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Growth of certain combinations of entire solutions of higher order linear differential equations

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Abstract. The main purpose of this paper is to study the growth of certain combinations of entire solutions of higher order complex linear differential equations.

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1. Introduction and main results

In this paper, we assume that the reader is familiar with the fundamental results and standard notations of the Nevanlinna theory [4, 13] . In addition, we will use $\rho(f)$ to denote the order and $\rho_2(f)$ to denote the hyper-order of f. See, [4, 6, 13] for notations and definitions.
We consider the differential equation

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_0(z) f = 0, \tag{1}$$

where $A_j(z)$ $(j = 0, \cdots, k-1)$ are entire functions. Suppose that $\{f_1, f_2, \cdots, f_k\}$ is the set of fundamental solutions of (1). It is clear that $f = c_1 f_1 + c_2 f_2 + \cdots + c_k f_k$ where c_i $(i = 1, \cdots, k)$ are complex numbers is a solution of (1), but what about the properties of $f = c_1 f_1 + c_2 f_2 + \cdots + c_k f_k$ if c_i $(i = 1, \cdots, k)$ are non-constant entire functions? In [7] , the authors gave answer to this question for the case $k = 2$, and obtained the following results.

Theorem 1.1. [7] *Let $A(z)$ be transcendental entire function of finite order. Let $d_j(z)$ $(j = 1, 2)$ be finite order entire functions that are not all vanishing identically such that* $\max\{\rho(d_1), \rho(d_2)\} < \rho(A)$. *If f_1 and f_2 are two linearly independent solutions of*

$$f'' + A(z) f = 0, \tag{2}$$

then the polynomial of solutions $g_f = d_1 f_1 + d_2 f_2$satisfies

$$\rho(g_f) = \rho(f_j) = \infty \ \ (j = 1, 2)$$

and

$$\rho_2(g_f) = \rho_2(f_j) = \rho(A) \quad (j = 1, 2).$$

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Theorem 1.2. [7] *Let $A(z)$ be a polynomial of $\deg A = n$. Let $d_j(z)$ $(j = 1, 2)$ be finite order entire functions that are not all vanishing identically such that $\max\{\rho(d_1), \rho(d_2)\} < \frac{n+2}{2}$ and $h \not\equiv 0$, where*

$$h = \begin{vmatrix} d_1 & 0 & d_2 & 0 \\ d_1' & d_1 & d_2' & d_2 \\ d_1'' - d_1 A & 2d_1' & d_2'' - d_2 A & 2d_2' \\ d_1''' - 3d_1' A - d_1 A' & d_1'' - d_1 A + 2d_1'' & d_2''' - 3d_2' A - d_2 A' & d_2'' - d_2 A + 2d_2'' \end{vmatrix}.$$

If f_1 and f_2 are two linearly independent solutions of (2), then the polynomial of solutions $g_f = d_1 f_1 + d_2 f_2$ satisfies

$$\rho(g_f) = \rho(f_j) = \frac{n+2}{2} \quad (j = 1, 2).$$

The aim of this paper is to study the growth of

$$g_k = d_1 f_1 + d_2 f_2 + \cdots + d_k f_k,$$

where $\{f_1, f_2, \cdots, f_k\}$ is any set of fundamental solutions of (1) and $d_j(z)$ $(j = 1, 2, \cdots, k)$ are finite order entire functions that are not all vanishing identically. In fact, we give sufficient conditions on $A_j(z)$ $(j = 0, \cdots, k-1)$ and $d_j(z)$ $(j = 1, 2)$ to prove that for any two solutions f_1 and f_2 of (1), the growth of $g_2 = d_1 f_1 + d_2 f_2$ is the same as the growth of f_j $(j = 1, 2)$, and we obtain the following results.

Theorem 1.3. *Let $A_j(z)$ $(j = 0, \cdots, k-1)$ be entire functions of finite order such that $\max\{\rho(A_j) : j = 1, \cdots, k-1\} < \rho(A_0)$. Let $d_j(z)$ $(j = 1, 2)$ be finite order entire functions that are not all vanishing identically such that $\max\{\rho(d_1), \rho(d_2)\} < \rho(A_0)$. If f_1 and f_2 are any two linearly independent solutions of (1), then the combination of solutions $g_2 = d_1 f_1 + d_2 f_2$ satisfies*

$$\rho(g_2) = \rho(f_j) = \infty \quad (j = 1, 2)$$

and

$$\rho_2(g_f) = \rho_2(f_j) = \rho(A) \quad (j = 1, 2).$$

Theorem 1.4. *Let $A_0(z)$ be transcendental entire function with $\rho(A_0) = 0$, and let $A_1, \cdots, A_{k-1}$ be polynomials. Let $d_j(z)$ $(j = 1, 2)$ be finite order entire functions that are not all vanishing identically. If f_1 and f_2 are any two linearly independent solutions of (1), then the combination of solutions $g_2 = d_1 f_1 + d_2 f_2$ satisfies*

$$\rho(g_2) = \rho(f_j) = \infty \quad (j = 1, 2).$$

Return now to the differential equation

$$f^{(k)} + p_{k-1}(z) f^{(k-1)} + \cdots + p_0(z) f = 0, \tag{3}$$

where $p_j(z)$ $(j = 0, \cdots, k-1)$ are polynomials with $p_0(z) \not\equiv 0$. It is well-known that every solution f of (3) is an entire function of finite rational order; see, [10], [11], [5, pp. 199 – 209], [9, pp. 106 – 108], [12, pp. 65 – 67]. For equation (3), set

$$\lambda = 1 + \max_{0 \le j \le k-1} \frac{\deg p_j}{k - j}. \tag{4}$$

It is known [6, p. 127] that for any solution f of (3), we have

$$\rho(f) \le \lambda.$$

As we have seen in Theorem 1.3 and [7], it is clear that the study of the growth of g_k where $k > 2$, is more difficult than the case where $k = 2$. For that, we give in the following result some sufficient conditions to prove that g_k keeps the same order of growth of solutions of (3) for $k \ge 2$, and we obtain the following result.

Theorem 1.5. *Let $p_j(z)$ $(j=0,\cdots,k-1)$ be polynomials, and let $d_i(z)$ $(1\leq i\leq k)$ be entire functions that are not all vanishing identically such that* $\max\{\rho(d_i):1\leq i\leq k\}<\lambda$. *If $\{f_1,f_2,\cdots,f_k\}$ is any set of fundamental solutions of* (3), *then the combination of solutions g_k satisfies*

$$\rho(g_k)=1+\max_{0\leq j\leq k-1}\frac{\deg p_j}{k-j}.$$

Remark 1.1. The proof of Theorems 1.3-1.5 is quite different from that in the proof of Theorems 1.1-1.2 (see, [7]) . The main ingredient in the proof is Lemma 2.1. By the proof of Theorem 1.5, we can deduce that Theorem 1.2 holds without the additional condition $h\not\equiv 0$.

Corollary 1.6. *Let $A(z)$ be a nonconstant polynomial and let $d_i(z)$ $(1\leq i\leq k)$ be entire functions that are not all vanishing identically such that*

$$\max\{\rho(d_i):\ 1\leq i\leq k\}<\frac{\deg(A)+k}{k}.$$

If $\{f_1,f_2,\cdots,f_k\}$ is any set of fundamental solutions of

$$f^{(k)}+A(z)f=0, \tag{5}$$

then the combination of solutions g_k satisfies

$$\rho(g_k)=\frac{\deg(A)+k}{k}.$$

2. Preliminary lemmas

Lemma 2.1. [8] *(i) Let $f(z)$ be an entire function with $\rho_2(f)=\alpha>0$, and let $L(f)=a_kf^{(k)}+a_{k-1}f^{(k-1)}+\cdots+a_0f$, where $a_0,\ a_1,\cdots,\ a_k$ are entire functions which are not all equal zero and satisfy $b=\max\{\rho(a_j):\ j=0,\cdots,k\}<\alpha$. Then $\rho_2(L(f))=\alpha$.*
(ii) Let $f(z)$ be an entire function with $\rho(f)=\alpha\leq\infty$, and let $L(f)=a_kf^{(k)}+a_{k-1}f^{(k-1)}+\cdots+a_0f$, where $a_0,\ a_1,\cdots,\ a_k$ are entire functions which are not all equal zero and satisfy $b=\max\{\rho(a_j):\ j=0,\cdots,k\}<\alpha$. Then $\rho(L(f))=\alpha$.

Lemma 2.2. [3] *For any given equation of the form* (3), *there must exists a solution of* (3) *that satisfies $\rho(f)=\lambda$, where λ is the constant in* (4).

Lemma 2.3. [1] *Let $A_j(z)$ $(j=0,\cdots,k-1)$ be entire functions of finite order such that*

$$\max\{\rho(A_j):j=1,\cdots,k-1\}<\rho(A_0).$$

Then every solution $f\not\equiv 0$ of (1) satisfies $\rho(f)=\infty$ and $\rho_2(f)=\rho(A_0)$.

Lemma 2.4. [2] *Let $A_0(z)$ be transcendental entire function with $\rho(A_0)=0$, and let $A_1,\cdots,A_{k-1}$ be polynomials. Then every solution $f\not\equiv 0$ of* (1) *satisfies $\rho(f)=\infty$.*

By using similar proofs as in the proofs of Proposition 1.5 and Proposition 5.5 in [6] , we easily obtain the following lemma.

Lemma 2.5. *For all non-trivial solutions f of* (5). *If A is a polynomial with* $\deg A=n\geq 1$, *then we have*

$$\rho(f)=\frac{n+k}{k}.$$

Lemma 2.6. *Let f be any nontrivial solution of* (1). *Then the following identity holds*

$$\sum_{j=0}^{k}\left(A_j\sum_{i=0}^{j}C_j^i\left(\frac{d_1}{d_2}\right)^{(i)}f^{(j-i)}\right)=\sum_{j=1}^{k}\left(A_j\sum_{i=1}^{j}C_j^i\left(\frac{d_1}{d_2}\right)^{(i)}f^{(j-i)}\right),$$

where $A_k(z)\equiv 1$ and $C_j^i=\dfrac{j!}{i!\,(j-i)!}$.

Proof. We have

$$\begin{aligned}\sum_{j=0}^{k}\left(A_j\sum_{i=0}^{j}C_j^i\left(\frac{d_1}{d_2}\right)^{(i)}f^{(j-i)}\right)&=A_0\frac{d_1}{d_2}f+\sum_{j=1}^{k}\left(A_j\sum_{i=0}^{j}C_j^i\left(\frac{d_1}{d_2}\right)^{(i)}f^{(j-i)}\right)\\&=A_0\frac{d_1}{d_2}f+\sum_{j=1}^{k}\left(A_jC_j^0\left(\frac{d_1}{d_2}\right)f^{(j)}+A_j\sum_{i=1}^{j}C_j^i\left(\frac{d_1}{d_2}\right)^{(i)}f^{(j-i)}\right)\\&=A_0\frac{d_1}{d_2}f+\sum_{j=1}^{k}A_j\left(\frac{d_1}{d_2}\right)f^{(j)}+\sum_{j=1}^{k}\left(A_j\sum_{i=1}^{j}C_j^i\left(\frac{d_1}{d_2}\right)^{(i)}f^{(j-i)}\right)\\&=\frac{d_1}{d_2}\left(f^{(k)}+A_{k-1}f^{(k-1)}+\cdots+A_0f\right)+\sum_{j=1}^{k}\left(A_j\sum_{i=1}^{j}C_j^i\left(\frac{d_1}{d_2}\right)^{(i)}f^{(j-i)}\right)\\&=\sum_{j=1}^{k}\left(A_j\sum_{i=1}^{j}C_j^i\left(\frac{d_1}{d_2}\right)^{(i)}f^{(j-i)}\right).\end{aligned}$$

□

Lemma 2.7. *Let f be any nontrivial solution of* (1). *Then the following identity holds*

$$\sum_{j=0}^{k}\left(A_j\sum_{i=0}^{j}C_j^i\left(\frac{d_1}{d_2}\right)^{(i)}f^{(j-i)}\right)=\frac{\sum\limits_{i=0}^{k-1}D_if^{(i)}}{d_2^{2^k}},$$

where $D_i\ \ (i=0,\cdots,k-1)$ are entire functions depending on d_1,d_2 and $A_j\ \ (j=1,\cdots,k-1)$, $A_k(z)\equiv 1$.

Proof. It is clear that we can express the double sum

$$\sum_{j=0}^{k}\left(A_j\sum_{i=0}^{j}C_j^i\left(\frac{d_1}{d_2}\right)^{(i)}f^{(j-i)}\right)=\sum_{j=1}^{k}A_j\left(\sum_{i=1}^{j}C_j^i\left(\frac{d_1}{d_2}\right)^{(i)}f^{(j-i)}\right)$$

in the form of differential polynomial in f of order $k-1$. By mathematical induction we can prove that

$$\sum_{j=1}^{k}A_j\left(\sum_{i=1}^{j}C_j^i\left(\frac{d_1}{d_2}\right)^{(i)}f^{(j-i)}\right)=\sum_{i=0}^{k-1}\alpha_if^{(i)},\tag{6}$$

where

$$\alpha_i=\sum_{p=i+1}^{k}A_pC_p^{p-i}\left(\frac{d_1}{d_2}\right)^{(p-i)}.\tag{7}$$

Also, we have

$$\left(\frac{d_1}{d_2}\right)^{(j)} = \frac{\beta_j}{d_2^{2^j}}, \tag{8}$$

where β_j is entire function. Hence, we deduce from (6)-(8) that

$$\sum_{i=0}^{k-1}\alpha_i f^{(i)} = \frac{\sum\limits_{i=0}^{k-1} D_i f^{(i)}}{d_2^{2^k}},$$

where D_i $(i = 0, \cdots, k-1)$ are entire functions depending on d_1, d_2 and A_j $(j = 1, \cdots, k-1)$, $A_k(z) \equiv 1$. □

3. Proof of Theorem 1.3

Proof. In the case when $d_1(z) \equiv 0$ or $d_2(z) \equiv 0$, then the conclusions of Theorem 1.3 are trivial. Suppose that f_1 and f_2 are two nontrivial linearly independent solutions of (1) such that $d_i(z) \not\equiv 0$ $(i = 1, 2)$ and let

$$g_2 = d_1 f_1 + d_2 f_2. \tag{9}$$

Then, by Lemma 2.3 we have $\rho(f_j) = \infty$ $(j = 1, 2)$ and $\rho_2(f_j) = \rho(A_0)$ $(j = 1, 2)$. Suppose that $d_1 = c d_2$, where c is a complex number. Then, by (9) we obtain

$$g_2 = c d_2 f_1 + d_2 f_2 = (c f_1 + f_2) d_2.$$

Since $f = c f_1 + f_2$ is a solution of (1) and $\rho(d_2) < \rho(A_0)$, then we have

$$\rho(g_2) = \rho(c f_1 + f_2) = \infty$$

and

$$\rho_2(g_2) = \rho_2(c f_1 + f_2) = \rho(A_0).$$

Suppose now that $d_1 \not\equiv c d_2$ where c is a complex number. Dividing both sides of (9) by d_2, we obtain

$$F_2 = \frac{g_2}{d_2} = f_2 + \frac{d_1}{d_2} f_1. \tag{10}$$

Differentiating both sides of equation (10), k times for all integers $j = 1, \cdots, k$, we get

$$F_2^{(j)} = f_2^{(j)} + \sum_{i=0}^{j} C_j^i f_1^{(i)} \left(\frac{d_1}{d_2}\right)^{(j-i)}. \tag{11}$$

Equations (10) and (11) are equivalent to

$$\begin{cases} F_2 = f_2 + \frac{d_1}{d_2} f_1, \\ F_2' = f_2' + \left(\frac{d_1}{d_2}\right) f_1' + \left(\frac{d_1}{d_2}\right)' f_1, \\ F_2'' = f_2'' + \left(\frac{d_1}{d_2}\right) f_1'' + 2\left(\frac{d_1}{d_2}\right)' f_1' + \left(\frac{d_1}{d_2}\right)'' f_1, \\ \cdots \\ F_2^{(k-1)} = f_2^{(k-1)} + \sum\limits_{i=0}^{k-1} C_{k-1}^i \left(\frac{d_1}{d_2}\right)^{(k-1-i)} f_1^{(i)}, \\ F_2^{(k)} = f_2^{(k)} + \sum\limits_{i=0}^{k} C_k^i \left(\frac{d_1}{d_2}\right)^{(k-i)} f_1^{(i)} \end{cases}$$

which is also equivalent to

$$\begin{cases} A_0F_2 = A_0f_2 + A_0\frac{d_1}{d_2}f_1, \\ A_1F_2' = A_1f_2' + A_1\left(\left(\frac{d_1}{d_2}\right)f_1' + \left(\frac{d_1}{d_2}\right)'f_1\right), \\ A_2F_2'' = A_2f_2'' + A_2\left(\left(\frac{d_1}{d_2}\right)f_1'' + 2\left(\frac{d_1}{d_2}\right)'f_1' + \left(\frac{d_1}{d_2}\right)''f_1\right), \\ \cdots \\ A_{k-1}F_2^{(k-1)} = A_{k-1}f_2^{(k-1)} + A_{k-1}\sum\limits_{i=0}^{k-1}C_{k-1}^i\left(\frac{d_1}{d_2}\right)^{(k-1-i)}f_1^{(i)}, \\ F_2^{(k)} = f_2^{(k)} + \sum\limits_{i=0}^{k}C_k^i\left(\frac{d_1}{d_2}\right)^{(k-i)}f_1^{(i)}. \end{cases} \tag{12}$$

By (12) we can obtain

$$F_2^{(k)} + A_{k-1}(z)F_2^{(k-1)} + \cdots + A_0(z)F_2 = \left(f_2^{(k)} + A_{k-1}(z)f_2^{(k-1)} + \cdots + A_0(z)f_2\right)$$
$$+\sum_{j=0}^{k}\left(A_j\sum_{i=0}^{j}C_j^i\left(\frac{d_1}{d_2}\right)^{(i)}f_1^{(j-i)}\right) = \sum_{j=0}^{k}\left(A_j\sum_{i=0}^{j}C_j^i\left(\frac{d_1}{d_2}\right)^{(i)}f_1^{(j-i)}\right), \tag{13}$$

where $A_k(z) \equiv 1$. By using Lemma 2.6, we have

$$F_2^{(k)} + A_{k-1}(z)F_2^{(k-1)} + \cdots + A_0(z)F_2 = \sum_{j=1}^{k}A_j\left(\sum_{i=1}^{j}C_j^i\left(\frac{d_1}{d_2}\right)^{(i)}f_1^{(j-i)}\right). \tag{14}$$

By Lemma 2.7, we get

$$\sum_{j=1}^{k}A_j\left(\sum_{i=1}^{j}C_j^i\left(\frac{d_1}{d_2}\right)^{(i)}f_1^{(j-i)}\right) = k\frac{(d_1'd_2 - d_2'd_1)\,d_2^{\sum_{n=0}^{k-1}2^n-1}}{d_2^{2^k}}f_1^{(k-1)} + \frac{1}{d_2^{2^k}}\sum_{i=0}^{k-2}D_if_1^{(i)}, \tag{15}$$

where D_i $(i = 0, \cdots, k-2)$ are entire functions depending on d_1, d_2 and A_j $(j = 1, \cdots, k-1)$, $A_k(z) \equiv 1$. By using (14) and (15), we obtain

$$F_2^{(k)} + A_{k-1}(z)F_2^{(k-1)} + \cdots + A_0(z)F_2 = \frac{L_{k-1}(f_1)}{d_2^{2^k}},$$

where

$$L_{k-1}(f_1) = \sum_{i=0}^{k-1}D_if_1^{(i)}$$

is differential polynomial with entire coefficients D_i $(i = 0, \cdots, k-1)$ of order $\rho(D_i) < \rho(A_0)$ $(i = 0, \cdots, k-1)$ and $D_{k-1} = k\frac{(d_1'd_2 - d_2'd_1)d_2^{\sum_{n=0}^{k-1}2^n-1}}{d_2^{2^k}} \not\equiv 0$ because $d_1 \not\equiv cd_2$. By Lemma 2.1 (i), we have

$$\rho_2\left(F_2^{(k)} + A_{k-1}(z)F_2^{(k-1)} + \cdots + A_0(z)F_2\right) = \rho_2(L_{k-1}(f_1)) = \rho_2(f_1).$$

Since

$$\rho_2(f_1) = \rho_2\left(F_2^{(k)} + A_{k-1}(z)F_2^{(k-1)} + \cdots + A_0(z)F_2\right) \le \rho_2(F_2) = \rho_2(g_2) \le \rho_2(f_1),$$

then

$$\rho_2(g_2) = \rho_2(f_1).$$

□

4. Proof of Theorem 1.4

Proof. By using a similar reasoning as in the proof of Theorem 1.3, Lemma 2.4 and Lemma 2.1 (ii) we obtain Theorem 1.4. □

5. Proof of Theorem 1.5

Proof. Without loss of generality, by using Lemma 2.2, we suppose that

$$\max\{\rho(f_j),\ j=1,\cdots,k\}=\rho(f_1)=\lambda=1+\max_{0\leq j\leq k-1}\frac{\deg p_j}{k-j}$$

and there exist at least two integers p and q such that $d_p\not\equiv cd_q$ where c is a complex number and $1\leq p\leq q\leq k$. By the same proof as Theorem 1.3 we obtain

$$F_2^{(k)}+p_{k-1}(z)F_2^{(k-1)}+\cdots+p_0(z)F_2=\sum_{j=1}^{k}\left(p_j\sum_{i=1}^{j}C_j^i\left(\frac{d_1}{d_2}\right)^{(i)}f_1^{(j-i)}\right) \tag{16}$$

and by Lemma 2.7, we get

$$\sum_{j=1}^{k}\left(p_j\sum_{i=1}^{j}C_j^i\left(\frac{d_1}{d_2}\right)^{(i)}f_1^{(j-i)}\right)=k\frac{(d_1'd_2-d_2'd_1)\,d_2^{\sum_{n=0}^{k-1}2^n-1}}{d_2^{2^k}}f_1^{(k-1)}+\frac{1}{d_2^{2^k}}\sum_{i=0}^{k-2}D_if_1^{(i)}, \tag{17}$$

where $p_k(z)\equiv 1$ and D_i $(i=0,\cdots,k-2)$ are entire functions. By using (16) and (17), we have

$$F_2^{(k)}+A_{k-1}(z)F_2^{(k-1)}+\cdots+A_0(z)F_2=\frac{L_{k-1}(f_1)}{d_2^{2^k}},$$

where

$$L_{k-1}(f_1)=\sum_{i=0}^{k-1}D_if_1^{(i)}$$

is differential polynomial with entire coefficients D_i $(i=0,\cdots,k-1)$ of order $\rho(D_i)<\lambda$ $(i=0,\cdots,k-1)$ and there exists $0\leq i\leq k-1$ such that $D_i\not\equiv 0$. By Lemma 2.1 (ii), we have

$$\rho\left(F_2^{(k)}+p_{k-1}(z)F_2^{(k-1)}+\cdots+p_0(z)F_2\right)=\rho(L_{k-1}(f_1))=\rho(f_1).$$

Since

$$\rho(f_1)=\rho\left(F_2^{(k)}+p_{k-1}(z)F_2^{(k-1)}+\cdots+p_0(z)F_2\right)\leq\rho(F_2)=\rho(g_2)\leq\rho(f_1),$$

then

$$\rho(g_2)=\rho(f_1).$$

Now, we suppose that

$$\rho(g_n)=\rho(f_1)$$

is true for all $n=1,\cdots,k-1$ and we show that

$$\rho(g_k)=\rho(f_1).$$

We have

$$g_k=d_1f_1+d_2f_2+\cdots+d_kf_k=g_{k-1}+d_kf_k. \tag{18}$$

Suppose that $d_k \not\equiv 0$, and dividing both sides of (18) by d_k, we get

$$F_k = \frac{g_k}{d_k} = \frac{g_{k-1}}{d_k} + f_k.$$

By the same reasoning as before, we obtain

$$\begin{cases} p_0 F_k = p_0 f_k + p_0 \frac{1}{d_k} g_{k-1}, \\ p_1 F_k' = p_1 f_k' + p_1 \left(\left(\frac{1}{d_k}\right) g_{k-1}' + \left(\frac{1}{d_k}\right)' g_{k-1} \right), \\ p_2 F_k'' = p_2 f_k'' + p_2 \left(\left(\frac{1}{d_k}\right) g_{k-1}'' + 2\left(\frac{1}{d_k}\right)' g_{k-1}' + \left(\frac{1}{d_k}\right)'' g_{k-1} \right), \\ \cdots \\ p_{k-1} F_k^{(k-1)} = p_{k-1} f_k^{(k-1)} + p_{k-1} \sum\limits_{i=0}^{k-1} C_{k-1}^i \left(\frac{1}{d_k}\right)^{(k-1-i)} g_{k-1}^{(i)}, \\ F_k^{(k)} = f_k^{(k)} + \sum\limits_{i=0}^{k} C_k^i \left(\frac{1}{d_k}\right)^{(k-i)} g_{k-1}^{(i)}. \end{cases} \tag{19}$$

By (19) we can deduce

$$F_k^{(k)} + p_{k-1}(z) F_k^{(k-1)} + \cdots + p_0(z) F_k = \left(f_k^{(k)} + p_{k-1}(z) f_k^{(k-1)} + \cdots + p_0(z) f_k \right)$$
$$+ \sum_{j=0}^{k} \left(p_j \sum_{i=0}^{j} C_j^i \left(\frac{1}{d_k}\right)^{(i)} g_{k-1}^{(j-i)} \right) = \sum_{j=0}^{k} \left(p_j \sum_{i=0}^{j} C_j^i \left(\frac{1}{d_k}\right)^{(i)} g_{k-1}^{(j-i)} \right). \tag{20}$$

By Lemma 2.6, we have

$$\sum_{j=0}^{k} \left(p_j \sum_{i=0}^{j} C_j^i \left(\frac{1}{d_k}\right)^{(i)} g_{k-1}^{(j-i)} \right) = \sum_{j=1}^{k} \left(p_j \sum_{i=1}^{j} C_j^i \left(\frac{1}{d_k}\right)^{(i)} g_{k-1}^{(j-i)} \right)$$
$$= -k \frac{d_k' d_k^{\sum_{n=0}^{k-1} 2^n - 1}}{d_k^{2^k}} g_{k-1}^{(k-1)} + \frac{1}{d_k^{2^k}} \sum_{i=0}^{k-2} B_i g_{k-1}^{(i)}, \tag{21}$$

where $p_k(z) \equiv 1$ and B_i $(i = 0, \cdots, k-1)$ are entire functions. By using (20) and (21), we obtain

$$F_k^{(k)} + A_{k-1}(z) F_k^{(k-1)} + \cdots + A_0(z) F_k = \frac{M_{k-1}(g_{k-1})}{d_k^{2^k}},$$

where

$$M_{k-1}(g_{k-1}) = \sum_{i=0}^{k-1} B_i g_{k-1}^{(i)}$$

is differential polynomial with entire coefficients B_i $(i = 0, \cdots, k-1)$ of order $\rho(B_i) < \lambda$ $(i = 0, \cdots, k-1)$. By Lemma 2.1 (ii), we have

$$\rho\left(F_k^{(k)} + p_{k-1}(z) F_k^{(k-1)} + \cdots + p_0(z) F_k \right) = \rho(M_{k-1}(g_{k-1})) = \rho(f_1).$$

Since

$$\rho(f_1) \le \rho\left(F_k^{(k)} + p_{k-1}(z) F_k^{(k-1)} + \cdots + p_0(z) F_k \right) \le \rho(F_k) = \rho(g_{k-1}) \le \rho(f_1),$$

then

$$\rho(F_k) = \rho(g_{k-1}) = \rho(f_1),$$

which implies that

$$\rho(g_k) = \rho(g_{k-1}) = \rho(f_1) = \lambda.$$

This completes the proof of Theorem 1.5. □

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