Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 41(1), 2014, Pages 59–68
ISSN: 1223-6934

Discrete Emden-Fowler problems driven by nonhomogeneous differential operators

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Abstract. In this paper, we prove the existence of positive homoclinic solutions for the $p(\cdot)$-Laplacian difference equation with periodic coefficients on the set of integers. The proof of the main result is obtained by using critical point theory combined with adequate variational techniques, which are mainly based on the mountain pass theorem.

2010 Mathematics Subject Classification. 39A14, 44A55, 34C375.
Key words and phrases. difference equations, discrete $p(\cdot)$-Laplacian, homoclinic solution, mountain pass lemma.

1. Introduction and main result

This paper is concerned with the study of the difference non-homogeneous equation of type

$$\begin{cases} -\Delta\phi_{p(k-1)}(\Delta u(k-1)) + a(k)\phi_{p(k)}(u(k)) = f(k, u(k)) & \text{for all } k \in \mathbb{Z} \\ u(k) \to 0 & \text{as } |k| \to \infty. \end{cases} \tag{1}$$

Here $p(\cdot) : \mathbb{Z} \to (1, \infty)$, $a(\cdot) : \mathbb{Z} \to \mathbb{R}$ is two T-periodic functions, where $T > 0$ is a given natural number, $\phi_{p(k)}(t) = |t|^{p(k)-2}t$ for all $t \in \mathbb{R}$ and for each $k \in \mathbb{Z}$, while $f(k, t) : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ is a continuous function in $t \in \mathbb{R}$ and T-periodic in k. We have denoted by Δ the difference operator, which is defined by

$$\Delta u(k-1) = u(k) - u(k-1) \text{ for each } k \in \mathbb{Z}.$$

Moreover,

$$\Delta\phi_{p(k-1)}(\Delta u(k-1)) = |\Delta u(k)|^{p(k)-2}\Delta u(k) - |\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1) \tag{2}$$

for each $k \in \mathbb{Z}$.

In the present paper, our goal is to establish the existence of positive homoclinic solutions for problem (1).

The presence of the nonconstant potential $p(\cdot)$ is an important feature of this paper. The study of difference equations involving non-homogeneous difference operators of type (2) was initiated by M. Mihăilescu, V. Rădulescu and S. Tersian in [11], where some eigenvalue problems were investigated.

The study of discrete boundary value problems has captured special attention in the last decade. In this context we point out the results obtained in the papers of R. P. Agarwal, K. Perera and D. O'Regan [1], A. Cabada, A. Iannizzotto and S. Tersian [3], H. Fang and D. Zhao [5], M. Ma and Z. Guo [10], A. Kristály, M. Mihăilescu, V. Rădulescu and S. Tersian [9]. The studies regarding such type of problems can be placed at the interface of certain mathematical fields such as nonlinear partial differential equations and numerical analysis.

Received February 28, 2014.

Throughout this paper we will use the notation

$$p^+ := \sup_{k\in\mathbb{Z}} p(k),\ \ p^- := \inf_{k\in\mathbb{Z}} p(k).$$

We assume that

$$1 < p^- \leqslant p(\cdot) < p^+ < \infty. \tag{3}$$

We also assume that the T-periodic function $a(\cdot) : \mathbb{Z} \to \mathbb{R}$ and the continuous function $f = f(k,t) : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ which is assumed to be T-periodic in k satisfy the following hypotheses:

(A_1) $a(k) \geqslant a_0 := \min\{a(0), \dots, a(T-1)\} > 0$ for all $k \in \mathbb{Z}$;

(A_2) $a_0 < p^+$;

(F_1) $\lim\limits_{|t|\to 0} \dfrac{f(k,t)}{|t|^{p^+-1}} = 0$ uniformly for all $k \in \mathbb{Z}$;

(F_2) there exist $\alpha > p^+$ and $r > 0$ such that $0 < \alpha F(k,t) \leqslant f(k,t)t$ for all $k \in \mathbb{Z}$, $t \geqslant r > 0$, where $F : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ is defined by $F(k,t) = \int_0^t f(k,s)ds$ for all $k \in \mathbb{Z}, t \in \mathbb{R}$;

(F_3) $f(k,t) \geqslant 0$ for any $t < 0$ and all $k \in \mathbb{Z}$.

The main result of this paper is the following.

Theorem 1.1. *Assume hypotheses (3), $(A_1) - (A_2)$ and $(F_1) - (F_3)$ are satisfied. Then problem (1) admits at least a positive homoclinic solution.*

The rest of the paper is organized as follows: in Section 2 we collect some preliminary results and in Section 3 we prove Theorem 1.1, employing the mountain pass theorem of Ambrosetti & Rabinowitz [2].

2. Preliminaries

In this section we define the functional spaces and some of their useful properties which will be used later.

We introduce for each $p(\cdot) : \mathbb{Z} \to (1,\infty)$ the space

$$l^{p(\cdot)} := \left\{ u : \mathbb{Z} \to \mathbb{R};\ \rho_{p(\cdot)}(u) := \sum_{k\in\mathbb{Z}} |u(k)|^{p(k)} < \infty \right\}.$$

On $l^{p(\cdot)}$ we introduce the Luxemburg norm

$$|u|_{p(\cdot)} := \inf \left\{ \lambda > 0 : \sum_{k\in\mathbb{Z}} \left| \frac{u(k)}{\lambda} \right|^{p(k)} \leqslant 1 \right\}.$$

We also consider the space

$$l^\infty = \left\{ u : \mathbb{Z} \to \mathbb{R};\ |u|_\infty := \sup_{k\in\mathbb{Z}} |u(k)| < \infty \right\}.$$

We recall some useful properties of the space $l^{p(\cdot)}$. Firstly, by clasical results of functional analysis we know that, for all $1 < p^- \leqslant p^+ < \infty$, $(l^{p(\cdot)}, |\cdot|_{p(\cdot)})$ is a reflexive Banach space whose dual is $(l^{p^*(\cdot)}, |\cdot|_*)$ with $\frac{1}{p(x)} + \frac{1}{p^*(x)} = 1$.

Proposition 2.1. [6, Proposition 2.3] *If $u \in l^{p(\cdot)}$, $(u_n) \subset l^{p(\cdot)}$ and $p^+ < +\infty$, then the following properties hold:*

$$|u|_{p(\cdot)} > 1 \quad \textit{implies } |u|_{p(\cdot)}^{p^-} \leqslant \rho_{p(\cdot)}(u) \leqslant |u|_{p(\cdot)}^{p^+}; \tag{4}$$

$$|u|_{p(\cdot)} < 1 \qquad \textit{implies } |u|_{p(\cdot)}^{p^+} \leqslant \rho_{p(\cdot)}(u) \leqslant |u|_{p(\cdot)}^{p^-}; \tag{5}$$

$$|u_n|_{p(\cdot)} \to 0 \quad \textit{if and only if } \rho_{p(\cdot)}(u_n) \to 0 \textit{ as } n \to \infty; \tag{6}$$

We denote by $l_0^{p(\cdot)}$ the set of compact support functions. Consider $u \in l^{p(\cdot)}$ has compact support, hence there exist $a, b \in \mathbb{Z}$, $a < b$ such that $u(k) = 0$ if $k \in \mathbb{Z} \setminus [a, b]$ and $u(k) \neq 0$ if $k \in \{a+1, b-1\}$.

Remark 2.1. The space $l_0^{p(\cdot)}$ is dense in $l^{p(\cdot)}$.

Indeed, for each $u \in l^{p(\cdot)}$ we can define $u_n \in l_0^{p(\cdot)}$ by $u_n(j) = 0$, if $|j| \geqslant n+1$ and $u_n(j) = u(j)$, if $|j| \neq n$ and we have

$$\sum_{j\in\mathbb{Z}} |u(j) - u_n(j)| \to 0 \text{ as } n \to \infty,$$

or, by relation (6), $|u - u_n|_{p(\cdot)} \to 0$ as $n \to \infty$.

On the other hand, in order to facilitate further computations, it is useful to introduce other norm on $l^{p(\cdot)}$, namely

$$\|u\|_{p(\cdot)} := \inf \left\{ \lambda > 0 : \sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)} \left| \frac{u(k)}{\lambda} \right|^{p(k)} \leqslant 1 \right\}.$$

We point out that $\|\cdot\|_{p(\cdot)}$ is an equivalent norm with $|\cdot|_{p(\cdot)}$. Moreover, we have the following properties of the above norm, which are similarly to the properties of the norm $|\cdot|_{p(\cdot)}$.

Proposition 2.2. *Let $u \in l^{p(\cdot)}$, $(u_n) \subset l^{p(\cdot)}$ and $p^+ < +\infty$. Then the following properties hold:*

$$\|u\|_{p(\cdot)} > 1 \qquad \textit{implies } \|u\|_{p(\cdot)}^{p^-} \leqslant \sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)} |u(k)|^{p(k)} \leqslant \|u\|_{p(\cdot)}^{p^+}; \tag{7}$$

$$\|u\|_{p(\cdot)} < 1 \qquad \textit{implies } \|u\|_{p(\cdot)}^{p^+} \leqslant \sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)} |u(k)|^{p(k)} \leqslant \|u\|_{p(\cdot)}^{p^-}; \tag{8}$$

$$\|u_n\|_{p(\cdot)} \to 0 \quad \textit{if and only if } \sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)} |u_n(k)|^{p(k)} \to 0 \textit{ as } n \to \infty. \tag{9}$$

Proof. (7) Let $\|u\|_{p(\cdot)} > 1$. Then

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)} |u(k)|^{p(k)} &= \sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)} \left| \frac{u(k)}{\|u\|_{p(\cdot)}} \times \|u\|_{p(\cdot)} \right|^{p(k)} \\ &\geqslant \|u\|_{p(\cdot)}^{p^-} \sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)} \left| \frac{u(k)}{\|u\|_{p(\cdot)}} \right|^{p(k)} = \|u\|_{p(\cdot)}^{p^-}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)} |u(k)|^{p(k)} &= \sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)} \left| \frac{u(k)}{\|u\|_{p(\cdot)}} \times \|u\|_{p(\cdot)} \right|^{p(k)} \\ &\leqslant \|u\|_{p(\cdot)}^{p^+} \sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)} \left| \frac{u(k)}{\|u\|_{p(\cdot)}} \right|^{p(k)} = \|u\|_{p(\cdot)}^{p^+}. \end{aligned}$$

Thus,

$$\|u\|_{p(\cdot)}^{p^-} \leqslant \sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)}|u(k)|^{p(k)} \leqslant \|u\|_{p(\cdot)}^{p^+};$$

(8) The proof is similar to that for (7).

(9)Case 1. $\|u_n\|_{p(\cdot)} \to 0$ as $n \to \infty$, then

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)}|u_n(k)|^{p(k)} &= \sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)}\left|\frac{u_n(k)}{\|u_n\|_{p(\cdot)}} \times \|u_n\|_{p(\cdot)}\right|^{p(k)} \\ &\leqslant \|u_n\|_{p(\cdot)}^{\alpha} \sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)}\left|\frac{u_n(k)}{\|u_n\|_{p(\cdot)}}\right|^{p(k)} = \|u_n\|_{p(\cdot)}^{\alpha},\end{aligned}$$

where $\alpha = \begin{cases} p^+ \text{ if } \|u_n\|_{p(\cdot)} \geqslant 1 \\ p^- \text{ if } \|u_n\|_{p(\cdot)} < 1.\end{cases}$ So, $\sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)}|u_n(k)|^{p(k)} \to 0$ as $n \to \infty$.

Case 2. $\sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)}|u_n(k)|^{p(k)} \to 0$ as $n \to \infty$, then

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)}|u_n(k)|^{p(k)} &= \sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)}\left|\frac{u_n(k)}{\|u_n\|_{p(\cdot)}} \times \|u_n\|_{p(\cdot)}\right|^{p(k)} \\ &\geqslant \|u_n\|_{p(\cdot)}^{\alpha} \sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)}\left|\frac{u_n(k)}{\|u_n\|_{p(\cdot)}}\right|^{p(k)} = \|u_n\|_{p(\cdot)}^{\alpha},\end{aligned}$$

where $\alpha = \begin{cases} p^- \text{ if } \|u_n\|_{p(\cdot)} \geqslant 1 \\ p^+ \text{ if } \|u_n\|_{p(\cdot)} < 1.\end{cases}$ So, $\|u_n\|_{p(\cdot)} \to 0$ as $n \to \infty$. $\square$

Next, if X is a real Banach space and X^* is its dual, we recall that a functional $J \in C^1(X,\mathbb{R})$, is said to satisfies the *Palais-Smale condition at the level* c, where c is a given real number, ($(PS)_c$ for short) if every sequence (u_n) in X satisfying $J(u_n) \to c$ and $J'(u_n) \to 0$ in X^* contains a convergent subsequence. Such condition is an essential hypothesis in the following mountain pass theorem, due to Ambrosetti & Rabinowitz [2].

Theorem 2.3. *Let X be a real Banach space and $J \in C^1(X,\mathbb{R})$ satisfies the following geometric conditions:*

(H_1) *there exist two numbers $R > 0$ and $c_0 > 0$ such that $J(u) \geqslant c_0$ for all $u \in X$ with $\|u\| = R$;*

(H_2) *$J(0) < c_0$ and $J(e) < 0$ for some $e \in X$ with $\|e\| > R$.*

With an additional compactness condition of Palais-Smale type it then follows that the functional J has a critical point $u_0 \in X\backslash\{0,e\}$ with critical value $c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t))$, $c \geqslant c_0$, where $\Gamma := \{\gamma \in C([0,1],X); \gamma(0) = 0, \gamma(1) = e\}$.

3. Existence of positive homoclinic solutions

In this section, we are interested in finding homoclinic solutions for problem of type (1).

Definition 3.1. We say that a function $u \in l^{p(\cdot)}$ is a *weak homoclinic solution* for the problem (1) if

$$\sum_{k\in\mathbb{Z}} \phi_{p(k-1)}(\Delta u(k-1))\Delta v(k-1) + \sum_{k\in\mathbb{Z}} a(k)\phi_{p(k)}(u(k))v(k) - \sum_{k\in\mathbb{Z}} f(k,u(k))v(k) = 0,$$

for all $v \in l^{p(\cdot)}$ and $\lim_{|k|\to\infty} u(k) = 0$.

The basic idea in proving Theorem 1.1 is to consider the associate energetic functional of problem (1) and to show that it possesses a nontrivial critical point by using Theorem 2.3.

Note that $f : \mathbb{Z}\times\mathbb{R} \to \mathbb{R}$ is continuous as $f(k, 0) = 0$ for all $k \in \mathbb{Z}$ (by (F_1)). Next, we introduce an energetic functional corresponding to problem (1), $J : l^{p(\cdot)} \to \mathbb{R}$ defined by

$$J(u) := \sum_{k\in\mathbb{Z}} \frac{1}{p(k-1)}|\Delta u(k-1)|^{p(k-1)} + \sum_{k\in\mathbb{Z}} \frac{1}{p(k)} a(k)|u(k)|^{p(k)} - \sum_{k\in\mathbb{Z}} F(k, u(k)).$$

Standard arguments assure that J is well-defined on the space $l^{p(\cdot)}$ and is of class $C^1(l^{p(\cdot)},\mathbb{R})$, with the derivative given by

$$\begin{aligned}\langle J'(u), v\rangle &= \sum_{k\in\mathbb{Z}} \phi_{p(k-1)}(\Delta u(k-1))\Delta v(k-1) + \sum_{k\in\mathbb{Z}} a(k)\phi_{p(k)}(u(k))v(k) \\ &\quad - \sum_{k\in\mathbb{Z}} f(k, u(k))v(k), \text{ for all } u, v \in l^{p(\cdot)}. \end{aligned} \tag{10}$$

Remark 3.1. [6, Remark 2.1] If $u \in l^{p(\cdot)}$, then $\lim_{|k|\to+\infty} u(k) = 0$.

Proposition 3.1. *Suppose that the functions $a : \mathbb{Z} \to \mathbb{R}$ and $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ satisfy hypotheses of Theorem 1.1 and $u \in l^{p(\cdot)}$ is a critical point of J. Then u is a homoclinic solution of (1) and $u(k) \geqslant 0$ for all $k \in \mathbb{Z}$.*

Proof. We know that the critical points of J correspond to the weak homoclinic solutions of problem (1) (see (10)). Fix $h \in \mathbb{Z}$ and define $e_h \in l^{p(\cdot)}$ by setting $e_h(k) = \delta_{h,k}$ (where $\delta_{h,k} = 1$ if $h = k$ and $\delta_{h,k} = 0$ if $h \neq k$) for all $k \in \mathbb{Z}$. Taking $v = e_h$ in (10) we obtain

$$-\Delta\phi_{p(h-1)}(\Delta u(h-1)) + a(h)\phi_{p(h)}(u(h)) = f(h, u(h)).$$

Moreover, $u(h) \to 0$ as $|h| \to +\infty$ (see Remark 3.1). So, u is in fact a solution of problem (1).

Now, arguing by contradiction, suppose that $u(k_0) < 0$ for some $k_0 \in \mathbb{Z}$ and let k_1 be such that $u(k_1) = \min\{u(k), k \in \mathbb{Z}\} < 0$. In consequence $\Delta\phi_{p(k_1-1)}(\Delta u(k_1-1)) \geqslant 0$, which by equation (1) implies that

$$f(k_1, u(k_1)) = -\Delta\phi_{p(k_1-1)}(\Delta u(k_1 - 1)) + a(k_1)\phi_{p(k_1)}(u(k_1)) < 0,$$

a contradiction with (F_3). So, $u(k) \geqslant 0$ for all $k \in \mathbb{Z}$. □

Proposition 3.2. *If the hypotheses of Theorem 1.1 are satisfied and $u \in l^{p(\cdot)}$ is a homoclinic solution of (1) such that $u(k) \geqslant 0$ for all $k \in \mathbb{Z}$ and $u \neq 0$, then $u(k) > 0$ for all $k \in \mathbb{Z}$.*

Proof. Arguing by contradiction, assume that $u(k_2) = 0$ for some $k_2 \in \mathbb{Z}$. By (1) we have

$$\phi_{p(k_2)}(\Delta u(k_2)) = \phi_{p(k_2-1)}(\Delta u(k_2 - 1))$$

(recall that $f(k, 0) = 0$). Note that if $u(k_2 + 1) = 0$ or $u(k_2 - 1) = 0$, the solution is identically zero by a recursion, which is a contradiction with $u \neq 0$. So, $u(k) > 0$ for all $k \in \mathbb{Z}$. □

Proposition 3.3. *If (3), $(A_1) - (A_2)$ and $(F_1) - (F_3)$ are satisfied, then J satisfies $(PS)_c$.*

Proof. We follow [12]. Let (u_n) be a sequence in $l^{p(\cdot)}$ such that

$$J(u_n) \to c > 0 \text{ and } J'(u_n) \to 0 \text{ in } l^{p^*(\cdot)}, \text{ as } n \to \infty. \tag{11}$$

The existence of such a sequence is given in the proof of Theorem 1.1.

Firstly, we prove that (u_n) is bounded in $l^{p(\cdot)}$. Assume that $\|u_n\|_{p(\cdot)} > 1$ for each n and by condition (F_2) and relations (3) and (7) we deduce that

$$\begin{aligned}
\alpha J(u_n) - \langle J'(u_n), u_n\rangle &= \sum_{k\in\mathbb{Z}} \left(\frac{\alpha}{p(k-1)} - 1\right) |\Delta u_n(k-1)|^{p(k-1)} \\
&\quad + \sum_{k\in\mathbb{Z}} \left(\frac{\alpha}{p(k)} - 1\right) a(k)|u_n(k)|^{p(k)} \\
&\quad - \sum_{k\in\mathbb{Z}} [\alpha F(k, u_n(k)) - f(k, u_n(k))u_n(k)] \\
&\geqslant \sum_{k\in\mathbb{Z}} \left[\frac{\alpha}{p(k)} a(k)|u_n(k)|^{p(k)} - \frac{1}{p(k)} a(k)|u_n(k)|^{p(k)} p(k)\right] \\
&\quad \text{(see } (F_2)) \\
&\geqslant \alpha \|u_n\|_{p(\cdot)}^{p^-} - p^+ \|u_n\|_{p(\cdot)}^{p^-} \text{ (see (7) and (3))} \\
&= (\alpha - p^+)\|u_n\|_{p(\cdot)}^{p^-},
\end{aligned}$$

for all n. The above estimates and condition (11) implies that (u_n) is bounded in $l^{p(\cdot)}$. This information combined with the fact that $l^{p(\cdot)}$ is a reflexive Banach space guarantees that, up to a subsequence, (u_n) is weakly convergent in $l^{p(\cdot)}$.

So, for each $n \in \mathbb{N}$ the sequence $(|u_n(k)|)_{k\in\mathbb{Z}} \subset l^{p(\cdot)}$ is bounded and $|u_n(k)| \to 0$ as $|k| \to \infty$. Suppose that $(|u_n(k)|)_{k\in\mathbb{Z}}$ achieves its maximum in $k_n \in \mathbb{Z}$. Hence, there exists $j_n \in \mathbb{Z}$ such that

$$j_n T \leqslant k_n < (j_n + 1)T.$$

Define $w_n(k) := u_n(k - j_n T)$. Then $(|w_n(k)|)_{k\in\mathbb{Z}}$ attains its maximum in $i_n := k_n - j_n T \in [0, T]$. The T-periodicity of $p(\cdot)$ and $a(\cdot)$ implies

$$\sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)} |u_n(k)|^{p(k)} = \sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)} |w_n(k)|^{p(k)} \text{ and } J(u_n) = J(w_n). \tag{12}$$

Since $(u_n)_{n\in\mathbb{N}}$ is bounded in $l^{p(\cdot)}$ the relations (7), (8) and (12) yield that $(w_n)_{n\in\mathbb{N}}$ is bounded in $l^{p(\cdot)}$, too. Then, passing if necessary to a subsequence, there exists $w \in l^{p(\cdot)}$ such that w_n converges weakly to w in $l^{p(\cdot)}$ as $n \to \infty$.

Now, we verify that $w_n(k) \to w(k)$ as $n \to \infty$ for all $k \in \mathbb{Z}$. Indeed, defining the test function $v_m \in l^{p(\cdot)}$ by $v_m(j) = \begin{cases} 1, & \text{if } j = m \\ 0, & \text{if } j \neq m \end{cases}$ and taking into account the weak convergence of w_n to w in $l^{p(\cdot)}$ we find

$$\lim_{n\to\infty} w_n(k) = \lim_{n\to\infty} \langle w_n, v_k\rangle = \langle w, v_k\rangle = w(k), \text{ for all } k \in \mathbb{Z}.$$

We point out that for each $v \in l^{p(\cdot)}$ we have

$$|\langle J'(w_n), v\rangle| = |\langle J'(u_n), v(\cdot + j_n T)\rangle| \leqslant \|J'(u_n)\|_* \|v\|_{p(\cdot)}.$$

So, by above relation and relations (11) and (12) we obtain

$$J(w_n) \to c > 0 \text{ and } J'(w_n) \to 0 \text{ in } l^{p^*(\cdot)}, \text{ as } n \to \infty. \tag{13}$$

Hence, for each $v \in l^{p(\cdot)}$, we have

$$\sum_{k\in\mathbb{Z}} \big[\phi_{p(k-1)}(\Delta w_n(k-1))\Delta v(k-1) + a(k)\phi_{p(k)}(w_n(k))v(k) - f(k, w_n(k))v(k)\big] \to 0, \text{ as } n\to\infty. \tag{14}$$

By Remark 2.1 we know that the space $l_0^{p(\cdot)}$ is dense in $l^{p(\cdot)}$. So, for each $v \in l_0^{p(\cdot)}$ in (14) taking into account the finite sums and the continuity of $f(k,\cdot)$ we obtain by passing to the limit as $n\to\infty$ that

$$\sum_{k\in\mathbb{Z}} \big[\phi_{p(k-1)}(\Delta w(k-1))\Delta v(k-1) + a(k)\phi_{p(k)}(w(k))v(k) - f(k, w(k))v(k)\big] \to 0.$$

Therefore, w is a critical point of J and consequently a solution of (1).

Next, we show that w is a nontrivial solution of problem (1). Arguing by contradiction, assume that $w = 0$. Then we have

$$|u_n|_\infty = |w_n|_\infty = \max\{|w_n(k)|; k\in\mathbb{Z}\} \to 0, \text{ as } n\to\infty.$$

Fix $\varepsilon > 0$. By (F_1), there exists $\delta \in (0,1)$ such that

$$\begin{cases} |f(k,t)t| \leqslant \varepsilon |t|^{p^+}, \\ |F(k,t)| \leqslant \frac{\varepsilon}{p^+}|t|^{p^+}. \end{cases} \tag{15}$$

for all $k\in\{0,\dots,T-1\}$ and all $|t|<\delta$. The above inequalities show that for every $k\in\{0,\dots,T-1\}$ there exists M_k such that for $n>M_k$ we have $|w_n(k)|<\delta$.

Since $i_n\in\{0,\dots,T-1\}$ it follows that, for $n > M := \max\{M_n;\ n\in\{0,\dots,T-1\}\}$ and every $k\in\mathbb{Z}$, we have

$$|w_n(k)| \leqslant |w_n(i_n)| < \delta < 1.$$

That fact and relation (15) imply

$$\begin{cases} |f(k,w_n(k))w_n(k)| \leqslant \varepsilon |w_n(k)|^{p^+} \leqslant \varepsilon |w_n(k)|^{p(k)}, \\ |F(k,w_n(k))| \leqslant \frac{\varepsilon}{p^+}|w_n(k)|^{p^+} \leqslant \frac{\varepsilon}{p(k)}|w_n(k)|^{p(k)}. \end{cases} \tag{16}$$

Finally, it follows that for each $n > M$ and every $k\in\mathbb{Z}$ we obtain the following estimates

$$\begin{aligned} 0 &< p^-J(w_n) = p^-\sum_{k\in\mathbb{Z}} \frac{1}{p(k-1)}|\Delta w_n(k-1)|^{p(k-1)} \\ &\quad + p^-\sum_{k\in\mathbb{Z}}\frac{1}{p(k)}a(k)|w_n(k)|^{p(k)} - p^-\sum_{k\in\mathbb{Z}}F(k,w_n(k)) \\ &\leqslant \sum_{k\in\mathbb{Z}}|\Delta w_n(k-1)|^{p(k-1)} + \sum_{k\in\mathbb{Z}}a(k)|w_n(k)|^{p(k)} - \sum_{k\in\mathbb{Z}}f(k,w_n(k))w_n(k) \\ &\quad - \sum_{k\in\mathbb{Z}}\big[p^-F(k,w_n(k)) - f(k,w_n(k))w_n(k)\big] \text{ (see (3))} \\ &\leqslant \langle J'(w_n), w_n\rangle + p^-\sum_{k\in\mathbb{Z}}|F(k,w_n(k))| + \sum_{k\in\mathbb{Z}}|f(k,w_n(k))w_n(k)| \end{aligned}$$

$$
\begin{aligned}
&\leqslant \langle J'(w_n), w_n\rangle + p^-\varepsilon \sum_{k\in\mathbb{Z}} \frac{|w_n(k)|^{p(k)}}{p(k)} + \varepsilon \sum_{k\in\mathbb{Z}} |w_n(k)|^{p(k)} \text{ (see (16))}\\
&\leqslant \langle J'(w_n), w_n\rangle + \frac{p^-}{a_0}\varepsilon \sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)}|w_n(k)|^{p(k)} + \frac{p^+}{a_0}\varepsilon \sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)}|w_n(k)|^{p(k)}\\
&\leqslant \|J'(w_n)\|_* \|w_n\|_{p(\cdot)} + \varepsilon\frac{p^- + p^+}{a_0}\|w_n\|_{p(\cdot)}^{p^-} \text{ (see (8))}.
\end{aligned}
$$

We know that $\|w_n\|_{p(\cdot)}$ is bounded and $\varepsilon > 0$ is arbitrary. Taking into account this informations and relations (12) and (13) we find by the above estimates a contradiction with $J(w_n) \to c > 0$ as $n \to \infty$. So, we have that w is a nontrivial solution of problem (1).

Finally, taking into account that $\|u_n\|_{p(\cdot)}$ is bounded and $w_n(k) = u_n(k - j_n T)$ and relations (12) and (13) holds, it follows that there exists u in $l^{p(\cdot)}$ such that $u_n \to u$ as $n \to \infty$. $\square$

Now we are in a suitable position to do the proof of our main result.

Proof of Theorem 1.1.

We are going to apply Theorem 2.3. Firstly, we show that J satisfies the geometric conditions (H_1) and (H_2) of Theorem 2.3.

(H_1) there exist two numbers $R > 0$ and $c_0 > 0$ such that $J(u) \geqslant c_0$ for all $u \in l^{p(\cdot)}$ with $\|u\|_{p(\cdot)} = R$;

Indeed, fix $0 < \varepsilon < \frac{a_0}{2}$. By (F_1), there exists $\delta \in (0, 1)$ such that

$$
F(k,t) \leqslant \frac{\varepsilon}{p^+}|t|^{p^+} \leqslant \frac{a_0}{2p^+}|t|^{p^+} \leqslant \frac{a_0}{2p^+}|t|^{p(k)}, \tag{17}
$$

for all $k \in \mathbb{Z}$ and all $|t| \leqslant \delta$. Define

$$
R := \left(\frac{a_0}{p^+}\right)^{\frac{1}{p^-}} \delta^{\frac{p^+}{p^-}}.
$$

By condition (A_2) we deduce that $R \in (0, 1)$. Then for all $u \in l^{p(\cdot)}$ with $\|u\|_{p(\cdot)} = R$ relation (8) implies

$$
\begin{aligned}
R^{p^-} &= \frac{a_0}{p^+}\delta^{p^+} = \|u\|_{p(\cdot)}^{p^-}\\
&\geqslant \sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)}|u(k)|^{p(k)}\\
&\geqslant \frac{a_0}{p^+}|u(k)|^{p(k)},
\end{aligned}
$$

for all $k \in \mathbb{Z}$. It follows that

$$
1 > \delta^{p^+} \geqslant |u(k)|^{p(k)}, \text{ for all } k \in \mathbb{Z}.
$$

Therefore, $|u(k)| < 1$ for every $k \in \mathbb{Z}$ and thus

$$
|u(k)|^{p(k)} \geqslant |u(k)|^{p^+}, \text{ for all } k \in \mathbb{Z}.
$$

The above inequalities show that $\delta \geqslant |u(k)|$, for all $k \in \mathbb{Z}$. Next, by (17) we deduce

$$
\sum_{k\in\mathbb{Z}} F(k, u(k)) \leqslant \frac{a_0}{2p^+}\sum_{k\in\mathbb{Z}} |u(k)|^{p(k)} \leqslant \frac{1}{2}\sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)}|u(k)|^{p(k)}, \text{ with } \|u\|_{p(\cdot)} = R. \tag{18}
$$

Define $c_0 := \frac{R^{p^+}}{2}$ and for each u with $\|u\|_{p(\cdot)} = R$, by (8) and (18) we deduce

$$\begin{aligned} J(u) &= \sum_{k\in\mathbb{Z}} \frac{1}{p(k-1)}|\Delta u(k-1)|^{p(k-1)} + \sum_{k\in\mathbb{Z}} \frac{1}{p(k)} a(k)|u(k)|^{p(k)} - \sum_{k\in\mathbb{Z}} F(k,u(k)) \\ &\geqslant \sum_{k\in\mathbb{Z}} \frac{1}{p(k)} a(k)|u(k)|^{p(k)} - \sum_{k\in\mathbb{Z}} F(k,u(k)) \\ &\geqslant \sum_{k\in\mathbb{Z}} \frac{1}{p(k)} a(k)|u(k)|^{p(k)} - \frac{1}{2}\sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)}|u(k)|^{p(k)} \\ &= \frac{1}{2}\sum_{k\in\mathbb{Z}} \frac{a(k)}{p(k)}|u(k)|^{p(k)} \\ &\geqslant \frac{1}{2}\|u\|_{p(\cdot)}^{p^+} = \frac{R^{p^+}}{2} = c_0. \end{aligned}$$

(H_2) $J(0) < c_0$ and $J(e) < 0$ for some $e \in l^{p(\cdot)}$ with $\|e\|_{p(\cdot)} > R$.

Indeed, clearly $J(0) = 0 < c_0$. By standard integration, (F_2) implies that there exist two constants $c_1 > 0$ and $c_2 > 0$ such that

$$F(k,t) \geqslant c_1|t|^\alpha - c_2, \text{ for all } k \in \mathbb{Z} \text{ and all } t \in \mathbb{R},\ \alpha > p^+. \tag{19}$$

Defining $v \in l^{p(\cdot)}$ by $v(k) = \begin{cases} a > 0 \text{ if } k = 0 \\ 0 \text{ if } k \neq 0 \end{cases}$ and using (19), for each $\eta > 0$ we obtain

$$\begin{aligned} J(\eta v) &= \sum_{k\in\mathbb{Z}} \frac{1}{p(k-1)}|\Delta(\eta v)(k-1)|^{p(k-1)} + \sum_{k\in\mathbb{Z}} \frac{1}{p(k)} a(k)|\eta v(k)|^{p(k)} - \sum_{k\in\mathbb{Z}} F(k,\eta v(k)) \\ &\leqslant (2 + a(0))\frac{\eta^{p(0)} a^{p(0)}}{p(0)} - c_1\eta^\alpha a^\alpha + c_2, \end{aligned}$$

which goes to $-\infty$ as $\eta \to +\infty$ (since by relation (3) we have $\alpha > p^+ \geqslant p(0)$). So, we can choose $\eta > 0$ big enough and set $e = \eta v$, such that $\|e\|_{p(\cdot)} > c_0$ and $J(e) < 0$.

Hence, J satisfies the geometric conditions (H_1) and (H_2).

By the above informations and the mountain pass theorem, namely Theorem 2.3, we deduce the existence of a sequence $(u_n) \subset l^{p(\cdot)}$ such that

$$J(u_n) \to c > 0 \text{ and } J'(u_n) \to 0 \text{ in } l^{p^*(\cdot)}, \text{ as } n \to \infty.$$

We also know by Proposition 3.3 that J satisfies $(PS)_c$ condition. Then there exists a subsequence, still denoted by (u_n), and $u_0 \in l^{p(\cdot)}$ such that (u_n) converges to u_0 in $l^{p(\cdot)}$. So, we have $J(u_0) = c > 0$ and $J'(u_0) = 0$ and we conclude that u_0 is a critical point of J. From Proposition 3.1 we have that u_0 is a homoclinic solution of (1) and $u_0(k) \geqslant 0$ for all $k \in \mathbb{Z}$. Moreover, since $J(u_0) > 0$ we also have $u_0 \neq 0$, which, by Proposition 3.2, it follows that $u_0(k) > 0$ for all $k \in \mathbb{Z}$. □

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