s<|u_n|\}}|\nabla u_n|^p dx\,dt\right)^{\frac{1}{p}}.\tag{13}$$

Using (13), we have

$$\begin{aligned}
&\frac{-d}{ds}\int_{\{s<|u_n|\}} \alpha|\nabla u_n|^p\,dx\,dt\\
&\quad= \alpha\left(\frac{-d}{ds}\int_{\{s<|u_n|\}} |\nabla u_n|^p\,dx\,dt\right)^{\frac{1}{p'}}\left(\frac{-d}{ds}\int_{\{s<|u_n|\}} |\nabla u_n|^p\,dx\,dt\right)^{\frac{1}{p}}
\end{aligned}$$

$$\leq \left(\frac{-d}{ds}\int_{\{s<|u_n|\}}|f|^{p'}\,dx\,dt\right)^{\frac{1}{p'}}\left(\frac{-d}{ds}\int_{\{s<|u_n|\}}|\nabla u_n|^p\,dx\,dt\right)^{\frac{1}{p}}$$
$$+\left(NC_N^{\frac{1}{N}}\right)^{-1}(\mu(s))^{\frac{1}{N}-1}(-\mu'(s))^{\frac{1}{p'}}\left(\frac{-d}{ds}\int_{\{s<|u_n|\}}|\nabla u_n|^p\,dx\,dt\right)^{\frac{1}{p}}$$
$$\times\int_s^{+\infty}\left(\frac{-d}{d\sigma}\int_{\{\sigma<|u_n|\}}h^p\,dx\,dt\right)^{\frac{1}{p}}\left(\frac{-d}{d\sigma}\int_{\{\sigma<|u_n|\}}|\nabla u_n|^p\,dx\,dt\right)^{\frac{1}{p'}}d\sigma.$$

Which implies that,

$$\alpha\left(\frac{-d}{ds}\int_{\{s<|u_n|\}}|\nabla u_n|^p\,dx\,dt\right)^{\frac{1}{p'}}\leq\left(\frac{-d}{ds}\int_{\{s<|u_n|\}}|f|^{p'}\,dx\,dt\right)^{\frac{1}{p'}}+\left(NC_N^{\frac{1}{N}}\right)^{-1}(\mu(s))^{\frac{1}{N}-1}$$
$$\times(-\mu'(s))^{\frac{1}{p'}}\int_s^{+\infty}\left(\frac{-d}{d\sigma}\int_{\{\sigma<|u_n|\}}h^p\,dx\,dt\right)^{\frac{1}{p}}\left(\frac{-d}{d\sigma}\int_{\{\sigma<|u_n|\}}|\nabla u_n|^p\,dx\,dt\right)^{\frac{1}{p'}}d\sigma. \tag{14}$$

Now, we consider two functions $B(s)$ and $F(s)$ (see Lemma 2.2 of [1]) defined by

$$\int_{\{s<|u_n|\}}h^p(x,t)dxdt=\int_0^{\mu(s)}B^p(\sigma)d\sigma, \tag{15}$$

$$\int_{\{s<|u_n|\}}|f|^{p'}dxdt=\int_0^{\mu(s)}F^{p'}(\sigma)d\sigma. \tag{16}$$

$$||B||_{L^p(0,T;W_0^{1,p}(\Omega))}\leq||h||_{L^p(0,T;W_0^{1,p}(\Omega))}\ \text{ and }\ ||F||_{L^{p'}(0,T;W^{-1,p'}(\Omega))}\leq||f||_{L^{p'}(0,T;W^{-1,p'}(\Omega))}. \tag{17}$$

From (14), (15) and (16) we get

$$\alpha\left(\tfrac{-d}{ds}\int_{\{s<|u_n|\}}|\nabla u_n|^p dxdt\right)^{\frac{1}{p'}}\leq F(\mu(s))(-\mu'(s))^{\frac{1}{p'}}$$
$$+(NC_N^{\frac{1}{N}})^{-1}(\mu(s))^{\frac{1}{N}-1}(-\mu'(s))^{\frac{1}{p'}}\int_s^{+\infty}B(\mu(\nu))(-\mu'(\nu))^{\frac{1}{p}}\left(-\frac{d}{d\nu}\int_{\{\nu<|u_n|\}}|\nabla u_n|^p dxdt\right)^{\frac{1}{p'}}d\nu.$$

From Lemma 2.1, we obtain

$$\alpha\left(\tfrac{-d}{ds}\int_{\{s<|u_n|\}}|\nabla u_n|^p\,dx\,dt\right)^{\frac{1}{p'}}\leq F(\mu(s))(-\mu'(s))^{\frac{1}{p'}}+(NC_N^{\frac{1}{N}})^{-1}(\mu(s))^{\frac{1}{N}-1}(-\mu'(s))^{\frac{1}{p'}}$$
$$\times\int_s^{+\infty}F(\mu(\sigma))B(\mu(\sigma))(-\mu'(\sigma))\exp\left(\int_s^{\sigma}(NC_N^{\frac{1}{N}})^{-1})B(\mu(r))(\mu(r))^{\frac{1}{N}-1}(-\mu'(r))dr\right)d\sigma.$$

Raising to the power $p'$, integrating between 0 and $+\infty$ and by a variable change we have

$$\alpha^{p'}\int_Q|\nabla u_n|^p\,dx\,dt\leq c_0\int_0^{|Q|}F^{p'}(\lambda)\,d\lambda$$
$$+c_0\int_0^{|Q|}\lambda^{(\frac{1}{N}-1)p'}\left[\int_0^{\lambda}F(z)B(z)\exp\left(\int_z^{\lambda}(NC_N^{\frac{1}{N}})^{-1}B(v)v^{\frac{1}{N}-1}dv\right)dz\right]^{p'}d\lambda.$$

Using Hölder's inequality and (17), then we get

$$||u_n||_{L^p(0,T;W_0^{1,p}(\Omega))}\ \leq\ c_1, \tag{18}$$

where $c_i$ is some positive constant not depending of $n$. Then there exists $u \in L^p(0,T;W_0^{1,p}(\Omega))$ such that, for some subsequence

$$u_n \rightharpoonup u \text{ weakly in } L^p(0,T;W_0^{1,p}(\Omega)), \tag{19}$$

we conclude that

$$||T_k(u_n)||^p_{L^p(0,T;W_0^{1,p}(\Omega))} \leq c_2 k. \tag{20}$$

Then for each $k$, the sequence $T_k(u_n)$ converges almost everywhere in $Q$, which implies that $u_n$ converges almost everywhere to some measurable function $u$ in $Q$. Thus by using the same argument as in [5, 6], we can show that

$$u_n \to u \text{ a.e. in } Q, \tag{21}$$

and we can deduce from (20) that

$$T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } L^p(0,T;W_0^{1,p}(\Omega)). \tag{22}$$

Which implies, by using (2), for all $k > 0$ that there exists a function $\overline{a} \in (L^{p'}(Q))^N$, such that

$$a(x,t,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup \overline{a} \text{ weakly in } (L^{p'}(Q))^N. \tag{23}$$

Finally, denoting $u_n' = f + \text{div}\Big(a(x,t,u_n,\nabla u_n)\Big) - g_n(x,t,u_n,\nabla u_n) - H_n(x,t,\nabla u_n)$ we observe that, $f + \text{div}\Big(a(x,t,u_n,\nabla u_n)\Big)$ is bounded in $L^{p'}(0,T;W^{-1,p'}(\Omega))$ and $-g_n(x,t,u_n,\nabla u_n) - H_n(x,t,\nabla u_n)$ is bounded in $L^1(Q)$. Then we can conclude that $(u_n)_n$ is relatively compact in $L^p_{loc}(Q)$, thus we can deduce $u_n \to u$ in $L^p_{loc}(Q)$, and $u_n \to u$ strongly in $L^1(Q)$.

**Step 2: Almost everywhere convergence of the gradients.** This step is devoted to introduce for $k \geq 0$ fixed a time regularization of the function $T_k(u)$ in order to perform the monotonicity method. This kind of regularization has been first introduced by R. Landes (see Lemma 6 and Proposition 3, p. 230, and Proposition 4, p. 231, in [14]). For $k > 0$ fixed, and let $\varphi(t) = te^{\gamma t^2}$, $\gamma > 0$. It is well known that when $\gamma > \left(\frac{L_1(k)}{2\alpha}\right)^2$, one has

$$\varphi'(s) - (\tfrac{L_1(k)}{\alpha})|\varphi(s)| \geq \tfrac{1}{2}, \text{ for all } s \in \mathbb{R}. \tag{24}$$

Let $\psi_i \in \mathcal{D}(\Omega)$ be a sequence which converges strongly to $u_0$ in $L^1(\Omega)$.

Set $w^i_\mu = (T_k(u))_\mu + e^{-\mu t} T_k(\psi_i)$ where $(T_k(u))_\mu$ is the mollification with respect to time of $T_k(u)$. Note that $w^i_\mu$ is a smooth function having the following properties:

$$\frac{\partial w^i_\mu}{\partial t} = \mu(T_k(u) - w^i_\mu), \quad w^i_\mu(0) = T_k(\psi_i), \quad \left|w^i_\mu\right| \leq k, \tag{25}$$

$$w^i_\mu \to T_k(u) \text{ strongly in } L^p(0,T;W_0^{1,p}(\Omega)), \text{ as } \mu \to \infty. \tag{26}$$

We introduce the following function:

$$h_m(s) = \begin{cases} 1 & \text{if } |s| \leq m, \\ 0 & \text{if } |s| \geq m+1, \\ m+1-|s| & \text{if } m \leq |s| \leq m+1, \end{cases}$$

where $m > k$. Let $\theta^{\mu,i}_n = T_k(u_n) - w^i_\mu$ and $z^{\mu,i}_{n,m} = \varphi(\theta^{\mu,i}_n) h_m(u_n)$.

Using in (10) the test function $z_{n,m}^{\mu,i}$, we obtain

$$\begin{aligned}
&\int_0^T \langle \frac{\partial u_n}{\partial t} \ ; \ \varphi(T_k(u_n) - w_\mu^i)h_m(u_n)\rangle \, dt \\
&\quad + \int_Q a(x,t,u_n,\nabla u_n)[\nabla T_k(u_n) - \nabla w_\mu^i]\varphi'(\theta_n^{\mu,i})h_m(u_n)\, dx\, dt \\
&\quad + \int_Q a(x,t,u_n,\nabla u_n)\nabla u_n \varphi(\theta_n^{\mu,i})h'_m(u_n)\, dx\, dt \\
&\quad + \int_Q \Big(g_n(x,t,u_n,\nabla u_n) + H_n(x,t,\nabla u_n)\Big) z_{n,m}^{\mu,i}\, dx\, dt \\
&= \int_0^T \langle f \ ; \ z_{n,m}^{\mu,i}\rangle \, dt,
\end{aligned}$$

which implies since $g_n(x,t,u_n,\nabla u_n)\varphi(T_k(u_n) - w_\mu^i)h_m(u_n) \geq 0$ on $\{|u_n| > k\}$

$$\begin{aligned}
&\int_0^T \langle \frac{\partial u_n}{\partial t} \ ; \ \varphi(T_k(u_n) - w_\mu^i)h_m(u_n)\rangle \, dt \\
&\quad + \int_Q a(x,t,u_n,\nabla u_n)[\nabla T_k(u_n) - \nabla w_\mu^i]\varphi'(\theta_n^{\mu,i})h_m(u_n)\, dx\, dt \\
&\quad + \int_Q a(x,t,u_n,\nabla u_n)\nabla u_n \varphi(\theta_n^{\mu,i})h'_m(u_n)\, dx\, dt \qquad (27)\\
&\quad + \int_{\{|u_n|\leq k\}} g_n(x,t,u_n,\nabla u_n)\varphi(T_k(u_n) - w_\mu^i)h_m(u_n)\, dx\, dt \\
&\leq \int_0^T \langle f \ ; \ z_{n,m}^{\mu,i}\rangle \, dt + \int_Q |H_n(x,t,\nabla u_n) z_{n,m}^{\mu,i}|\, dx\, dt.
\end{aligned}$$

In the sequel and throughout the paper, we will omit for simplicity the denote $\varepsilon(n,\mu,i,m)$ all quantities (possibly different) such that

$$\lim_{m\to\infty}\lim_{i\to\infty}\lim_{\mu\to\infty}\lim_{n\to\infty}\varepsilon(n,\mu,i,m) = 0,$$

and this will be the order in which the parameters we use will tend to infinity, that is, first $n$, then $\mu, i$ and finally $m$. Similarly we will write only $\varepsilon(n)$, or $\varepsilon(n,\mu)$,... to mean that the limits are made only on the specified parameters.

We will deal with each term of (27). First of all, observe that

$$\int_0^T \langle f \ ; \ z_{n,m}^{\mu,i}\rangle \, dt + \int_Q |H_n(x,t,\nabla u_n) z_{n,m}^{\mu,i}|\, dx\, dt = \varepsilon(n,\mu), \qquad (28)$$

since $\varphi(T_k(u_n) - w_\mu^i)h_m(u_n)$ converges to $\varphi\big(T_k(u) - (T_k(u))_\mu + e^{-\mu t}T_k(\psi_i)\big)h_m(u)$ strongly in $L^p(Q)$ and weakly$-*$ in $L^\infty(Q)$ as $n \to \infty$ and finally $\varphi\big(T_k(u) - (T_k(u))_\mu + e^{-\mu t}T_k(\psi_i)\big)h_m(u)$ converges to 0 strongly in $L^p(Q)$ and weakly$-*$ in $L^\infty(Q)$ as $\mu \to \infty$.

On the one hand, the definition of the sequence $w_\mu^i$ makes it possible to establish the following Lemma 2.3.

**Lemma 2.3.** *For $k \geq 0$ we have*

$$\int_0^T \langle \frac{\partial u_n}{\partial t} \ ; \ \varphi(T_k(u_n) - w_\mu^i)h_m(u_n)\rangle \, dt \geq \varepsilon(n,m,\mu,i). \qquad (29)$$

*Proof.* (see Blanchard, Murat and Redwane [6]). □

Then, the second term of the left hand side of (27) can be written

$$\begin{aligned}
&\int_Q a(x,t,u_n,\nabla u_n)(\nabla T_k(u_n)-\nabla w_\mu^i)\varphi'(T_k(u_n)-w_\mu^i)h_m(u_n)\,dx\,dt\\
&\quad=\int_{\{|u_n|\le k\}} a(x,t,u_n,\nabla u_n)(\nabla T_k(u_n)-\nabla w_\mu^i)\varphi'(T_k(u_n)-w_\mu^i)h_m(u_n)\,dx\,dt\\
&\qquad+\int_{\{|u_n|>k\}} a(x,t,u_n,\nabla u_n)(\nabla T_k(u_n)-\nabla w_\mu^i)\varphi'(T_k(u_n)-w_\mu^i)h_m(u_n)\,dx\,dt\\
&\quad=\int_Q a(x,t,u_n,\nabla u_n)(\nabla T_k(u_n)-\nabla w_\mu^i)\varphi'(T_k(u_n)-w_\mu^i)\,dx\,dt\\
&\qquad+\int_{\{|u_n|>k\}} a(x,t,u_n,\nabla u_n)(\nabla T_k(u_n)-\nabla w_\mu^i)\varphi'(T_k(u_n)-w_\mu^i)h_m(u_n)\,dx\,dt,
\end{aligned}$$

since $m>k$ and $h_m(u_n)=1$ on $\{|u_n|\le k\}$, we deduce that

$$\begin{aligned}
&\int_Q a(x,t,u_n,\nabla u_n)(\nabla T_k(u_n)-\nabla w_\mu^i)\varphi'(T_k(u_n)-w_\mu^i)h_m(u_n)\,dx\,dt\\
&\quad=\int_Q\Big(a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(u))\Big)\\
&\qquad\qquad(\nabla T_k(u_n)-\nabla T_k(u))\varphi'(T_k(u_n)-w_\mu^i)\,dx\,dt\\
&\qquad+\int_Q a(x,t,T_k(u_n),\nabla T_k(u))(\nabla T_k(u_n)-\nabla T_k(u))\varphi'(T_k(u_n)-w_\mu^i)h_m(u_n)\,dx\,dt\\
&\qquad+\int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u)\varphi'(T_k(u_n)-w_\mu^i)h_m(u_n)\,dx\,dt\\
&\qquad-\int_Q a(x,t,u_n,\nabla u_n)\nabla w_\mu^i\varphi'(T_k(u_n)-w_\mu^i)h_m(u_n)\,dx\,dt\\
&\quad=K_1+K_2+K_3+K_4.
\end{aligned}$$

Using (2), (23) and Lebesgue theorem we have $a(x,t,T_k(u_n),\nabla T_k(u))$ converges to $a(x,t,T_k(u),\nabla T_k(u))$ strongly in $(L^{p'}(Q))^N$ and $\nabla T_k(u_n)$ converges to $\nabla T_k(u)$ weakly in $(L^p(Q))^N$, then $K_2=\varepsilon(n)$. Using (23) and (26) we have $K_3=\int_Q \overline{a}\,\nabla T_k(u)\,dx\,dt+\varepsilon(n,\mu)$. For what concerns $K_4$ can be written, since $h_m(u_n)=0$ on $\{|u_n|>m+1\}$

$$\begin{aligned}
K_4&=-\int_Q a(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\nabla w_\mu^i\varphi'(T_k(u_n)-w_\mu^i)h_m(u_n)\,dx\,dt\\
&=-\int_{\{|u_n|\le k\}} a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla w_\mu^i\varphi'(T_k(u_n)-w_\mu^i)h_m(u_n)\,dx\,dt\\
&\quad-\int_{\{k<|u_n|\le m+1\}} a(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\nabla w_\mu^i\varphi'(T_k(u_n)-w_\mu^i)h_m(u_n)\,dx\,dt,
\end{aligned}$$

and, as above, by letting $n$ to $+\infty$ we get

$$\begin{aligned}
K_4=&-\int_{\{|u|\le k\}}\overline{a}\nabla w_\mu^i\varphi'(T_k(u)-w_\mu^i)\,dx\,dt\\
&-\int_{\{k<|u|\le m+1\}}\overline{a}\nabla w_\mu^i\varphi'(T_k(u)-w_\mu^i)h_m(u)\,dx\,dt+\varepsilon(n),
\end{aligned}$$

so that, by letting $\mu$ to $+\infty$ we get

$$K_4 \;=\; -\int_Q \overline{a}\nabla T_k(u)\,dx\,dt+\varepsilon(n,\mu).$$

We conclude then that

$$\begin{aligned}&\int_Q a(x,t,u_n,\nabla u_n)(\nabla T_k(u_n)-\nabla w_\mu^i)\varphi'(T_k(u_n)-w_\mu^i)h_m(u_n)\,dx\,dt\\&=\int_Q\Big(a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(u))\Big)\\&\qquad(\nabla T_k(u_n)-\nabla T_k(u))\varphi'(T_k(u_n)-w_\mu^i)\,dx\,dt+\varepsilon(n,\mu).\end{aligned}\tag{30}$$

To deal with the third term of the left hand side of (27), observe that

$$\begin{aligned}&\left|\int_Q a(x,t,u_n,\nabla u_n)\nabla u_n\varphi(\theta_n^{\mu,i})h'_m(u_n)\,dx\,dt\right|\\&\quad\le\varphi(2k)\int_{\{m\le|u_n|\le m+1\}}a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt.\end{aligned}$$

By (2) and (18), we obtain

$$\left|\int_Q a(x,t,u_n,\nabla u_n)\nabla u_n\varphi(\theta_n^{\mu,i})h'_m(u_n)\,dx\,dt\right|\quad\le\quad\varepsilon(n,m).\tag{31}$$

We now turn to fourth term of the left hand side of (27), can be written

$$\begin{aligned}&\left|\int_{\{|u_n|\le k\}}g(x,t,u_n,\nabla u_n)\varphi(T_k(u_n)-w_\mu^i)h_m(u_n)dxdt\right|\\&\le\int_{\{|u_n|\le k\}}L_1(k)(L_2(x,t)+|\nabla T_k(u_n)|^p|\varphi(T_k(u_n)-w_\mu^i)h_m(u_n)\,dx\,dt\\&\le L_1(k)\int_Q L_2(x,t)|\varphi(T_k(u_n)-w_\mu^i)|\,dx\,dt\\&\quad+\frac{L_1(k)}{\alpha}\int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)|\varphi(T_k(u_n)-w_\mu^i)|\,dx\,dt,\end{aligned}\tag{32}$$

since $L_2(x,t)$ belong to $L^1(Q)$ it is easy to see that

$$L_1(k)\int_Q L_2(x,t)|\varphi(T_k(u_n)-w_\mu^i)|\,dx\,dt\quad=\quad\varepsilon(n,\mu).$$

On the other hand, the second term of the right hand side of (32), can be written

$$\begin{aligned}&\frac{L_1(k)}{\alpha}\int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)|\varphi(T_k(u_n)-w_\mu^i)|\,dx\,dt\\&=\frac{L_1(k)}{\alpha}\int_Q\Big(a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(u))\Big)\\&\qquad\qquad(\nabla T_k(u_n)-\nabla T_k(u))|\varphi(T_k(u_n)-w_\mu^i)|\,dx\,dt\\&\quad+\frac{L_1(k)}{\alpha}\int_Q a(x,t,T_k(u_n),\nabla T_k(u))(\nabla T_k(u_n)-\nabla T_k(u))|\varphi(T_k(u_n)-w_\mu^i)|\,dx\,dt\\&\quad+\frac{L_1(k)}{\alpha}\int_Q a(x,t,T_k(u_n),\nabla T_k(u))\nabla T_k(u)|\varphi(T_k(u_n)-w_\mu^i)|\,dx\,dt,\end{aligned}$$

and, as above, by letting first $n$ then finally $\mu$ to infinity, we can easily see, that each one of last two integrals is of the form $\varepsilon(n,\mu)$. This implies that

$$\begin{aligned}&\left|\int_{\{|u_n|\leq k\}} g(x,t,u_n,\nabla u_n)\varphi(T_k(u_n)-w_\mu^i)h_m(u_n)dxdt\right| \\ &\quad\leq \frac{L_1(k)}{\alpha}\int_Q\Big(a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(u))\Big) \\ &\qquad\qquad (\nabla T_k(u_n)-\nabla T_k(u))|\varphi(T_k(u_n)-w_\mu^i)|\,dx\,dt+\varepsilon(n,\mu).\end{aligned} \tag{33}$$

Combining (27), (29), (30), (31) and (33), we get

$$\begin{aligned}&\int_Q\Big(a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(u))\Big) \\ &\qquad (\nabla T_k(u_n)-\nabla T_k(u))\left(\varphi'(T_k(u)-w_\mu^i)-\tfrac{L_1(k)}{\alpha}|\varphi(T_k(u_n)-w_\mu^i)|\right)\,dx\,dt \\ &\quad\leq \varepsilon(n,\mu,i,m),\end{aligned}$$

and so, thanks to (24), we have

$$\int_Q\Big(a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(u))\Big)(\nabla T_k(u_n)-\nabla T_k(u))\,dx\,dt\leq\varepsilon(n). \tag{34}$$

Hence by passing to the limit sup over $n$, we get

$$\limsup_{n\to\infty}\int_Q\Big(a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))\Big)(\nabla T_k(u_n)-\nabla T_k(u))\,dx\,dt=0$$

This implies that

$$T_k(u_n)\to T_k(u)\text{ strongly in }L^p(0,T;W_0^{1,p}(\Omega))\text{ for all }k. \tag{35}$$

Now, observe that for every $\sigma>0$,

$$\begin{aligned}&\operatorname{meas}\Big\{(x,t)\in Q:\ |\nabla u_n-\nabla u|>\sigma\Big\} \\ &\quad\leq\operatorname{meas}\Big\{(x,t)\in Q:\ |\nabla u_n|>k\Big\}+\operatorname{meas}\Big\{(x,t)\in Q:\ |u|>k\Big\} \\ &\qquad+\operatorname{meas}\Big\{(x,t)\in Q:\ \big|\nabla T_k(u_n)-\nabla T_k(u)\big|>\sigma\Big\}\end{aligned}$$

then as a consequence of (35) we have that $\nabla u_n$ converges to $\nabla u$ in measure and therefore, always reasoning for a subsequence,

$$\nabla u_n\to\nabla u\text{ a. e. in }Q. \tag{36}$$

Which implies

$$a(x,t,T_k(u_n),\nabla T_k(u_n))\rightharpoonup a(x,t,T_k(u),\nabla T_k(u))\text{ weakly in }(L^{p'}(Q))^N. \tag{37}$$

**Step 3: Equi-integrability of $H_n$ and $g_n$.** We shall now prove that $H_n(x,t,\nabla u_n)$ converges to $H(x,t,\nabla u)$ and $g_n(x,t,u_n,\nabla u_n)$ converges to $g(x,t,u,\nabla u)$ strongly in $L^1(Q)$ by using Vitali's theorem. Since $H_n(x,t,\nabla u_n)\to H(x,t,\nabla u)$ a.e. $Q$ and $g_n(x,t,u_n,\nabla u_n)\to g(x,t,u,\nabla u)$ a.e. $Q$, thanks to (5) and (7), it suffices to prove that $H_n(x,t,\nabla u_n)$ and $g_n(x,t,u_n,\nabla u_n)$ are uniformly equi-integrable in $Q$. We will now prove that $H_n(x,\nabla u_n)$ is uniformly equi-integrable, we use Hölder's inequality and (18), we have

$$\begin{aligned}\int_E |H_n(x,\nabla u_n)| &\leq \left(\int_E h^p(x,t)\,dx\,dt\right)^{\frac{1}{p}} \left(\int_Q |\nabla u_n|^p\right)^{\frac{1}{p'}} \\ &\leq c_1 \left(\int_E h^p(x,t)\,dx\,dt\right)^{\frac{1}{p}}.\end{aligned} \tag{38}$$

which is small uniformly in $n$ when the measure of $E$ is small.

To prove the uniform equi-integrability of $g_n(x,t,u_n,\nabla u_n)$. For any measurable subset $E \subset Q$ and $m \geq 0$,

$$\int_E |g(x,t,u_n,\nabla u_n)|\,dx\,dt = \int_{E\cap\{|u_n|\leq m\}} |g(x,t,u_n,\nabla u_n)|\,dx\,dt + \int_{E\cap\{|u_n|>m\}} |g(x,t,u_n,\nabla u_n)|\,dx\,dt$$

$$\begin{aligned}&\leq L_1(m)\int_{E\cap\{|u_n|\leq m\}} [L_2(x,t)+|\nabla u_n|^p]\,dx\,dt + \int_{E\cap\{|u_n|>m\}} |g(x,t,u_n,\nabla u_n)|\,dx\,dt \quad (39)\\ &\leq L_1(m)\int_{E\cap\{|u_n|\leq m\}} [L_2(x,t)+|\nabla T_m(u_n)|^p]\,dx\,dt + \int_{E\cap\{|u_n|>m\}} |g(x,t,u_n,\nabla u_n)|\,dx\,dt \\ &= K_1 + K_2.\end{aligned}$$

For fixed $m$, we get

$$K_1 \leq L_1(m)\int_E [L_2(x,t)+|\nabla T_m(u_n)|^p]\,dx\,dt,$$

which is thus small uniformly in $n$ for $m$ fixed when the measure of $E$ is small (recall that $T_m(u_n)$ tends to $T_m(u)$ strongly in $L^p(0,T;W_0^{1,p}(\Omega))$). We now discuss the behavior of the second integral of the right hand side of (39), let $\psi_m$ be a function such that

$$\begin{cases} \psi_m(s) = 0 & \text{if } \ |s| \leq m-1, \\ \psi_m(s) = \text{sign}(s) & \text{if } \ |s| \geq m, \\ \psi'_m(s) = 1 & \text{if } \ m-1 < |s| < m. \end{cases} \tag{40}$$

We choose $\psi_m(u_n)$ as a test function for $m > 1$ in (10), we obtain

$$\begin{aligned}&\Big[\int_\Omega B_m^n(u_n)dx\Big]_0^T + \int_Q a(x,t,u_n,\nabla u_n)\nabla u_n \psi'_m(u_n)\,dx\,dt \\ &\qquad + \int_Q g_n(x,t,u_n,\nabla u_n)\psi_m(u_n)\,dx\,dt + \int_Q H_n(x,t,\nabla u_n)\psi_m(u_n)\,dx\,dt \\ &\quad = \int_0^T \langle f \ ; \ \psi_m(u_n)\rangle\,dt,\end{aligned}$$

where $B_m^n(r) = \int_0^r \psi_m(s)ds$, which implies, since $B_m^n(r) \geq 0$ and using (4), Hölder's inequality

$$\begin{aligned}\int_{\{m-1\leq|u_n|\}} |g_n(x,t,u_n,\nabla u_n)|\,dx\,dt &\leq \int_E |H_n(x,t,\nabla u_n)|\,dx\,dt \\ &\quad + \|f\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))}\Big(\int_{\{m-1\leq|u_n|\leq m\}} |\nabla u_n|^p\,dx\,dt\Big)^{\frac{1}{p}}.\end{aligned}$$

By (18), we have

$$\lim_{m\to\infty}\sup_{n\in\mathbb{N}}\int_{\{|u_n|>m-1\}}|g_n(x,t,u_n,\nabla u_n)|\,dx\,dt=0.$$

Thus we proved that the second term of the right hand side of (39) is also small, uniformly in $n$ and in $E$ when $m$ is sufficiently large. Which shows that $g_n(x,t,u_n,\nabla u_n)$ and $H_n(x,t,\nabla u_n)$ are uniformly equi-integrable in $Q$ as required, we conclude that

$$\begin{cases} H_n(x,t,\nabla u_n)\to H(x,t,\nabla u) & \text{strongly in} \quad L^1(Q),\\ g_n(x,t,u_n,\nabla u_n)\to g(x,t,u,\nabla u) & \text{strongly in} \quad L^1(Q). \end{cases} \tag{41}$$

**Step 4: Passing to the limit.** Going back to approximate equations (10) and using $v\in L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(Q)$ as the test function, one has

$$\begin{aligned} &\int_\Omega S_k(u_n-v)(T)\,dx+\Big\langle\frac{\partial v}{\partial t},T_k(u_n-v)\Big\rangle+\int_Q a(x,t,u_n,\nabla u_n)\nabla T_k(u_n-v)\,dx\,dt\\ &\quad+\int_Q g(x,t,u_n,\nabla u_n)T_k(u_n-v)\,dx\,dt+\int_Q H(x,t,\nabla u_n)T_k(u_n-v)\,dx\,dt\\ &=\int_Q fT_k(u_n-v)\,dx\,dt+\int_\Omega S_k(u_{n0}-v(0))\,dx, \end{aligned}$$

in which we can pass to the limit thanks to the previous results, we prove the existence of a solution $u$ of the nonlinear parabolic problems (9). This completes the proof of Theorem 2.2. □

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