Solving nonlinear fractional differential equations using multi-step homotopy analysis method

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Abstract. This paper presents a numerical technique for solving fractional differential equation by employing the multi-step homotopy analysis method (MHAM). It is known that the corresponding numerical solution obtained using the HAM is valid only for a short time. On the contrary, the results obtained using the MHAM are more valid and accurate during a long time, and are highly agreement with the exact solutions in the case of integer-order systems. The objective of the present paper is to modify the HAM to provide symbolic approximate solution for linear and nonlinear of fractional differential equations. The efficient and accuracy of the method used in this paper will be demonstrated by comparison with the known methods and with the known exact solutions in the non fractional case. The fractional derivatives are described in the Caputo sense.

1. Introduction

The solutions of fractional differential equations are much involved. In general, there exists no method that yields an exact solution for fractional differential equations. Only approximate solution can be derived using linearization or perturbation methods. In recent years, many researchers have focused on the numerical solution of ordinary differential equations of fractional order and some numerical methods have been developed such as Fourier transform method [1], Adomian decomposition method [2, 3] and Homotopy perturbation method [4, 5]. Recently, the Homotopy analysis method (HAM) has been proposed by Liao [6, 7, 8, 9, 10, 11]. Based on homotopy of topology, the validity of the HAM is independent of whether there exist small parameters or not in the considered equation. The HAM has been used to investigate a variety of mathematical and physical problems [7]. The homotopy analysis method contains a certain auxiliary parameter h and auxiliary linear operator L which provides us with a simple way to control and adjust the rate of convergence of the series solution [12, 13]. The objective of the present paper is to modify the HAM to provide approximate solution for linear and nonlinear fractional differential equations. In this paper we investigate the applicability and effectiveness of the HAM when treated as an algorithm in a sequence of intervals (i.e. time step) for finding accurate approximate solutions to the fractional differential equation of the form

$$D_*^\alpha u(t) + a_n u^{(n)}(t) + a_{n-1} u^{(n-1)}(t) + ... + a_0 u(t) + N(u(t), u'(t)) = f(t),$$

$$t \geq 0, \quad n - 1 < \alpha \leq n, \tag{1.1}$$

subject to the initial conditions

$$u^{(i)}(0) = b_i, \qquad i = 0, 1, ..., n-1. \qquad (1.2)$$

Here D_*^α is the fractional derivative in the Caputo sense. This modified method is the multi-step homotopy analysis method. It can be found that the corresponding numerical solutions obtained by using the HAM are valid only for a short time [13]. While the ones obtained by using the MHAM are more valid and accurate during a long time and are highly agreement with the exact solutions in the case of integer-order systems. Some examples are given to illustrate this method. The paper is organized as follows. A brief review of the fractional calculus is given in Section 2. In Section 3, the proposed method is described. In Section 4, we investigate the applicability and effectiveness of the multi-step homotopy analysis method for finding accurate approximate solutions to the fractional differential equation and we will present a comparison between our results with the exact solution by plotting the exact solution and the approximate solution. Conclusions are presented in Section 5.

2. Fractional calculus

In this section, we introduce the linear operators of fractional integration and fractional differentiation in the framework of the Riemann-Liouville and Caputo fractional calculus [14, 15, 16, 17].

Definition 2.1. A real function $f(x)$, $x > 0$, is said to be in the space C_α, $\alpha \in \mathbb{R}$ if it can be written as $f(x) = x^p f_1(x)$, for some $p > \alpha$ where $f_1(x)$ is continous in $[0, \infty)$, and it is said to be in the space C_α^m if $f^{(m)} \in C_\alpha$, $m \in \mathbb{N}$.

Definition 2.2. A function $f(x)$, $x > 0$, is said to be in the space C_μ^m, $m \in \mathbb{N} \cup \{0\}$, if $f^m \in C_\mu$.

Definition 2.3. The Riemann-Liouville fractional integral of order $\alpha > 0$ is

$$J^\alpha u(t) = u_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} u(\tau) d\tau, \quad t > 0, \ \alpha \in \mathbb{R}^+,$$

$$J^0 u(t) = u(t).$$

Definition 2.4. The Caputo fractional derivative of $u(t)$ is defined as:

$$D_*^\alpha \ u(t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t \dfrac{u^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} \, d\tau, & m-1 < \alpha < m, \\ \dfrac{d^m u(t)}{dt^m}, & \alpha = m, \end{cases}$$

Hence, we have the following properties

1. $\qquad J^\alpha J^\beta u(t) = J^{\alpha+\beta} u(t) = J^\alpha J^\beta u(t), \ \alpha, \ \beta \geq 0, \ u \in C\mu,$

2. $\qquad J^\alpha t^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}, \ \alpha > 0, \ \gamma > -1, \ t > 0,$

3. $\qquad D_*^\alpha J^\alpha u(t) = u(t),$

4. $\qquad J^\alpha D_*^\alpha u(t) = u(t) - \displaystyle\sum_{k=0}^{m-1} u^{(k)}(0^+)\dfrac{t^k}{k!}, \ t > 0, \ m-1 < \alpha \leq m.$

3. Multi-step homotopy analysis method algorithm

The HAM has been extended by many authors to solve linear and nonlinear problems in terms of convergent series with easily computable components, it has some drawbacks: the series solution always converges in a small region and it has slow convergent rate or completely divergent in the wider region. In this section, we present the basic ideas of the MHAM that have been developed for the numerical solution of our problem (1.1), (1.2). It is only a simple modification of the standard HAM and can ensure the validity of the approximate solution for large time t. To extend this solution over the interval $[0, \ t]$, we divide the interval I into $r-$subintervals of equal length $\triangle t$, $[t_0, \ t_1)$, $[t_1, \ t_2)$,..., $[t_{r-1}, \ t_r]$ with $t_0 = 0$, $t_r = t$. Let t^* be the initial value and $u_j(t)$ be approximate solutions in each subinterval $[t_{j-1}, \ t_j)$, $j = 1, 2, \ldots, r$, then the equations (1.1), (1.2) can be transformed into the following system

$$D_*^\alpha \ u_j(t) + a_n \ u_j^{(n)}(t) + a_{n-1} \ u_j^{(n-1)}(t) + ... + a_0 \ u_j(t) + N(u_j(t), \ u_j'(t)) = f(t),$$

$$t \geq 0, \quad n - 1 < \alpha \leq n, \ , j = 1, 2, \ldots, r, \tag{3.1}$$

subject to the initial conditions

$$u_1^{(i)}(t^*) = b_i, \ \ u_j^{(i)}(t^*) = u_{j-1}^{(i)}(t_{j-1}) = c_{j,i} \quad i = 0, 1, ..., n-1, \ , j = 1, 2, \ldots, r. \tag{3.2}$$

The zero-order deformation equation of system (3.1) has the form

$$(1 - q)L[\phi_j(t, q) - u_j(t^*)] = qh[D_*^\alpha \phi_j(t, q) + a_n \ \frac{d^n}{dt^n}\phi_j(t, q) \tag{3.3}$$

$$+a_{n-1}\frac{d^{n-1}}{dt^{n-1}}\phi_j(t, q) + ... + a_0\phi_j(t, q) + N(\phi_j(t, q), \ \frac{d}{dt}\phi_j(t, q)) - f(t)].$$

Where $q \in [0, 1]$ is an embedding parameter, L is an auxiliary linear operator, $h \neq 0$ is an auxiliary parameter, $\phi_j(t; q)$ is unknown function, $u_j(t^*)$ be the initial guess of $u_j(t)$ which satisfy the initial condition and $f(t)$ is known function. Obviously, when $q = 0$ we have

$$\phi_1(t, 0) = u_1(t^*), \ \ \phi_j(t, 0) = u_j(t^*), \quad j = 2, 3, ..., r. \tag{3.4}$$

When $q = 1$, we have

$$\phi_j(t, 1) = u_j(t), \quad j = 1, 2, ..., r. \tag{3.5}$$

Expanding $\phi_j(t, q), \ j = 1, 2, ..., r$, in Taylor series with respect to q, we get

$$\phi_j(t, q) = u_j(t^*) + \sum_{m=1}^{\infty} u_{j,m}(t)q^m, \quad j = 1, 2, ..., r, \tag{3.6}$$

where

$$u_{j,m}(t) = \frac{1}{m!}\frac{\partial^m\phi_j(t, q)}{\partial q^m}|_{q=0}. \tag{3.7}$$

If the initial guess $u_j(t^*)$, the auxiliary linear operator L and the nonzero auxiliary parameter h are properly chosen so that the power series (3.6) converges at $q = 1$, one has

$$u_j(t) = \phi_j(t; 1) = u_j(t^*) + \sum_{m=1}^{\infty} u_{j,m}(t).$$

For brevity, define the vector

$$\overrightarrow{u}_{j,m}(t) = \{u_{j,0}(t), u_{j,1}(t), \ldots, u_{j,m}(t)\},$$

Differentiating the zero-order deformation equation (3.3) m times with respective to q and then dividing by $m!$ and finally setting $q = 0$, we have the so-called $mth-$order deformation equations

$$L[u_{j,m}(t) - \chi_m u_{j,m-1}(t)] = h\ \Re_{j,m}(\overrightarrow{u}_{j,m-1}(t)), \tag{3.8}$$

where

$$\Re_{j,m}(\overrightarrow{u}_{j,m-1}(t)) = D_*^\alpha\ u_{j,m-1}(t) + a_n\ u_{j,m-1}^{(n)}(t) + a_{n-1}\ u_{j,m-1}^{(n-1)}(t) + ... + a_0\ u_{j,m-1}(t)$$

$$+ \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}N(\phi_j(t,q),\frac{d}{dt}\phi_j(t,q))|_{q=0} - f(t)(1 - \chi_m), \tag{3.9}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1 \\ 1, & m > 1 \end{cases}. \tag{3.10}$$

Select the auxiliary linear operator $L = D_*^\alpha$, then the mth-order deformation equations (3.8) can be written in the form

$$u_{j,m}(t) = \chi_m u_{j,m-1}(t) + h\ J^\alpha[\ \Re_{j,m}(\overrightarrow{u}_{j,m-1}(t))], \tag{3.11}$$

and a power series solution has the form

$$u_j(t) = \sum_{m=0}^{\infty} u_{j,m}(t), \quad j = 1, 2, ..., r.$$

Finally, the solutions of system (1.1) has the form

$$u(t) = \begin{cases} u_1(t), & t \in [t_0, t_1] \\ u_2(t), & t \in [t_1, t_2] \\ \quad \vdots & \\ u_r(t), & t \in [t_{r-1}, t_r] \end{cases}.$$

4. Numerical results

To demonstrate the effectiveness of the method for solving nonlinear fractional differential equations, we consider here the following four examples.

4.1. Example 1. Consider the following simple harmonic fractional oscillator

$$D_*^\alpha u(t) + (0.5)^\alpha\ u(t) = 0, \quad t \ge 0, \quad 1 < \alpha \le 2, \tag{4.1}$$

subject to the initial condition

$$u(0) = 1,\ u^{'}(0) = 0. \tag{4.2}$$

The exact solutions of this equation when $\alpha = 2$ is $u(t) = \cos 0.5t$. Let $u_j(t)$ be the approximate solutions in the subinterval $[t_{j-1}, t_j)$, then equation (4.1), is transformed into the following system

$$D_*^\alpha u_j(t) + (0.5)^\alpha\ u_j(t) = 0, \quad t \ge 0, \quad 1 < \alpha \le 2,\ j = 1, 2, ..., r. \tag{4.3}$$

Let $u_j(t^*) = c_j$, with $c_1 = 1$ are the initial guesses of $u_j(t)$, then we can construct the MHAM (3.11) where

$$\Re_{j,m}(\overrightarrow{u}_{j,m-1}(t)) = D_*^\alpha u_{j,m-1}(t) + (0.5)^\alpha u_{j,m-1}(t). \tag{4.4}$$

The series solution for equation (4.3) is given by,

$$u_j(t) = c_j + \frac{h(0.5)^\alpha c_j(1 + (1+h) + (1+h)^2)}{\Gamma(\alpha+1)}(t-t^*)^\alpha$$
$$+\frac{h^2(0.5)^{2\alpha}c_j(1+2(1+h))}{\Gamma(2\alpha+1)}(t-t^*)^{2\alpha} + \frac{h^3(0.5)^{3\alpha}c_j}{\Gamma(3\alpha+1)}(t-t^*)^{3\alpha} + ...$$

To demonstrate the effectiveness of the proposed algorithm as an approximate tool for solving the fractional differential equations (4.1), (4.2) for larger t, we apply the proposed algorithm on the interval $[0, 100]$. We choose to divide the interval $[0, 100]$ to subintervals with time step $\triangle t = 0.1$. Figure 1 shows the series solution exhibit the periodic behavior which is the characteristic of the simple harmonic Equations (4.1), (4.2) obtained for $\alpha = 2$, 1.6, 1.5 and when $h = -1$. It can be seen that the results obtained using the MHAM (when $\alpha = 2$) match the results of the exact solution $(u(t) = \cos 0.5t)$ very well, and are highly in agreement during a long time. It is clear that the numerical results obtained using the MHAM has the same trajectories for various values of α and all its solutions are expected to oscillate with decreasing to zero when the value of α is decreasing.
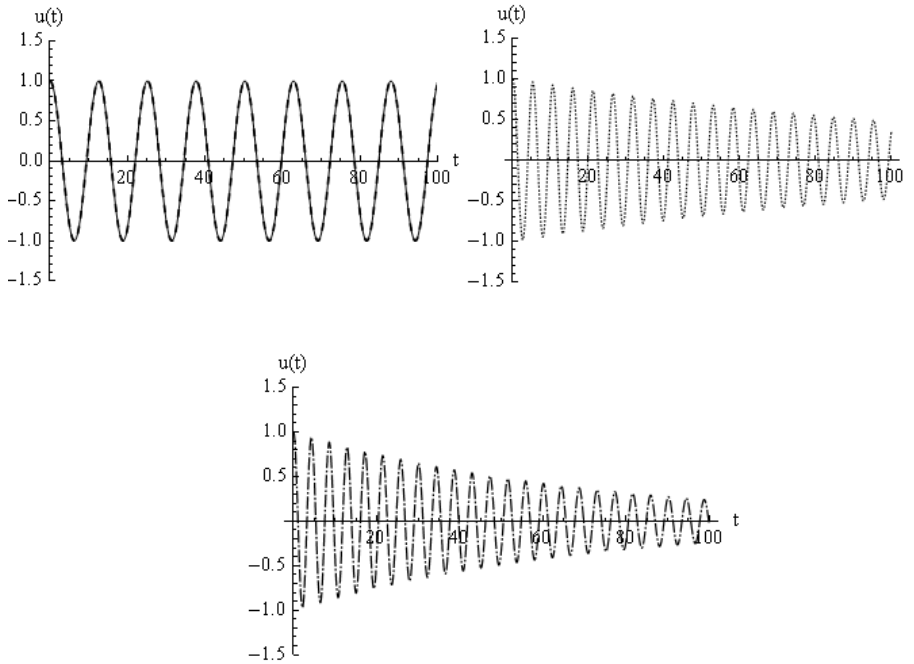

FIGURE 1. The displacement for Example 1: Solid line: exact solution, Dashed line: MHAM solution when $\alpha = 2$, Dotted line: MHAM solution when $\alpha = 1.6$, Dashed dotted line: MHAM solution when $\alpha = 1.5$.

4.2. Example 2. Consider the following nonlinear fractional Riccati equation

$$D_*^\alpha u(t) + u^2(t) - 1 = 0, \quad t \geq 0, \quad 0 < \alpha \leq 1, \tag{4.5}$$

subject to the initial condition

$$u(0) = 0. \tag{4.6}$$

The exact solutions of this equation when $\alpha = 1$ is $u(t) = \frac{e^{2t}-1}{e^{2t}+1}$. Let $u_j(t)$ be the approximate solutions in the subinterval $[t_{j-1}, t_j)$, then equation (4.5), is transformed into the following system

$$D_*^\alpha u_j(t) + u_j^2(t) - 1 = 0, \quad t \geq 0, \quad 0 < \alpha \leq 1, \ j = 1, 2, ..., r. \qquad (4.7)$$

Let $u_j(t^*) = c_j$, with $c_1 = 0$ are the initial guesses of $u_j(t)$, then we can construct the MHAM (3.11) where

$$\Re_{j,m}(\overrightarrow{u}_{j,m-1}(t)) = D_*^\alpha u_{j,m-1}(t) + \sum_{i=0}^{m-1} u_{j,i}(t)u_{j,m-i-1}(t) - (1 - \chi_m). \qquad (4.8)$$

The series solution for equation (4.7) is given by

$$u_j(t) = c_j + \frac{h(c_j^2 - 1)(1 + (1 + h) + (1 + h)^2)}{\Gamma(\alpha + 1)}(t - t^*)^\alpha$$
$$+ \frac{2h^2 c_j(c_j^2 - 1)(1 + 2(1 + h))}{\Gamma(2\alpha + 1)}(t - t^*)^{2\alpha} + \frac{h^3(c_j^2 - 1)^2 \Gamma(2\alpha + 1)}{\Gamma(3\alpha + 1)\Gamma^2(\alpha + 1)}(t - t^*)^{3\alpha} + ...$$

In this example we apply the proposed algorithm on the interval $[0, \ 20]$. We choose to divide the interval $[0, \ 20]$ to subintervals with time step $\triangle t = 0.1$. Figure 2 shows the series solution of the MHAM of the nonlinear fractional Riccati equations (4.5), (4.6) (when $\alpha = 1$, 0.9, 0.7 and $h = -1$) and the displacement of the exact solution $(u(t) = \frac{e^{2t}-1}{e^{2t}+1})$. From the graphical results it can be seen that the results obtained using the MHAM (when $\alpha = 1$) match the results of the exact solution very well. Therefore, the proposed method is very effcient and accurate method that can be used to provide analytical solutions for linear and nonlinear fractional differential equations. Also as the previous example, the numerical results obtained using the MHAM has the same trajectories for various values of α.
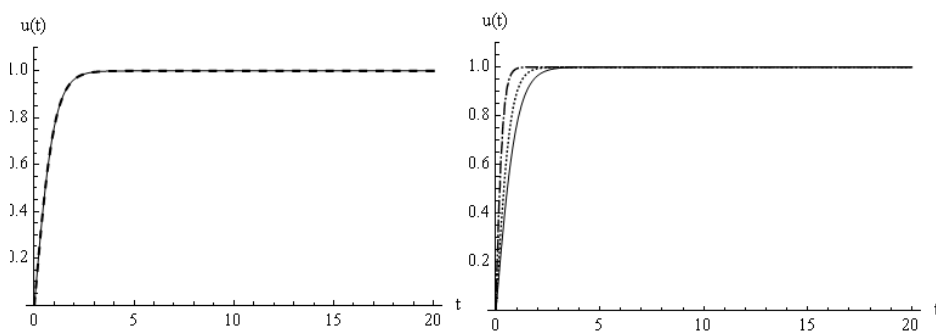

FIGURE 2. The displacement for Example 2: Solid line: exact solution, Dashed line: MHAM solution when $\alpha = 1$, Dotted line: MHAM solution when $\alpha = 0.9$, Dashed dotted line: MHAM solution when $\alpha = 0.7$.

4.3. Example 3.

$$D_*^\alpha u(t) - 2u(t) + u^2(t) - 1 = 0, \quad t \geq 0, \quad 0 < \alpha \leq 1, \qquad (4.9)$$

subject to the initial condition

$$u(0) = 0. \qquad (4.10)$$

The exact solutions of this equation when $\alpha = 1$ is $u(t) = 1 + \sqrt{2} \ \tanh(\sqrt{2} \ t + \frac{1}{2} \log(\frac{\sqrt{2}-1}{\sqrt{2}+1}))$. Let $u_j(t)$ be the approximate solutions in the subinterval $[t_{j-1}, t_j)$, then equation (4.9), is transformed into the following system

$$D_*^\alpha u_j(t) - 2u_j(t) + u_j^2(t) - 1 = 0, \quad t \geq 0, \quad 0 < \alpha \leq 1, \ j = 1, 2, ..., r. \quad (4.11)$$

Let $u_j(t^*) = c_j$, with $c_1 = 0$ are the initial guesses of $u_j(t)$, then we can construct the MHAM (3.11) where

$$\Re_{j,m}(\overrightarrow{u}_{j,m-1}(t)) = D_*^\alpha u_{j,m-1}(t) - 2 \ u_{j,m-1}(t) + \sum_{i=0}^{m-1} u_{j,i}(t)u_{j,m-i-1}(t) - (1 - \chi_m).$$
$$(4.12)$$

Then the analytic solution for system (4.11) is derived as followsis

$$u_j(t) = c_j + \frac{h(c_j^2 - 2c_j - 1)(1 + (1+h) + (1+h)^2)}{\Gamma(\alpha+1)}(t - t^*)^\alpha$$
$$+ \frac{2h^2(c_j^2 - 2c_j - 1)(c_j - 1)(1 + 2(1+h))}{\Gamma(2\alpha+1)}(t - t^*)^{2\alpha}$$
$$+ \frac{h^3(c_j^2 - 2c_j - 1)}{\Gamma(3\alpha+1)}(\frac{(c_j^2 - 2c_j - 1)\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} + (2c_j^2 - 6c_j + 4))(t - t^*)^{3\alpha} + ...$$

In this example we apply the proposed algorithm on the interval $[0, \ 20]$. We choose to divide the interval $[0, \ 20]$ to subintervals with time step $\triangle t = 0.1$. Figure 3 shows the series solution of the MHAM of the nonlinear fractional Riccati equations (4.9), (4.10) (when $\alpha = 1$, 0.9, 0.7 and $h = -1$) and the displacement of the exact solution $(u(t) = 1 + \sqrt{2} \ \tanh(\sqrt{2} \ t + \frac{1}{2} \log(\frac{\sqrt{2}-1}{\sqrt{2}+1})))$. Also the results of our computations (when $\alpha = 1$) are in excellent agreement with the results obtained by the exact solution and are highly in agreement during a long time. Therefore, the proposed method is very efficient and accurate method that can be used to provide analytic solutions for linear and nonlinear fractional differential equations. Also as example (2), the numerical results obtained using the MHAM has the same trajectories for various values of α.
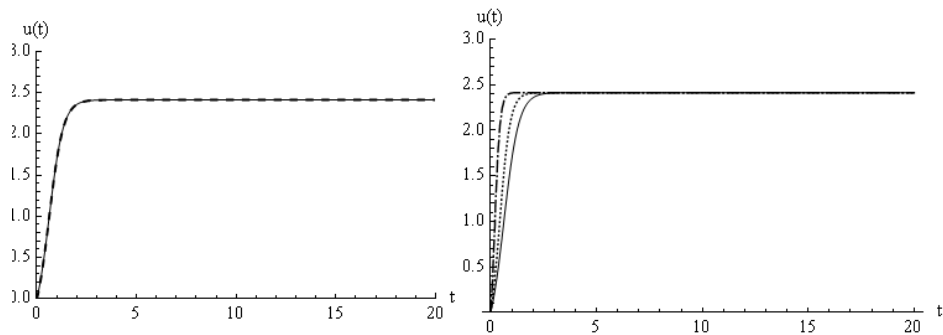

FIGURE 3. The displacement for Example 3: Solid line: exact solution, Dashed line: MHAM solution when $\alpha = 1$, Dotted line: MHAM solution when $\alpha = 0.9$, Dashed dotted line: MHAM solution when $\alpha = 0.7$.

4.4. Example 4. Consider the following nonlinear fractional equation

$$D_*^\alpha u(t) + 2u(t) + u^2(t) = 0, \quad t \geq 0, \quad 1 < \alpha \leq 2, \tag{4.13}$$

subject to the initial condition

$$u(0) = 0.1, \ u'(0) = 0. \tag{4.14}$$

Let $u_j(t)$ be the approximate solutions in the subinterval $[t_{j-1}, t_j)$, then equation (4.13), is transformed into the following system

$$D_*^\alpha u_j(t) + 2 \ u_j(t) + u_j^2(t) = 0, \quad t \geq 0, \quad 1 < \alpha \leq 2, \ j = 1, 2, ..., r. \tag{4.15}$$

Let $u_j(t^*) = c_j$, with $c_1 = 0$ are the initial guesses of $u_j(t)$, then we can construct the MHAM (3.11) where

$$\Re_{j,m}(\overrightarrow{u}_{j,m-1}(t)) = D_*^\alpha u_{j,m-1}(t) + 2 \ u_{j,m-1}(t) + \sum_{i=0}^{m-1} u_{j,i}(t) u_{j,m-i-1}(t). \tag{4.16}$$

The analytic solutions for system (4.15) derived by

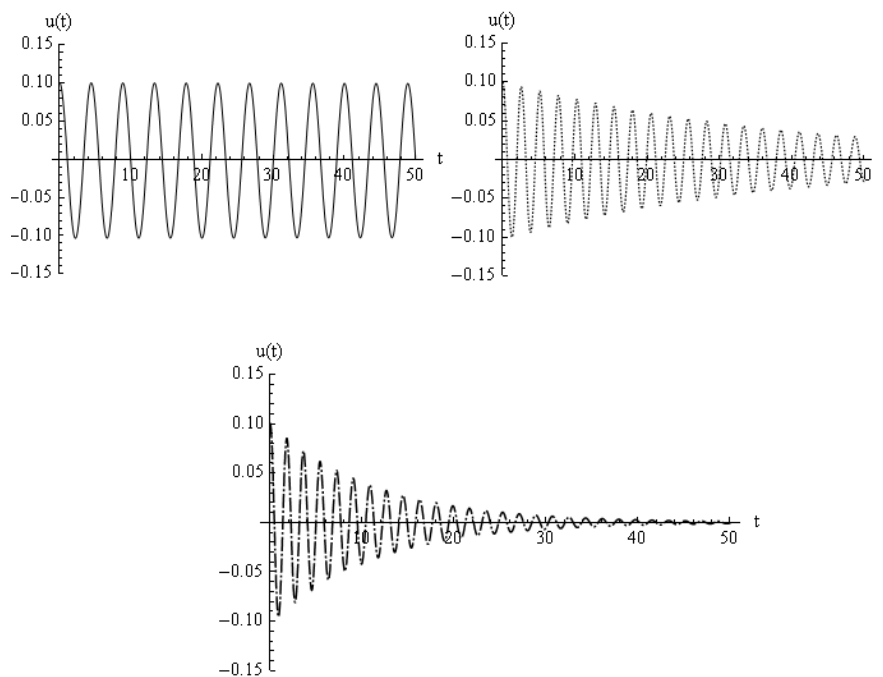

FIGURE 4. The displacement for Example 4: Solid line: MHAM solution when $\alpha = 2$, Dotted line: MHAM solution when $\alpha = 1.7$, Dashed dotted line: MHAM solution when $\alpha = 1.5$.

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\begin{aligned}
u_j(t) \quad = \quad & c_j + \frac{h(c_j^2 + 2c_j)(1 + (1+h) + (1+h)^2)}{\Gamma(\alpha+1)}(t - t^*)^\alpha \\
& + \frac{2h^2(c_j^2 + 2c_j)(c_j + 1)(1 + 2(1+h))}{\Gamma(2\alpha+1)}(t - t^*)^{2\alpha} \\
& + \frac{h^3(c_j^2 + 2c_j)}{\Gamma(3\alpha+1)}\left(\frac{(c_j^2 + 2c_j)\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} + (2c_j^2 + 6c_j + 4)\right)(t - t^*)^{3\alpha} + ...
\end{aligned}
$$

Let $[0, 50]$ be the interval over which we want to find the solution of the initial value problem (4.13), (4.14). Assume that the interval $[0, 50]$ is divided into subintervals of equal length $\triangle t = 0.1$. Figure 4 shows the series solution exhibit the periodic behavior which is the characteristic of the nonlinear fractional differential equations (4.13), (4.14) obtained for $\alpha = 2, 1.7, 1.5$ and when $h = -1$. It is clear that the numerical results obtained using the MHAM have the same trajectories for various values of α and all its solutions are expected to oscillate with decreasing to zero when the value of α is decreasing.

5. Conclusions

The fundamental goal of this work has been to propose an efficient algorithm for the solution of linear and nonlinear fractional differential equation. Based on some numerical and analytical techniques, we discussed in this paper the MHAM. The MHAM is an efficient modification of the HAM which introduces an efficient tool for calculating approximate solution for linear and nonlinear fractional differential equation. Our method is a direct method, further it is simple and accurate. It is a practical method and can easily be implemented on computer to solve such problems. We have used the method with four examples. The main advantage of the method is a fast convergence to the solution. Moreover, it avoids amount of calculations required by the other existing analytical methods. The new method leads to higher accuracy and simplicity, and in all cases the solutions obtained are easily programmable approximates to the analytic solution of the original problems with the accuracy required. The proposed scheme can be applied for other nonlinear equations. It can be found that the corresponding numerical solutions obtained by using HAM are valid only for a short time. While the ones obtained by using MHAM are more valid and accurate during a long time. A comparison between the graph of the numerical result with the graph of the exact solution indicates that the MHAM method is powerful analytic method for handling differential equations of fractional order.

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