Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 41(2), 2014, Pages 209–225
ISSN: 1223-6934

On fuzzy real valued asymptotically equivalent sequences and lacunary ideal convergence

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Abstract. In this paper we present some definitions which are the natural combination of the definition of asymptotic equivalence, statistical convergence, lacunary statistical convergence of fuzzy real numbers and ideal. In addition, we also present asymptotically ideal equivalent sequences of fuzzy real numbers and establish some relations related to this concept. Finally we introduce the notion of Cesaro Orlicz asymptotically equivalent sequences of fuzzy real numbers and establish their relationship with other classes.

2010 Mathematics Subject Classification. Primary 40A35; Secondary 40G15, 03E72.
Key words and phrases. Asymptotic equivalence, statistical convergence, lacunary sequence, ideal, ideal convergence, fuzzy number, Orlicz function.

1. Introduction

The concept of fuzzy set and fuzzy set operations were first introduced by Zadeh [30] and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming. Matloka [20] introduced bounded and convergent sequences of fuzzy numbers and studied their some properties. Later on sequences of fuzzy numbers have been discussed by Diamond and Kloeden [4], Mursaleen and Basarir [21], Altin et al. [1], Nanda [22] and many others.

Actually the idea of statistical convergence was formerly given under the name "almost convergence" by Zygmund in the first edition of his celebrated monograph published in Warsaw in 1935 [31]. The concept was formally introduced by Fast [6], Steinhaus [28] and later on it was reintroduced by Schoenberg [27]. A lot of developments have been made in this areas after the works of Šalát [26] and Fridy [8]. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Fridy and Orhan [9] introduced the concept of lacunary statistical convergence which is a generalization of statistical convergence. Mursaleen and Mohiuddine [23], introduced the concept of lacunary statistical convergence with respect to the intuitionistic fuzzy normed space. For details related to lacunary statistical convergence, we refer to [3, 10].

Marouf [19] peresented definitions for asymptotically equivalent and asymptotic regular matrices. Pobyvancts [25] introduced the concept of asymptotically regular matrices, which preserve the asymptotic equivalence of two nonnegative numbers sequences. Patterson [24] extended these concepts by presenting an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for

Received March 11, 2014.
This paper is in final form and no version of it will be submitted for publication elsewhere.

nonnegative summability matrices. Esi [5] introduced the concept of an asymptotically lacunary statistical equivalent sequences of interval numbers.

Kostyrko et. al [15] introduced the notion of I-convergence with the help of an admissible ideal I denote the ideal of subsets of $\mathbb{N}$, which is a generalization of statistical convergence. Kumar and Sharma [18] introduced the lacunary equivalent sequences of real numbers using ideals and studied some basic properties of this notion. More applications of ideals can be found in the works of [2, 11, 12, 13, 14, 29].

A family of sets $I \subset P(\mathbb{N})$ (power sets of $\mathbb{N}$) is called an *ideal* if and only if for each $A, B \in I$, we have $A \cup B \in I$ and for each $A \in I$ and each $B \subset A$, we have $B \in I$. A non-empty family of sets $\mathcal{F} \subset P(\mathbb{N})$ is a *filter* on $\mathbb{N}$ if and only if $\phi \notin \mathcal{F}$, for each $A, B \in \mathcal{F}$, we have $A \cap B \in \mathcal{F}$ and each $A \in \mathcal{F}$ and each $B \supset A$, we have $B \in \mathcal{F}$. An ideal I is called non-trivial ideal if $I \neq \phi$ and $\mathbb{N} \notin I$. Clearly $I \subset P(\mathbb{N})$ is a non-trivial ideal if and only if $\mathcal{F} = \mathcal{F}(I) = \{\mathbb{N} - A : A \in I\}$ is a filter on $\mathbb{N}$. A non-trivial ideal $I \subset P(\mathbb{N})$ is called *admissible* if and only if $\{\{x\} : x \in \mathbb{N}\} \subset I$. A non-trivial ideal I is *maximal* if there cannot exists any non-trivial ideal $J \neq I$ containing I as a subset. Recall from [15] that, a sequence $x = (x_k)$ of points in $\mathbb{R}$ is said to be I-convergent to a real number L if $\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\} \in I$ for every $\varepsilon > 0$. In this case we write $I - \lim x_k = L$.

Let D denote the set of all closed and bounded intervals $X = [x_1, x_2]$ on the real line $\mathbb{R}$. For $X = [x_1, x_2]$ and $Y = [y_1, y_2]$ in D, we define

$$X \leq Y \text{ if and only if } x_1 \leq y_1 \text{ and } x_2 \leq y_2.$$

Define a metric $\bar{d}$ on D by

$$\bar{d}(X, Y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}.$$

It can be easily proved that $\bar{d}$ is a metric on D and $(D, \bar{d})$ is a complete metric space. Also the relation $\leq$ is a partial order on D.

A *fuzzy number* X is a fuzzy subset of the real line $\mathbb{R}$ i.e. a mapping $X : \mathbb{R} \to J(= [0,1])$ associating each real number t with its grade of membership $X(t)$.

Let $L(\mathbb{R})$ denote the set of all fuzzy numbers. The linear structure of $L(\mathbb{R})$ induces the addition $X + Y$ and the scalar multiplication λX in terms of α-level sets, by

$$[X + Y]^\alpha = [X]^\alpha + [Y]^\alpha$$

and

$$[\lambda X]^\alpha = \lambda [X]^\alpha \text{ for each } 0 \leq \alpha \leq 1.$$

The set $\mathbb{R}$ of real numbers can be embedded in $L(\mathbb{R})$ if we define $\overline{r} \in L(\mathbb{R})$ by

$$\overline{r}(t) = \begin{cases} 1, & \text{if } t = r; \\ 0, & \text{if } t \neq r \end{cases}$$

The additive identity and multiplicative identity of $L(\mathbb{R})$ are denoted by $\overline{0}$ and $\overline{1}$, respectively.

For r in $\mathbb{R}$ and X in $L(\mathbb{R})$, the product rX is defined as follows:

$$rX(t) = \begin{cases} X(r^{-1}t), & \text{if } r \neq 0; \\ 0, & \text{if } r = 0 \end{cases}$$

Define a map $d : L(\mathbb{R}) \times L(\mathbb{R}) \to \mathbb{R}$ by

$$d(X, Y) = \sup_{0 \leq \alpha \leq 1} \bar{d}(X^\alpha, Y^\alpha).$$

For $X, Y \in L(\mathbb{R})$ define $X \leq Y$ if and only if $X^\alpha \leq Y^\alpha$ for any $\alpha \in [0, 1]$. It is known that $(L(\mathbb{R}), d)$ is complete metric space (see [20]).

A sequence $u = (u_k)$ of fuzzy numbers is a function X from the set $\mathbb{N}$ of natural numbers into $L(\mathbb{R})$. The fuzzy number u_k denotes the value of the function at $k \in \mathbb{N}$ (see [20]). We denote by w^F the set of all sequences $u = (u_k)$ of fuzzy numbers.

A sequence $u = (u_k)$ of fuzzy numbers is said to be bounded if the set $\{u_k : k \in \mathbb{N}\}$ of fuzzy numbers is bounded. We denote by ℓ_∞^F the set of all bounded sequences $u = (u_k)$ of fuzzy numbers.

A sequence $u = (u_k)$ of fuzzy numbers is said to be convergent to a fuzzy number u_0 if for every $\varepsilon > 0$ there is a positive integer k_0 such that $d(u_k, u_0) < \varepsilon$ for $k > k_0$. We denote by c^F the set of all convergent sequences $u = (u_k)$ of fuzzy numbers. It is straightforward to see that $c^F \subset \ell_\infty^F \subset w^F$.

2. Definitions and Notations

Throughout the paper, we denote I is an admissible ideal of subsets of $\mathbb{N}$, unless otherwise stated.

Now we recall the definitions which are using throughout the article.

Definition 2.1. A real or complex number sequence $x = (x_k)$ is said to be *statistically convergent* to L if for every $\varepsilon > 0$

$$\lim_n \frac{1}{n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0.$$

In this case, we write $S - \lim x = L$ or $x_k \to L(S)$ and S denotes the set of all statistically convergent sequences.

A lacunary sequence $\theta = (k_r)$ is an increasing sequence of non-negative integers where $k_0 = 0$, $k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $J_r = (k_{r-1}, k_r]$ and we let $h_r = k_r - k_{r-1}$. The space of lacunary strongly convergent sequences $\mathcal{N}_\theta$ was defined by Freedman et al., [7] as follows.

$$\mathcal{N}_\theta = \left\{ x = (x_k) : \lim_r \frac{1}{h_r} \sum_{k \in J_r} |x_k - L| = 0, \text{ for some } L \right\}.$$

Definition 2.2. [9] A sequence $x = (x_k)$ is said to be *lacunary statistically convergent* to the number L if for every $\varepsilon > 0$

$$\lim_{r \to \infty} \frac{1}{h_r} |\{k \in J_r : \ |x_k - L| \geq \varepsilon\}| = 0.$$

Let S_θ denote the set of all lacunary statistically convergent sequences. If $\theta = (2^r)$, then S_θ is the same as S.

Definition 2.3. [19] Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be *asymptotically equivalent* if

$$\lim_k \frac{x_k}{y_k} = 1,$$

denoted by $x \sim y$.

Definition 2.4. [24] Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be *asymptotically statistical equivalent* of multiple L provided that for every $\varepsilon > 0$

$$\lim_n \frac{1}{n} \left| \left\{ k \leq n : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \right| = 0,$$

denoted by $x \overset{S^L}{\sim} y$ and *simply asymptotically statistical equivalent* if $L = 1$.

Definition 2.5. Two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be *asymptotically lacunary statistical equivalent* of multiple L provided that for every $\varepsilon > 0$

$$\lim_r \frac{1}{h_r}\left|\left\{k \in J_r : \left|\frac{x_k}{y_k} - L\right| \geq \varepsilon\right\}\right| = 0,$$

denoted by $x \overset{S_\theta^L}{\sim} y$ and *simply asymptotically lacunary statistical equivalent* if $L = 1$. If we take $\theta = (2^r)$, then we get the definition 4.

Definition 2.6. [18] Two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be *strongly asymptotically I-equivalent* of multiple L provided that for each $\varepsilon > 0$

$$\left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}\left|\frac{x_k}{y_k} - L\right| \geq \varepsilon\right\} \in I$$

denoted by $x \overset{I[C_1]^L}{\sim} y$ and *simply strongly asymptotically I-equivalent* if $L = 1$.

Definition 2.7. [18] Two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically I-lacunary statistical equivalent of multiple L provided that for each $\varepsilon > 0$ and $\delta > 0$,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\left\{k \in J_r : \left|\frac{x_k}{y_k} - L\right| \geq \varepsilon\right\}\right| \geq \delta\right\} \in I$$

denoted by $x \overset{I(S_\theta)^L}{\sim} y$ and simply asymptotically I-lacunary statistical equivalent if $L = 1$.

3. Asymptotically lacunary statistical equivalent sequences using ideals

In this section, we define I-statistical convergence, asymptotically I-equivalent, asymptotically I-statistical equivalent and asymptotically I-lacunary statistical equivalent sequences of fuzzy real numbers and obtain some analogous results from these new definitions point of views.

Definition 3.1. Two sequences $u = (u_k)$ and $v = (v_k)$ of fuzzy real numbers are said to be *asymptotically statistical equivalent* of multiple L provided that for every $\varepsilon > 0$

$$\lim_n \frac{1}{n}\left|\left\{k \leq n : d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\right\}\right| = 0,$$

denoted by $x \overset{\mathcal{S}^L}{\sim} y$ and simply asymptotically statistical equivalent if $L = 1$.

Definition 3.2. [17] A sequence $u = (u_k)$ of fuzzy numbers is said to be *I-convergent* to a fuzzy number u_0 if for each $\epsilon > 0$

$$A = \{k \in \mathbb{N} : d(u_k, u_0) \geq \varepsilon\} \in I.$$

Definition 3.3. A sequence (u_k) of fuzzy real numbers is said to be *I-statistically convergent* to a fuzzy real number u_0 if for each $\varepsilon > 0$ and $\delta > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\{k \leq n : d(u_k, u_0) \geq \varepsilon\}\right| \geq \delta\right\} \in I.$$

In this case we write $I(\mathcal{S}) - \lim u_k = u_0$.

Definition 3.4. Two nonnegative sequences (u_k) and (v_k) of fuzzy real numbers are said to be *asymptotically I-equivalent* of multiple L provided that for every $\varepsilon > 0$

$$\left\{k \in \mathbb{N} : d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\right\} \in I,$$

denoted by $(u_k) \overset{I^L}{\sim} (v_k)$ and *simply asymptotically I-equivalent* if $L = 1$.

Lemma 3.1. *Let $I \subset P(\mathbb{N})$ be an admissible ideal. Let $(u_k), (v_k) \in \ell_\infty^F$ with $I - \lim_k u_k = \bar{0} = I - \lim_k v_k$ such that $(u_k) \overset{I^L}{\sim} (v_k)$. Then there exists a sequence $(w_k) \in \ell_\infty^F$ with $I - \lim_k w_k = \bar{0}$ such that $(u_k) \overset{I^L}{\sim} (w_k) \overset{I^L}{\sim} (v_k)$.*

Definition 3.5. Two sequences (u_k) and (v_k) of fuzzy numbers are said to be *asymptotically I-statistical equivalent* of multiple L provided that for every $\varepsilon > 0$ and for every $\delta > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\right\}\right| \geq \delta\right\} \in I,$$

denoted by $(u_k) \overset{I(\mathcal{S})^L}{\sim} (v_k)$ and *simply asymptotically I-statistical equivalent* if $L = 1$.

Definition 3.6. Two sequences (u_k) and (v_k) of fuzzy real numbers are said to be *strongly asymptotically Cesàro I-equivalent* (or *$I([C_1])$-equivalent*) of multiple L provided that for every $\delta > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n} d\left(\frac{u_k}{v_k}, L\right) \geq \delta\right\} \in I$$

denoted by $(u_k) \overset{I([C_1])^L}{\sim} (v_k)$ and *simply strongly asymptotically Cesàro I-equivalent* if $L = 1$.

Theorem 3.2. *Let $(u_k), (v_k)$ be two sequences of fuzzy real numbers. If $(u_k), (v_k) \in \ell_\infty^F$ and $(u_k) \overset{I(\mathcal{S})^L}{\sim} (v_k)$. Then $(u_k) \overset{I([C_1])^L}{\sim} (v_k)$.*

Proof. Suppose that $(u_k), (v_k) \in \ell_\infty^F$ and $(u_k) \overset{I(\mathcal{S})^L}{\sim} (v_k)$. Then we can assume that

$$d\left(\frac{u_k}{v_k}, L\right) \leq M \text{ for almost all } k.$$

Let $\varepsilon > 0$. Then we have

$$\begin{aligned}
\left|\frac{1}{n}\sum_{k=1}^{n} d\left(\frac{u_k}{v_k}, L\right)\right| &\leq \frac{1}{n}\sum_{k=1}^{n} d\left(\frac{u_k}{v_k}, L\right) \\
&\leq \frac{1}{n}\sum_{\substack{k=1 \\ d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon}}^{n} d\left(\frac{u_k}{v_k}, L\right) + \frac{1}{n}\sum_{\substack{k=1 \\ d\left(\frac{u_k}{v_k}, L\right) < \varepsilon}}^{n} d\left(\frac{u_k}{v_k}, L\right) \\
&\leq M.\frac{1}{n}\left|\left\{k \leq n : \ d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\right\}\right| + \frac{1}{n}.n.\varepsilon.
\end{aligned}$$

Consequently for any $\delta > \varepsilon > 0$, δ and ε are independent, put $\delta_1 = \delta - \varepsilon > 0$ we have

$$\left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n} d\left(\frac{u_k}{v_k}, L\right) \geq \delta\right\} \subseteq \left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\right\}\right| \geq \frac{\delta_1}{M}\right\} \in I.$$

This shows that $(u_k) \overset{I([C_1])^L}{\sim} (v_k)$. □

Definition 3.7. Two sequences (u_k) and (v_k) of fuzzy real numbers are said to be *strongly asymptotically* I*-lacunary equivalent* (or $I([N_\theta])$*-equivalent*) of multiple L provided that for every $\delta > 0$,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} d\left(\frac{u_k}{v_k}, L\right) \geq \delta\right\} \in I$$

denoted by $(u_k) \overset{I([N_\theta])^L}{\sim} (v_k)$ and *simply strongly asymptotically* I*-lacunary equivalent* if $L = 1$.

Definition 3.8. Two sequences (u_k) and (v_k) of fuzzy real numbers are said to be *asymptotically* I*-lacunary statistical equivalent* (or $I(\mathcal{S}_\theta)$*-equivalent*) of multiple L provided that for every $\varepsilon > 0$, for every $\delta > 0$,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} \left|\left\{k \in J_r : d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\right\}\right| \geq \delta\right\} \in I$$

denoted by $(u_k) \overset{I(\mathcal{S}_\theta)^L}{\sim} (v_k)$ and *simply asymptotically* I*-lacunary statistical equivalent* if $L = 1$.

Remark 3.1. If we take $I = I_{fin} = \{A \subset \mathbb{N} : A \text{ is finite set}\}$, then the asymptotically I-statistical equivalent, $I([N_\theta])$-equivalent and $I(\mathcal{S}_\theta)$-equivalent of sequences, respectively coincides with their statistically equivalent, lacunary-equivalent and lacunary statistically equivalent.

Theorem 3.3. *Let* $(u_k), (v_k)$ *be two sequences of fuzzy real numbers. Then*
(a) $(u_k) \overset{I([N_\theta])^L}{\sim} (v_k) \Rightarrow (u_k) \overset{I(\mathcal{S}_\theta)^L}{\sim} (v_k)$.
(b) *Let* $(u_k) \in \ell_\infty^F$ *and* $(u_k) \overset{I(\mathcal{S}_\theta)^L}{\sim} (v_k)$*, then* $(u_k) \overset{I([N_\theta])^L}{\sim} (v_k)$.
(c) $I(\mathcal{S}_\theta)^L \cap \ell_\infty^F = I([N_\theta])^L \cap \ell_\infty^F$.

Proof. (a) Let $\varepsilon > 0$ and $(u_k) \overset{I([N_\theta])^L}{\sim} (v_k)$. Then we can write

$$\sum_{k \in J_r} d\left(\frac{u_k}{v_k}, L\right) \geq \sum_{\substack{k \in J_r \\ d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon}} d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon \left|\left\{k \in J_r : \ d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\right\}\right|$$

$$\Rightarrow \frac{1}{\varepsilon . h_r} \sum_{k \in J_r} d\left(\frac{u_k}{v_k}, L\right) \geq \frac{1}{h_r} \left|\left\{k \in J_r : \ d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\right\}\right|.$$

Thus for any $\delta > 0$,

$$\frac{1}{h_r} \left|\left\{k \in J_r : \ d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\right\}\right| \geq \delta$$

implies that

$$\frac{1}{h_r} \sum_{k \in J_r} d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\delta.$$

Therefore we have

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} \left|\left\{k \in J_r : \ d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\right\}\right| \geq \delta\right\} \subset \left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\delta\right\}.$$

Since $(u_k) \overset{I([N_\theta])^L}{\sim} (v_k)$, so that

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\delta\right\} \in I$$

which implies that

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} \left|\left\{k \in J_r : \ d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\right\}\right| \geq \delta\right\} \in I.$$

This shows that $(u_k) \overset{I(\mathcal{S}_\theta)^L}{\sim} (v_k)$.

(b) Suppose that $(u_k) \overset{I(\mathcal{S}_\theta)^L}{\sim} (v_k)$ and $(u_k), (v_k) \in \ell_\infty^F$. We assume that $d\left(\frac{u_k}{v_k}, L\right) \leq M$ for all $k \in \mathbb{N}$. Given $\varepsilon > 0$, we get

$$\frac{1}{h_h} \sum_{k \in J_r} d\left(\frac{u_k}{v_k}, L\right) = \frac{1}{h_r} \sum_{\substack{k \in J_r \\ d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon}} d\left(\frac{u_k}{v_k}, L\right) + \frac{1}{h_r} \sum_{\substack{k \in J_r \\ d\left(\frac{u_k}{v_k}, L\right) < \varepsilon}} d\left(\frac{u_k}{v_k}, L\right)$$

$$\leq \frac{M}{h_r} \left|\left\{k \in J_r : \ d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\right\}\right| + \varepsilon.$$

If we put

$$A(\varepsilon) = \left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\right\}$$

and

$$B(\varepsilon_1) = \left\{r \in \mathbb{N} : \frac{1}{h_r} \left|\left\{k \in J_r : \ d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\right\}\right| \geq \frac{\varepsilon_1}{M}\right\},$$

where $\varepsilon_1 = \delta - \varepsilon > 0$ (δ and ε are independent), then we have $A(\varepsilon) \subset B(\varepsilon_1)$ and so $A(\varepsilon) \in I$. This shows that $(u_k) \overset{I([N_\theta])^L}{\sim} (v_k)$.

(c) It follows from (a)and (b).

□

If we let $\theta = (2^r)$ in Theorem 3.3, then we have the following corollary.

Corollary 3.4. *Let $(u_k), (v_k)$ be two sequences of fuzzy real numbers. Then*

(a) $(u_k) \overset{I([C_1])^L}{\sim} (v_k) \Rightarrow (u_k) \overset{I(\mathcal{S})^L}{\sim} (v_k)$.

(b) *Let $(u_k) \in \ell_\infty^F$ and $(u_k) \overset{I(\mathcal{S})^L}{\sim} (v_k)$, then $(u_k) \overset{I([C_1])^L}{\sim} (v_k)$.*

(c) $I(\mathcal{S})^L \cap \ell_\infty^F = I([C_1])^L \cap \ell_\infty^F$.

Theorem 3.5. *Let I be a non-trivial admissible ideal. Suppose for given $\delta > 0$ and every $\varepsilon > 0$*

$$\left\{n \in \mathbb{N} : \frac{1}{n} \left|\left\{0 \leq k \leq n-1 : d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\right\}\right| < \delta\right\} \in \mathcal{F}$$

then $(u_k) \overset{I(\mathcal{S})^L}{\sim} (v_k)$.

Proof. Let $\delta > 0$ be given. For every $\varepsilon > 0$, choose n_1 such that

$$\frac{1}{n} \left|\left\{0 \leq k \leq n-1 : d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\right\}\right| < \frac{\delta}{2}, \text{ for all } n \geq n_1. \tag{1}$$

It is sufficient to show that there exists n_2 such that for $n \geq n_2$

$$\frac{1}{n}\left|\left\{0\le k\le n-1 : d\left(\frac{u_k}{v_k},L\right)\ge\varepsilon\right\}\right|<\frac{\delta}{2}.$$

Let $n_0=\max\{n_1,n_2\}$. The relation (3.1) will be true for $n>n_0$. If m_0 chosen fixed, then we get

$$\left|\left\{0\le k\le m_0-1 : d\left(\frac{u_k}{v_k},L\right)\ge\varepsilon\right\}\right|=M.$$

Now for $n>m_0$ we have

$$\begin{aligned}\frac{1}{n}\left|\left\{0\le k\le n-1 : d\left(\frac{u_k}{v_k},L\right)\ge\varepsilon\right\}\right| &\le \frac{1}{n}\left|\left\{0\le k\le m_0-1 : d\left(\frac{u_k}{v_k},L\right)\ge\varepsilon\right\}\right|\\ &+\frac{1}{n}\left|\left\{m_0\le k\le n-1 : d\left(\frac{u_k}{v_k},L\right)\ge\varepsilon\right\}\right|\\ &\le \frac{M}{n}+\frac{1}{n}\left|\left\{m_0\le k\le n-1 : d\left(\frac{u_k}{v_k},L\right)\ge\varepsilon\right\}\right|\le\frac{M}{n}+\frac{\delta}{2}.\end{aligned}$$

Thus for sufficiently large n

$$\frac{1}{n}\left|\left\{m_0\le k\le n-1 : d\left(\frac{u_k}{v_k},L\right)\ge\varepsilon\right\}\right|\le\frac{M}{n}+\frac{\delta}{2}<\delta.$$

This established the result. □

Theorem 3.6. *Let (u_k) and (v_k) be two sequences of fuzzy real numbers. Let $\theta=(k_r)$ be a lacunary sequence with $\liminf_r q_r>1$. Then $(u_k)\overset{I(\mathcal{S})^L}{\sim}(v_k)\Rightarrow(u_k)\overset{I(\mathcal{S}_\theta)^L}{\sim}(v_k)$.*

Proof. Suppose that $\liminf_r q_r>1$ then there exists an $a>0$ such that $q_r\ge 1+a$ for sufficiently large r, which implies that

$$\frac{h_r}{k_r}\ge\frac{a}{1+a}.$$

Suppose that $(u_k)\overset{I(\mathcal{S})^L}{\sim}(v_k)$. For a given $\varepsilon>0$ and sufficiently large r, we have

$$\begin{aligned}\frac{1}{k_r}\left|\left\{k\le k_r : d\left(\frac{u_k}{v_k},L\right)\ge\varepsilon\right\}\right| &\ge \frac{1}{k_r}\left|\left\{k\in J_r : d\left(\frac{u_k}{v_k},L\right)\ge\varepsilon\right\}\right|\\ &\ge\left(\frac{a}{1+a}\right)\cdot\frac{1}{h_r}\left|\left\{k\in J_r : d\left(\frac{u_k}{v_k},L\right)\ge\varepsilon\right\}\right|.\end{aligned}$$

and for any $\delta>0$ we have

$$\begin{aligned}&\left\{r\in\mathbb{N}:\frac{1}{k_r}\left|\left\{k\in J_r : d\left(\frac{u_k}{v_k},L\right)\ge\varepsilon\right\}\right|\ge\delta\right\}\\ &\qquad\subseteq\left\{r\in\mathbb{N}:\frac{1}{k_r}\left|\left\{k\le k_r : d\left(\frac{u_k}{v_k},L\right)\ge\varepsilon\right\}\right|\ge\frac{a\delta}{1+a}\right\}\in I.\end{aligned}$$

This shows that $(u_k)\overset{I(\mathcal{S}_\theta)^L}{\sim}(v_k)$. □

Theorem 3.7. *Let $I=I_{fin}=\{A\subset\mathbb{N} : A$ is a finite set$\}$ be a non trivial ideal. Let $(u_k),(v_k)$ be two sequences of fuzzy real numbers and $\theta=(k_r)$ be a lacunary sequence with $\limsup_r q_r<\infty$, then $(u_k)\overset{I(\mathcal{S}_\theta)^L}{\sim}(v_k)\Rightarrow(u_k)\overset{I(\mathcal{S})^L}{\sim}(v_k)$.*

Proof. Suppose that $\limsup_r q_r < \infty$, then there exists a $H > 0$ such that $q_r < H$ for all r. Suppose that $(u_k) \overset{I(\mathcal{S}_\theta)^L}{\sim} (v_k)$ and for every $\varepsilon > 0$, we put

$$N_r = \left|\left\{k \in J_r : d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\right\}\right|.$$

Since $(u_k) \overset{I(\mathcal{S}_\theta)^L}{\sim} (v_k)$ it follws that for every $\varepsilon > 0$ and $\delta > 0$

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\left\{k \in J_r : d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\right\}\right| \geq \delta\right\} = \left\{r \in \mathbb{N} : \frac{N_r}{h_r} \geq \delta\right\} \in I$$

and therefore it is a finite set. We can choose an integer $r_0 \in \mathbb{N}$ such that

$$\frac{N_r}{h_r} < \delta \text{ for all } r > r_0. \tag{2}$$

Let $M = \max\{N_r : 1 \leq r \leq r_0\}$ and n be any integer satisfying $k_{r-1} < n \leq k_r$, then we have

$$\begin{aligned}
\frac{1}{n}\left|\left\{k \leq n : d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\right\}\right| &\leq \frac{1}{k_{r-1}}\left|\left\{k \leq k_r : d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\right\}\right| \\
&= \frac{1}{k_{r-1}}\{N_1 + N_2 + ... + N_{r_0} + N_{r_0+1} + ... + N_r\} \\
&\leq \frac{M}{k_{r-1}}.r_0 + \frac{1}{k_{r-1}}\left\{h_{r_0+1}\left(\frac{N_{r_0+1}}{h_{r_0+1}}\right) + ... + h_r\left(\frac{N_r}{h_r}\right)\right\} \\
&\leq \frac{M}{k_{r-1}}.r_0 + \frac{1}{k_{r-1}}\left(\sup_{r>r_0}\left(\frac{N_r}{h_r}\right)\right)\{H_{r_0+1} + ... + h_r\} \\
&\leq \frac{M}{k_{r-1}}.r_0 + \delta\left(\frac{k_r - k_{r_0}}{k_{r-1}}\right) \\
&\leq \frac{M}{k_{r-1}}.r_0 + \delta q_r \leq \frac{M}{k_{r-1}}.r_0 + \delta.H
\end{aligned}$$

This completes the proof of the theorem. □

Definition 3.9. Let $p \in (0, \infty)$. Two sequences (u_k) and (v_k) of fuzzy real numbers are said to be *asymptotically lacunary p-equivalent* provided that for every $\varepsilon > 0$

$$\lim_r \frac{1}{h_r}\sum_{k \in J_r} d\left(\frac{u_k}{v_k}, L\right)^p = 0$$

denoted by $(u_k) \overset{[N_{\theta_p}]^L}{\sim} (v_k)$ and *simply asymptotically lacunary p-equivalent* if $L = 1$.

Definition 3.10. Let $p \in (0, \infty)$. Two sequences (u_k) and (v_k) of fuzzy real numbers are said to be *asymptotically lacunary statistical p-equivalent* provided that for every $\varepsilon > 0$

$$\lim_r \frac{1}{h_r}\left|\left\{k \in J_r : d\left(\frac{u_k}{v_k}, L\right)^p \geq \varepsilon\right\}\right| = 0$$

denoted by $(u_k) \overset{\mathcal{S}^L_{\theta_p}}{\sim} (v_k)$ and *simply asymptotically lacunary statistical p-equivalent* if $L = 1$.

The proof of the following theorem is similar to Theorem 3.3 for $I = I_{fin}$.

Theorem 3.8. *Let $(u_k), (v_k)$ be two sequences of fuzzy real numbers. Then*

(a) $(u_k) \overset{[N_{\theta_p}]^L}{\sim} (v_k) \Rightarrow (u_k) \overset{\mathcal{S}^L_{\theta_p}}{\sim} (v_k)$.

(b) *Let* $(u_k) \in \ell_\infty^F$ *and* $(u_k) \overset{\mathcal{S}_{\theta_p}^L}{\sim} (v_k)$, *then* $(u_k) \sim^{[N_{\theta_p}]^L} (v_k)$.
(c) $\mathcal{S}_{\theta_p}^L \cap \ell_\infty^F = [N_{\theta_p}]^L \cap \ell_\infty^F$.

Definition 3.11. Let $p \in (0, \infty)$. Two sequences (u_k) and (v_k) of fuzzy real numbers are said to be *asymptotically* I*-lacunary* p*-equivalent* (or $I([N_{\theta_p}])$*-equivalent*) provided that for every $\varepsilon > 0$

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} d\left(\frac{u_k}{v_k}, L \right)^p \geq \varepsilon \right\} \in I$$

denoted by $(u_k) \overset{I([N_{\theta_p}])^L}{\sim} (v_k)$ and simply asymptotically I-lacunary p-equivalent if $L = 1$.

Definition 3.12. Let $p \in (0, \infty)$. Two sequences (u_k) and (v_k) of fuzzy real numbers are said to be *asymptotically* I*-lacunary statistical* p*-equivalent* provided that for every $\varepsilon > 0$, for every $\delta > 0$

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \left| \left\{ k \leq n : d\left(\frac{u_k}{v_k}, L \right)^p \geq \varepsilon \right\} \right| \geq \delta \right\} \in I$$

denoted by $(u_k) \overset{I(\mathcal{S}_{\theta_p})^L}{\sim} (v_k)$ and *simply asymptotically* I*-statistical* p*-equivalent* if $L = 1$.

The proof of the following theorem follows from Theorems 3.3 and 3.8.

Theorem 3.9. *Let* $(u_k), (v_k)$ *be two sequences of fuzzy real numbers. Then*
(a) $(u_k) \overset{I([N_{\theta_p}])^L}{\sim} (v_k) \Rightarrow (u_k) \overset{I(\mathcal{S}_{\theta_p})^L}{\sim} (v_k)$.
(b) *Let* $(u_k) \in \ell_\infty^F$ *and* $(u_k) \overset{I(\mathcal{S}_{\theta_p})^L}{\sim} (v_k)$, *then* $(u_k) \overset{I([N_{\theta_p}])^L}{\sim} (v_k)$.
(c) $I(\mathcal{S}_{\theta_p})^L \cap \ell_\infty^F = I([N_{\theta_p}])^L \cap \ell_\infty^F$.

4. Cesàro Orlicz asymptotically ϕ-statistical equivalent sequences

In this section we define the notion of Cesàro Orlicz asymptotically ϕ-statistical equivalent sequences of fuzzy real numbers.

Let P denote the space whose elements are finite sets of distinct positive integers. Given any element σ of P, we denote by $p(\sigma)$ the sequence $\{p_n(\sigma)\}$ such that $p_n(\sigma) = 1$ for $n \in \sigma$ and $p_n(\sigma) = 0$ otherwise. Further

$$P_s = \left\{ \sigma \in P : \sum_{n=1}^{\infty} p_n(\sigma) \leq s \right\},$$

i.e. P_s is the set of those σ whose support has cardinality at most s, and we get

$$\Phi = \left\{ \phi = (\phi_n) : 0 < \phi_1 \leq \phi_n \leq \phi_{n+1} \text{ and } n\phi_{n+1} \leq (n+1)\phi_n \right\}.$$

We define

$$\tau_s = \frac{1}{\phi_s} \sum_{k \in \sigma, \sigma \in P_s} x_k.$$

Now we give the following definitions.

Definition 4.1. A sequence $x = (x_k)$ is said to be ϕ*-summable* to ℓ if $\lim_s \tau_s = \ell$.

Definition 4.2. A sequence $x = (x_k)$ is said to be *strongly ϕ-summable* to ℓ if

$$\lim_{s\to\infty} \frac{1}{\phi_s} \sum_{k\in\sigma,\sigma\in P_s} |x_k - \ell| = 0.$$

In this case we write $x_k \overset{[\phi]}{\to} \ell$ and $[\phi]$ denote the set of all strongly ϕ-summable sequences.

Definition 4.3. Let $E \subseteq \mathbb{N}$. The number

$$\delta_\phi(E) = \lim_{s\to\infty} \frac{1}{\phi_s} |\{k \in \sigma, \sigma \in P_s : k \in E\}|$$

is said to be the *ϕ-density* of E.

Definition 4.4. A sequence $x = (x_k)$ is said to be *ϕ-statistical convergent* to $\ell \in \mathbb{R}$ if for each $\varepsilon > 0$

$$\lim_{s\to\infty} \frac{1}{\phi_s} |\{k \in \sigma, \sigma \in P_s : |x_k - \ell| \geq \varepsilon\}| = 0.$$

In this case we write $S_\phi - \lim_k x_k = \ell$ or $x_k \overset{S_\phi}{\to} \ell$ and S_ϕ denote the set of all ϕ-statisticallly convergent sequences.

An Orlicz function is a function $M : [0,\infty) \to [0,\infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. An Orlicz function M is said to satisfy the $\Delta_2 -$ *condition* for all values of u, if there exists a constant $K > 0$ such that $M(2u) \leq KM(u)$, $u \geq 0$. Note that, if $0 < \lambda < 1$,then $M(\lambda x) \leq \lambda M(x)$, for all $x \geq 0$(see [16]).

Now we define the following asymptotic ϕ-statistical equivalence sequences of fuzzy real numbers.

Definition 4.5. Two sequences (u_k) and (v_k) of fuzzy real numbers are said to be *Cesàro Orlicz asymptotically equivalent* of multiple L provided that

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} M\left(d\left(\frac{u_k}{v_k}, L\right)\right) = 0$$

denoted by $(u_k) \overset{[C_1]^L(M)}{\sim} (v_k)$ and *simply Cesàro Orlicz asymptotically equivalent* if $L = 1$.

Definition 4.6. Two sequences (u_k) and (v_k) of fuzzy real numbers are said to be *Cesàro Orlicz asymptotically I-equivalent* of multiple L provided that for every $\delta > 0$

$$\left\{n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} M\left(d\left(\frac{u_k}{v_k}, L\right)\right) \geq \delta\right\} \in I$$

denoted by $(u_k) \overset{I[C_1]^L(M)}{\sim} (v_k)$ and simply *Cesàro Orlicz asymptotically I-equivalent* if $L = 1$.

Definition 4.7. Two sequences (u_k) and (v_k) of fuzzy real numbers are said to be *Orlicz asymptotically ϕ-equivalent* of multiple L provided that

$$\lim_s \frac{1}{\phi_s} \sum_{k\in\sigma,\sigma\in P_s} M\left(d\left(\frac{u_k}{v_k}, L\right)\right) = 0$$

denoted by $(u_k) \overset{[\phi]^L(M)}{\sim} (v_k)$ and *simply Orlicz asymptotically ϕ-equivalent* if $L = 1$.

Definition 4.8. Two sequences (u_k) and (v_k) of fuzzy real numbers are said to be *asymptotically ϕ-statistical equivalent* of multiple L provided that for every $\varepsilon > 0$

$$\lim_s \frac{1}{\phi_s}\left|\left\{k \in \sigma, \sigma \in P_s : d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\right\}\right| = 0$$

denoted by $(u_k) \overset{\mathcal{S}_\phi^L}{\sim} (v_k)$ and *simply asymptotically ϕ-statistical equivalent* if $L = 1$.

Definition 4.9. Two sequences (u_k) and (v_k) of fuzzy real numbers are said to be *Orlicz asymptotically ϕ-statistical equivalent* of multiple L provided that for every $\varepsilon > 0$

$$\lim_s \frac{1}{\phi_s}\left|\left\{k \in \sigma, \sigma \in P_s : M\left(d\left(\frac{u_k}{v_k}, L\right)\right) \geq \varepsilon\right\}\right| = 0$$

denoted by $(u_k) \overset{\mathcal{S}_\phi^L(M)}{\sim} (v_k)$ and *simply Orlicz asymptotically ϕ-statistical equivalent* if $L = 1$.

Theorem 4.1. *Let $(u_k), (v_k)$ be two sequences of fuzzy real numbers and M be an Orlicz function. Then*

(a) $(u_k) \overset{I[C_1]^L(M)}{\sim} (v_k) \Rightarrow (u_k) \overset{I(\mathcal{S})^L}{\sim} (v_k)$.

(b) $(u_k) \overset{I(\mathcal{S})^L}{\sim} (v_k)$ *implies* $(u_k) \overset{I[C_1]^L(M)}{\sim} (v_k)$, *if M is bounded.*

Proof. (a) Suppose that $(u_k) \overset{I[C_1]^L(M)}{\sim} (v_k)$ and let $\varepsilon > 0$ be given, then we can write

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n} M\left(d\left(\frac{u_k}{v_k}, L\right)\right) &\geq \frac{1}{n}\sum_{\substack{k=1 \\ d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon}}^{n} M\left(d\left(\frac{u_k}{v_k}, L\right)\right) \\ &\geq \frac{M(\varepsilon)}{n}\left|\left\{k \leq n : d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\right\}\right|.\end{aligned}$$

Consequently for any $\eta > 0$, we have

$$\begin{aligned}&\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\right\}\right| \geq \frac{\eta}{M(\varepsilon)}\right\} \\ &\subseteq \left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n} M\left(d\left(\frac{u_k}{v_k}, L\right)\right) \geq \eta\right\} \in I.\end{aligned}$$

Hence $(u_k) \overset{I(\mathcal{S})^L}{\sim} (v_k)$.

(b) Suppose that M is bounded and $(u_k) \overset{I(\mathcal{S})^L}{\sim} (v_k)$. Since M is bounded then there exists a real number $K > 0$ such that $\sup_t M(t) \leq K$. Moreover for any $\varepsilon > 0$ we can write

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n} M\left(d\left(\frac{u_k}{v_k}, L\right)\right) &= \frac{1}{n}\left[\sum_{\substack{k=1 \\ d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon}}^{n} M\left(d\left(\frac{u_k}{v_k}, L\right)\right) + \sum_{\substack{k=1 \\ d\left(\frac{u_k}{v_k}, L\right) < \varepsilon}}^{n} M\left(d\left(\frac{u_k}{v_k}, L\right)\right)\right] \\ &\leq \frac{K}{n}\left|\left\{k \leq n : d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\right\}\right| + M(\varepsilon).\end{aligned}$$

Now applying $\varepsilon \to 0$, then the result follows. $\square$

Theorem 4.2. *Let $(u_k), (v_k)$ be two sequences of fuzzy real numbers and (ϕ_s) be a nondecreasing sequence of positive real numbers such that $\phi_s \to \infty$ as $s \to \infty$ and $\phi_s \le s$ for every $s \in \mathbb{N}$. Then $(u_k) \overset{\mathcal{S}^L}{\sim} (v_k) \Rightarrow (u_k) \overset{\mathcal{S}^L_\phi}{\sim} (v_k)$.*

Proof. By the definition of the sequences ϕ_s it follows that $\inf_s \frac{s}{s-\phi_s} \ge 1$. Then there exists a $a > 0$ such that

$$\frac{s}{\phi_s} \le \frac{1+a}{a}.$$

Suppose that $(u_k) \overset{\mathcal{S}^L}{\sim} (v_k)$, then for every $\varepsilon > 0$ and sufficiently large s we have

$$\begin{aligned}
\frac{1}{\phi_s}\left|\left\{k \in \sigma, \sigma \in P_s : d\left(\frac{u_k}{v_k}, L\right) \ge \varepsilon\right\}\right| &= \frac{1}{s}.\frac{s}{\phi_s}\left|\left\{k \le s : d\left(\frac{u_k}{v_k}, L\right) \ge \varepsilon\right\}\right| \\
&- \frac{1}{\phi_s}\left|\left\{k \in \{1,2,...s\} - \sigma, \sigma \in P_s : d\left(\frac{u_k}{v_k}, L\right) \ge \varepsilon\right\}\right| \\
&\le \frac{1+a}{a}\frac{1}{s}\left|\left\{k \le s : d\left(\frac{u_k}{v_k}, L\right) \ge \varepsilon\right\}\right| \\
&- \frac{1}{\phi_s}\left|\left\{k_0 \in \{1,2,...s\} - \sigma, \sigma \in P_s : d\left(\frac{u_{k_0}}{v_{k_0}}, L\right) \ge \varepsilon\right\}\right|.
\end{aligned}$$

This completes the proof of the theorem. □

Theorem 4.3. *Let $(u_k), (v_k)$ be two sequences of fuzzy real numbers and let M be an Orlicz function satifies the Δ_2-conditions. Then $(u_k) \overset{\mathcal{S}^L}{\sim} (v_k) \Rightarrow (u_k) \overset{\mathcal{S}^L_\phi(M)}{\sim} (v_k)$.*

Proof. By the definition of the sequences ϕ_s it follows that $\inf_s \frac{s}{s-\phi_s} \ge 1$. Then there exists an $a > 0$ such that

$$\frac{s}{\phi_s} \le \frac{1+a}{a}.$$

Suppose that $(u_k) \overset{\mathcal{S}^L}{\sim} (v_k)$, then for every $\varepsilon > 0$ and sufficiently large s we have

$$\begin{aligned}
\frac{1}{\phi_s}\left|\left\{k \in \sigma, \sigma \in P_s : M\left(d\left(\frac{u_k}{v_k}, L\right)\right) \ge \varepsilon\right\}\right| &= \frac{1}{s}.\frac{s}{\phi_s}\left|\left\{k \le s : M\left(d\left(\frac{u_k}{v_k}, L\right)\right) \ge \varepsilon\right\}\right| \\
&- \frac{1}{\phi_s}\left|\left\{k \in \{1,2,...s\} - \sigma, \sigma \in P_s : M\left(d\left(\frac{u_k}{v_k}, L\right)\right) \ge \varepsilon\right\}\right| \\
&\le \frac{1+a}{a}\frac{1}{s}\left|\left\{k \le s : M\left(d\left(\frac{u_k}{v_k}, L\right)\right) \ge \varepsilon\right\}\right| \qquad (3) \\
&- \frac{1}{\phi_s}\left|\left\{k_0 \in \{1,2,...s\} - \sigma, \sigma \in P_s : M\left(d\left(\frac{u_{k_0}}{v_{k_0}}, L\right)\right) \ge \varepsilon\right\}\right|.
\end{aligned}$$

Since M satisfies the Δ_2-conditions, it follows that

$$M\left(d\left(\frac{u_k}{v_k}, L\right)\right) \le K.d\left(\frac{u_k}{v_k}, L\right)$$

for some constant $K > 0$ in both the cases where $d\left(\frac{u_k}{v_k}, L\right) \le 1$ and $d\left(\frac{u_k}{v_k}, L\right) \ge 1$.

In first case it follows from the definition of Orlicz function and for the second case we have

$$d\left(\frac{u_k}{v_k}, L\right) = 2.L^{(1)} = 2^2.L^{(2)} = ... = 2^s.L^{(s)}$$

such that $L^{(s)} \leq 1$. Using the Δ_2-conditions of Orlicz functions we get the following estimation

$$M\left(d\left(\frac{u_k}{v_k}, L\right)\right) \leq T.L^{(s)}.M(1) = K.d\left(\frac{u_k}{v_k}, L\right), \tag{4}$$

where K and T are constants. The proof of the theorem follows from the relations (4.1) and (4.2). □

Remark 4.1. From the Theorems 4.2 and 4.3, we can concluded that $(u_k) \overset{\mathcal{S}^L}{\sim} (v_k) \Leftrightarrow (u_k) \overset{\mathcal{S}^L_\phi(M)}{\sim} (v_k)$.

Theorem 4.4. *Let $(u_k), (v_k)$ be two sequences of fuzzy real numbers. Let M be an Orlicz function and $k \in \mathbb{Z}$ such that $\phi_s \leq [\phi_s] + k$, $\sup_s \frac{[\phi_s]+k}{\phi_{s-1}} < \infty$. Then $(u_k) \overset{\mathcal{S}^L_\phi(M)}{\sim} (v_k) \Rightarrow (u_k) \overset{\mathcal{S}^L}{\sim} (v_k)$.*

Proof. If $\sup_s \frac{[\phi_s]+k}{\phi_{s-1}} < \infty$, then there exists $K > 0$ such that $\frac{[\phi_s]+k}{\phi_{s-1}} < K$ for all $s \geq 1$. Let n be an integer such that $\phi_{s-1} < n \leq \phi_s$. Then for every $\varepsilon > 0$, we have

$$\begin{aligned}
\frac{1}{n}\left|\left\{k \leq n : d\left(\frac{u_k}{v_k}, L\right) \geq \varepsilon\right\}\right| &\leq \frac{1}{n}\left|\left\{k \leq n : M\left(d\left(\frac{u_k}{v_k}, L\right)\right) \geq M(\varepsilon)\right\}\right| \\
&\leq \frac{1}{[\phi_s]+k}.\frac{[\phi_s]+k}{\phi_{s-1}}\left|\left\{k \leq \phi_s : M\left(d\left(\frac{u_k}{v_k}, L\right)\right) \geq M(\varepsilon)\right\}\right| \\
&\leq \frac{1}{[\phi_s]+k}.\frac{[\phi_s]+k}{\phi_{s-1}}\left|\left\{k \in \sigma, \sigma \in P_{[\phi_s]+k} : M\left(d\left(\frac{u_k}{v_k}, L\right)\right) \geq M(\varepsilon)\right\}\right| \\
&\leq \frac{K}{[\phi_s]+k}\left|\left\{k \in \sigma, \sigma \in P_{[\phi_s]+k} : M\left(d\left(\frac{u_k}{v_k}, L\right)\right) \geq M(\varepsilon)\right\}\right|.
\end{aligned}$$

This established the result. □

Theorem 4.5. *Let $(u_k), (v_k)$ be two sequences of fuzzy real numbers. Let M be an Orlicz function. Then*

(a) $(u_k) \overset{[C_1]^L(M)}{\sim} (v_k) \Rightarrow (u_k) \overset{[\phi]^L(M)}{\sim} (v_k)$.

(b) $\sup_s \frac{\phi_s}{\phi_{s-1}} < \infty$ *for every $s \in \mathbb{N}$, then* $(u_k) \overset{[\phi]^L(M)}{\sim} (v_k) \Rightarrow (u_k) \overset{[C_1]^L(M)}{\sim} (v_k)$.

Proof. (a) From definition of the sequence (ϕ_s) it follows that $\inf_s \frac{s}{s-\phi_s} \geq 1$. Then there exists $a > 0$ such that

$$\frac{s}{\phi_s} \leq \frac{1+a}{a}.$$

Then we get the following relation

$$\begin{aligned}
&\frac{1}{\phi_s}\sum_{k\in\sigma,\sigma\in P_s} M\left(d\left(\frac{u_k}{v_k}, L\right)\right) \\
&= \frac{s}{\phi_s}.\frac{1}{s}\sum_{k=1}^{n} M\left(d\left(\frac{u_k}{v_k}, L\right)\right) - \frac{1}{\phi_s}\sum_{k\in\{1,2,\ldots s\}-\sigma,\sigma\in P_s} M\left(d\left(\frac{u_k}{v_k}, L\right)\right) \\
&\leq \frac{1+a}{a}\frac{1}{s}\sum_{k=1}^{s} M\left(d\left(\frac{u_k}{v_k}, L\right)\right) - \frac{1}{\phi_s}\sum_{k_0\in\{1,2,\ldots s\}-\sigma,\sigma\in P_s} M\left(d\left(\frac{u_{k_0}}{v_{k_0}}, L\right)\right).
\end{aligned}$$

Since $(u_k) \overset{[C_1]^L(M)}{\sim} (v_k)$ and M is continuous, letting $s \to \infty$ on the last relation we get

$$\frac{1}{\phi_s}\sum_{k\in\sigma,\sigma\in P_s} M\left(d\left(\frac{u_k}{v_k},L\right)\right)\to 0.$$

Hence $(u_k)\overset{[\phi]^L(M)}{\sim}(v_k)$.
(b) Suppose that $\sup_s \frac{\phi_s}{\phi_{s-1}}<\infty$ then there exists $A>0$ such that $\frac{\phi_s}{\phi_{s-1}}<A$ for all $s\geq 1$. Suppose $(u_k)\overset{[\phi]^L(M)}{\sim}(v_k)$. Then for every $\varepsilon>0$ there exists $R>0$ such that for every $s\geq R$

$$\frac{1}{\phi_s}\sum_{k\in\sigma,\sigma\in P_s} M\left(d\left(\frac{u_k}{v_k},L\right)\right)<\varepsilon.$$

We can also find a constant $K>0$ such that

$$\frac{1}{\phi_s}\sum_{k\in\sigma,\sigma\in P_s} M\left(d\left(\frac{u_k}{v_k},L\right)\right)<K \text{ for all } s\in\mathbb{N}.$$

Let n be any integer with $\phi_{s-1}<n\leq[\phi_s]$ for every $s>R$. Then we have

$$\frac{1}{n}\sum_{k=1}^{n} M\left(d\left(\frac{u_k}{v_k},L\right)\right)\leq\frac{1}{\phi_{s-1}}\sum_{k=1}^{[\phi_s]} M\left(d\left(\frac{u_k}{v_k},L\right)\right)$$
$$=\frac{1}{\phi_{s-1}}\left(\sum_{k=1}^{[\phi_1]} M\left(d\left(\frac{u_k}{v_k},L\right)\right)+\sum_{[\phi_1]}^{[\phi_2]} M\left(d\left(\frac{u_k}{v_k},L\right)\right)+...+\sum_{[\phi_{s-1}]}^{[\phi_s]} M\left(d\left(\frac{u_k}{v_k},L\right)\right)\right)$$
$$\leq\frac{\phi_1}{\phi_{s-1}}\left(\frac{1}{\phi_1}\sum_{k\in\sigma,\sigma\in P^{(1)}} M\left(d\left(\frac{u_k}{v_k},L\right)\right)\right)+\frac{\phi_2}{\phi_{s-1}}\left(\frac{1}{\phi_2}\sum_{k\in\sigma,\sigma\in P^{(2)}} M\left(d\left(\frac{u_k}{v_k},L\right)\right)\right)+...$$
$$+\frac{\phi_R}{\phi_{s-1}}\left(\frac{1}{\phi_R}\sum_{k\in\sigma,\sigma\in P^{(R)}} M\left(d\left(\frac{u_k}{v_k},L\right)\right)\right)+...+\frac{\phi_s}{\phi_{s-1}}\left(\frac{1}{\phi_s}\sum_{k\in\sigma,\sigma\in P^{(s)}} M\left(d\left(\frac{u_k}{v_k},L\right)\right)\right),$$

where $P^{(t)}$ are sets of integer which have more than $[\phi_t]$ elements for $t\in\{1,2,...s\}$. By taking limit as $n\to\infty$ on the last relation we get

$$\frac{1}{n}\sum_{k=1}^{n} M\left(d\left(\frac{u_k}{v_k},L\right)\right)\to 0.$$

It follows that $(u_k)\overset{[C_1]^L(M)}{\sim}(v_k)$. □

Theorem 4.6. *Let $(u_k),(v_k)$ be two sequences of fuzzy real numbers. Let M be an Orlicz function. Then*

(a) $(u_k)\overset{[C_1]^L(M)}{\sim}(v_k)\Rightarrow(u_k)\overset{\mathcal{S}^L}{\sim}(v_k)$.
(b) *If M satisfies the Δ_2-condition and $(u_k)\in\ell_\infty^F(M)$ such that $(u_k)\overset{\mathcal{S}^L}{\sim}(v_k)$ then $(u_k)\overset{[C_1]^L(M)}{\sim}(v_k)$.*
(c) *If M satisfies the Δ_2-condition, then $[C_1]^L(M)\cap\ell_\infty^F(M)=\mathcal{S}^L\cap\ell_\infty^F(M)$, where $\ell_\infty^F(M)=\{(u_k)\in w^F: M(u_k)\in\ell_\infty^F\}$.*

Proof. (a) Suppose that $(u_k)\overset{[C_1]^L(M)}{\sim}(v_k)$. Then for every $\varepsilon>0$ we have

$$\frac{1}{n}\left|\left\{k\leq n: M\left(d\left(\frac{u_k}{v_k},L\right)\right)\geq M(\varepsilon)\right\}\right|$$

$$\leq \frac{1}{n} \sum_{\substack{k=1 \\ M\left(d\left(\frac{u_k}{v_k},L\right)\right) \geq M(\varepsilon)}}^{n} M\left(d\left(\frac{u_k}{v_k},L\right)\right) \leq \frac{1}{n}\sum_{k=1}^{n} M\left(d\left(\frac{u_k}{v_k},L\right)\right).$$

This established the result.
(b) The proof of this part follows from the same techniques used in the proofs of the Theorems 3.3 and 4.3.
(c) It follows from (a) and (b). □

Theorem 4.7. *Let $(u_k), (v_k)$ be two sequences of fuzzy real numbers. Let M be an Orlicz function. Then*

(a) $(u_k) \overset{[\phi]^L(M)}{\sim} (v_k) \Rightarrow (u_k) \overset{\mathcal{S}_\phi^L}{\sim} (v_k)$.

(b) *If M satisfies the Δ_2-condition and $(u_k) \in \ell_\infty^F(M)$ such that $(u_k) \overset{\mathcal{S}_\phi^L}{\sim} (v_k)$ then $(u_k) \overset{[\phi]^L(M)}{\sim} (v_k)$.*

(c) *If M satisfies the Δ_2-condition, then $[\phi]^L(M) \cap \ell_\infty^F(M) = \mathcal{S}_\phi^L \cap \ell_\infty^F(M)$.*

Proof. The proof of this theorem follows from the same techniques used in the proofs of the Theorems 3.3, 4.3 and 4.6. □

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