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Renormalized solutions for a class of nonlinear parabolic equations without sign condition involving nonstandard growth

Elhoussine Azroul, Mohamed Badr Benboubker, Hicham Redwane, and Chihab Yazough

Abstract. An existence result of a renormalized solution for a class of doubly nonlinear parabolic equations with variable exponents is established. The main contribution of our work is to prove the existence of a renormalized solution without the sign condition and the coercivity condition on the nonlinearity H. The second term f belongs to $L^1(Q)$ and $b(u_0) \in L^1(\Omega)$.

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1. Introduction

We consider a bounded open spatial domain $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ with a Lipschitz boundary denoted by $\partial\Omega$. Fixing a final time $T > 0$, we set $Q = \Omega\times]0, T[$.

The operator $Au = -\text{div}(a(x, t, u, \nabla u))$ is a Leray-Lions operator defined from the generalized Sobolev space V into its dual V^*(the two functionals spaces will be developed as bellow).

Our aim is to prove the existence of renormalized solutions u to the doubly nonlinear parabolic equation

$$\begin{cases} \frac{\partial b(u)}{\partial t} + Au + H(x,t,u,\nabla u) = f & \text{in } Q, \\ u = 0 & \text{on } \partial\Omega\times]0,T[, \\ b(u)(t=0) = b(u_0) & \text{on } \Omega. \end{cases} \tag{1.1}$$

where $f \in L^1(Q)$, the function b is assumed to be strictly increasing $\mathcal{C}^1$-function, and H is a nonlinear lower order term satisfying the growth condition of the form

$$|H(x,t,s,\xi)| \leq b(s)\Big(\sum_{i=1}^{N} |\xi_i|^{p(x)} + c(x,t)\Big). \tag{1.2}$$

Note that, the problem (1.1) was studied by Akdim et al. [3] in the framework of weighted Sobolev spaces where the nonlinearity H is just verified the growth condition with respect to ∇u, and in the case where $H(x, t, u, \nabla u) = \text{div}(\phi(x))$ the author proved the existence results in the classical Sobolev spaces and Orlicz spaces (see [19, 20]). Besides, Akdim et al. in [2] proved the existence of renormalized solutions in the weighted Sobolev spaces.

We recall that the notion of renormalized solutions was introduced in [13] by Diperna and Lions in their study of the Boltzman equation. This notion then adapted

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to the study of some nonlinear elliptic or parabolic problems and evolution problems in fluid mechanics. We refer to [10, 11, 16, 18] for more details.

In former paper (see [8]) we have already studied the corresponding unilateral elliptic problem with variable exponents involving lower order terms. In particular, we have established an existence result for entropy solutions of the stationary problem with L^1-data.

The aim of our paper is to extend the results in [7, 2] to the case of parabolic equations. Besides, this paper can be seen as a continuous of [10] in the case where $b(u) = u$, $a(x,t,s,\xi) = |\xi|^{p(x)-2}\xi$ and $H = 0$. As far as we know, there are no papers concerned with the doubly nonlinear parabolic equations with variable exponents. One of our motivations for studying (1.1) comes from applications to electrorheological fluids (see [21]), other important applications are related to image processing (see [12]) and elasticity see ([24]), etc. For the interested reader, we refer to [4, 5, 6, 7, 9, 8, 10] for the advances and the references in this area.

The paper is organized as follows. In section 2, we recall some basic notations and properties of Sobolev spaces with variable exponents. In section 3, we make precise all the assumptions on a, H, f and b, and we introduce the definition of a renormalized solution. In section 4, we give some technical results. In section 5, we prove the main result of this paper (Theorem 5.1) which is the existence of a renormalized solution.

2. Preliminaries

For each open bounded subset Ω of $I\!R^N$ $(N \geq 2)$, we denote

$$\mathcal{C}^+(\overline{\Omega}) = \{\text{continuous function} \quad p : \overline{\Omega} \longrightarrow I\!R^+ \quad \text{such that} \quad 1 < p_- \leq p_+ < \infty\},$$

where $p_- = \min_{x\in\Omega} p(x)$ and $p_+ = \max_{x\in\Omega} p(x)$. We define the variable exponent Lebesgue space for $p \in \mathcal{C}^+(\overline{\Omega})$ by:

$$L^{p(x)}(\Omega) = \{u : \Omega \longrightarrow I\!R \quad \text{measurable} \ / \ \int_\Omega |u(x)|^{p(x)}\,dx < \infty\},$$

endowed with the Luxemburg norm

$$\|u\|_{p(x)} = \inf\left\{\lambda > 0, \quad \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} \leq 1\right\}.$$

The variable exponent Lebesgue spaces resemble to the classical Lebesgue spaces in many respects: they are Banach spaces ([15]; Theorem 2.5), the Hölder inequality holds ([15]; Theorem 2.1), they are reflexive if and only if $1 < p_- \leq p_+ < \infty$ ([15]; Corollary 2.7) and continuous functions are dense, if $p_+ < \infty$ ([15]; Theorem 2.11). We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

Proposition 2.1. (see [14]) *If we denote*

$$\rho(u) = \int_\Omega |u|^{p(x)}\,dx, \quad \forall u \in L^{p(x)}(\Omega),$$

then the following assertions holds true:

(i) $\|u\|_{p(x)} < 1 \quad (resp, = 1, > 1) \quad \Leftrightarrow \quad \rho(u) < 1 \quad (resp, = 1, > 1)$,

(ii) $\|u\|_{p(x)} > 1 \quad \Rightarrow \quad \|u\|_{p(x)}^{p_-} \leq \rho(u) \leq \|u\|_{p(x)}^{p_+}$ *and* $\|u\|_{p(x)} < 1 \quad \Rightarrow \quad \|u\|_{p(x)}^{p_+} \leq \rho(u) \leq \|u\|_{p(x)}^{p_-}$,

(iii) $\|u_n\|_{p(x)} \to 0 \quad \Leftrightarrow \quad \rho(u_n) \to 0$ *and* $\|u_n\|_{p(x)} \to \infty \quad \Leftrightarrow \quad \rho(u_n) \to \infty$.

Extending a variable exponent $p : \bar{\Omega} \to [1, \infty)$ to $\bar{Q} = \Omega \times [0, T]$ by setting $p(t, x) := p(x)$ for all $(t, x) \in \bar{Q}$, we may also consider the generalized Lebesgue space

$$L^{p(x)}(Q) = \{u : Q \longrightarrow I\!\!R \quad \text{measurable } / \int_Q |u(t, x)|^{p(x)} \, d(t, x) < \infty\},$$

endowed with the norm

$$\|u\|_{L^{p(x)}(Q)} = \inf\left\{\lambda > 0, \quad \int\int_Q \left|\frac{u(t, x)}{\lambda}\right|^{p(x)} d(t, x) \leq 1\right\}$$

which, of course, shares the same type of properties as $L^{p(x)}(\Omega)$.

We define the variable Sobolev space by

$$W^{1,p(x)}(\Omega) = \{ u \in L^{p(x)}(\Omega) \text{ and } |\nabla u| \in L^{p(x)}(\Omega)\}.$$

It is endowed with the following norm,

$$\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)} \qquad \forall u \in W^{1,p(x)}(\Omega).$$

We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ and $p^*(x) = \frac{N\, p(x)}{N - p(x)}$ for $p(x) < N$.

Proposition 2.2. (see [14]) (***i***) *Assuming* $1 < p_- \leq p_+ < \infty$, *the spaces* $W^{1,p(x)}(\Omega)$ *and* $W_0^{1,p(x)}(\Omega)$ *are separable and reflexive Banach spaces.*
(***ii***) *If* $q \in \mathcal{C}^+(\bar{\Omega})$ *and* $q(x) < p^*(x)$ *for any* $x \in \overline{\Omega}$, *then the embedding* $W_0^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}(\Omega)$ *is compact and continuous.*
(***iii***) *There is a constant* $C > 0$, *such that*

$$\|u\|_{p(x)} \leq C\, \|\nabla u\|_{p(x)} \quad \forall u \in W_0^{1,p(x)}(\Omega).$$

We are naturally let introduce the functional space

$$V = \{f \in L^{p^-}(0, T, W_0^{1,p(x)}(\Omega)) : |\nabla f| \in L^{p(x)}(Q)\},$$

endowed with the norm

$$\|f\|_V := \|\nabla f\|_{L^{p(x)(Q)}}$$

or

$$|||f|||_V := \|f\|_{L^{p^-}(0,T,W_0^{1,p(x)}(\Omega))} + \|\nabla f\|_{L^{p(x)}(Q)}$$

We have used the standard notations for Bochner spaces, i.e. if X is a Banach space and $q \geq 1$, then $L^q(0, T, X)$ denotes the space of strongly measurable function $u : (0, T) \to X$ for which $t \to ||u(t)||_X \in L^q(0, T)$. Moreover, $\mathcal{C}([0, T]; X)$ denotes the space of continuous functions $u : [0, T] \to X$ endowed with the norm $||u||_{\mathcal{C}([0,T];X)} := \max_{t \in [0,T]} ||u(t)||_X$.

Lemma 2.1. (1) *In the preceding definition as well as in the following, we identify,* V *and its dual by* V^*, *then we have the following continuous embeddings*

$$L^{p^+}(0, T, W_0^{1,p(x)}(\Omega)) \hookrightarrow V \hookrightarrow L^{p^-}(0, T, W_0^{1,p(x)}(\Omega)).$$

In particular, since $\mathcal{D}(Q)$ *is dense in* $L^{p^+}(0, T, W_0^{1,p(x)}(\Omega))$, *it is dense in* V *and for the corresponding dual spaces, we have*

$$L^{(p^-)'}(0, T, W^{-1,p'(x)}(\Omega)) \hookrightarrow V^* \hookrightarrow L^{(p^+)'}(0, T, W^{-1,p'(x)}(\Omega)).$$

(2) *One can represent the elements of* V^* *as follows: if* $T \in V^*$, *then there exists* $F = (f_1, ..., f_N) \in (L^{p'(x)}(Q))^N$ *such that* $T = div\ F$ *and*

$< T,\zeta >_{V^*,V} = \int_0^T \int_\Omega F.\nabla\zeta dxdt$ *for any* $\zeta \in V$.
Moreover, we have

$$\|T\|_{V^*} := \max\{\|f_i\|_{L^{p'(x)}(Q)}, i = 1, ..., N\}.$$

3. Basic assumptions

Throughout the paper, we assume that the following assumptions hold true:

Assumption (A1). $b : I\!R \to I\!R$ is a strictly increasing $\mathcal{C}^1$ − function with $b(0) = 0$. (3.1)

Assumption (A2). $a : Q \times I\!R \times I\!R^N \to I\!R^N$ is a Carathéodory function satisfying the following conditions:
for almost every $(x,t) \in Q$, for every $s \in I\!R$,

$$|a(x,t,s,\xi)| \le \beta(k(x,t) + |s|^{p(x)-1} + |\xi|^{p(x)-1}), \tag{3.2}$$

$$[a(x,t,s,\xi) - a(x,t,s,\eta)](\xi - \eta) > 0, \text{ for all } \xi \neq \eta \in I\!R^N, \tag{3.3}$$

$$a(x,t,s,\xi)\xi \ge \alpha|\xi|^{p(x)}, \tag{3.4}$$

where $k(x,t)$ is a positive function lying in $L^{p'(x)}(Q)$ and $\alpha, \beta > 0$.
Assumption (A3). $H : \Omega \times [0,T] \times I\!R \times I\!R^N \to I\!R$ is a Carathéodory function such that for a.e. $(x,t) \in Q$ and for all $s \in I\!R, \xi \in I\!R^N$, the growth condition

$$|H(x,t,s,\xi)| \le \gamma(x,t) + g(s)\sum_{i=1}^{N}|\xi_i|^{p(x)}, \tag{3.5}$$

is satisfied, where $g : I\!R \to I\!R^+$ is a continuous positive function that belongs to $L^1(I\!R)$, while $\gamma(x,t)$ belongs to $L^1(Q)$.

4. Some technical results

Characterization of the time mollification of a function u**.** To deal with time derivative, we introduce a time mollification of a function u belonging to a some Lebesgue space. Thus we define for all $\mu \ge 0$ and all $(x,t) \in Q$,

$$u_\mu = \mu \int_\infty^t \tilde{u}(x,s)\exp(\mu(s-t))ds$$

where $\tilde{u}(x,s) = u(x,s)\chi_{(0,T)}(s)$.
Note that in this section, we omit the proof of each of the above proposition and lemmas, since it is a slight modification of its analogous in [1].

Proposition 4.1. (see [1]) (1) *If* $u \in L^{p(x)}(Q)$, *then* u_μ *is measurable in* Q, $\frac{\partial u_\mu}{\partial t} = \mu(u - u_\mu)$ *and*

$$\|u_\mu\|_{L^{p(x)}(Q)} \le \|u\|_{L^{p(x)}(Q)}.$$

(2) *If* $u \in W_0^{1,p(x)}(Q)$, *then* $u_\mu \to u$ *in* $W_0^{1,p(x)}(Q)$ *as* $\mu \to \infty$.
(3) *If* $u_n \to u$ *in* $W_0^{1,p(x)}(Q)$, *then* $(u_n)_\mu \to u_\mu$ *in* $W_0^{1,p(x)}(Q)$.

Some embedding and compactness results. In this section we establish some embedding and compactness results in generalized Sobolev spaces. Let $X = W_0^{1,p(x)}(\Omega)$, $H = L^2(\Omega)$ and let $X^* = W^{-1,p'(x)}(\Omega)$, with $(2 \leq p^- < \infty)$. Denoting the space $W^1_{p(x)}(0,T,X,H) = \{v \in V : v' \in V^*\}$ endowed with the norm

$$\|u\|_{W^1_{p(x)}} = \|u\|_V + \|u'\|_{V^*},$$

which is a Banach space. Here u' stands for the generalized derivative of u; i.e.,

$$\int_0^T u'(t)\varphi(t)dt = -\int_0^T u(t)\varphi'(t)dt \quad \text{for all } \varphi \in C_0^\infty(0,T).$$

Lemma 4.1. (see [23]) (1) *The evolution triple* $X \subseteq H \subseteq X^*$ *is satisfied.*
(2) *The embedding* $W^1_{p(x)}(0,T,X,H) \subseteq \mathcal{C}(0,T,H)$ *is continuous.*
(3) *The embedding* $W^1_{p(x)}(0,T,X,H) \subseteq L^{p(x)}(Q)$ *is compact.*
(4) *The evolution triple* $L^{p^+}(0,T;L^{p(x)}(\Omega)) \subseteq L^{p(x)}(Q) \subseteq L^{p^-}(0,T;L^{p(x)}(\Omega))$ *is satisfied.*

Lemma 4.2. (see [1]) *Let* $g \in L^{r(x)}(Q)$ *and* $g_n \in L^{r(x)}(Q)$ *with* $\|g_n\|_{L^{r(x)}(Q)} \leq C$ *for* $1 < r < \infty$. *If* $g_n(x) \to g(x)$ *a.e. in* Q, *then* $g_n \rightharpoonup g$ *in* $L^{r(x)}(Q)$.

Lemma 4.3. (see [1]) *Assume that*

$$\frac{\partial v_n}{\partial t} = \alpha_n + \beta_n \quad \text{in } \mathcal{D}'(Q)$$

where α_n *and* β_n *are bounded respectively in* V^* *and in* $L^1(Q)$. *If* v_n *is bounded in* V, *then* $v_n \to u$ *in* $L^{p(x)}_{\text{loc}}(Q)$. *Further* $v_n \to v$ *strongly in* $L^1(Q)$ *where* $n \to \infty$.

Lemma 4.4. (see [1]) *Assume that* (3.2) - (3.4) *are satisfied and let* (u_n) *be a sequence in* V *such that* $u_n \rightharpoonup u$ *weakly in* V *and*

$$\int_Q [a(x,t,u_n,\nabla u_n) - a(x,t,u,\nabla u)][\nabla u_n - \nabla u]\,dx\,dt \to 0. \tag{4.1}$$

Then, $u_n \to u$ *in* V.

5. Existence result

Definition 5.1. Let $f \in L^1(Q)$ and $b(u_0) \in L^1(\Omega)$. A real-valued function u defined on Q is a renormalized solution of problem 1.1 if

$$T_k(u) \in V \quad \text{for all } k \geq 0 \text{ and } b(u) \in L^\infty(0,T;L^1(\Omega)), \tag{5.1}$$

$$\int_{\{m \leq |u| \leq m+1\}} a(x,t,u,\nabla u)\nabla u\,dx\,dt \to 0 \quad \text{as } m \to +\infty, \tag{5.2}$$

$$\begin{aligned}\frac{\partial B_S(u)}{\partial t} - \operatorname{div}(S'(u)a(x,t,u,\nabla u)) + S''(u)a(x,t,u,\nabla u)\nabla u + H(x,t,u,\nabla u)S'(u)\\ = fS'(u) \quad \text{in } \mathcal{D}'(Q),\end{aligned} \tag{5.3}$$

for all functions $S \in W^{2,\infty}(I\!R)$ which is piecewise $\mathcal{C}^1$ and such that S' has a compact support in $I\!R$, where $B_S(z) = \int_0^z \frac{\partial b(r)}{\partial r} S'(r)dr$ and

$$B_S(u)(t=0) = B_S(u_0) \quad \text{in } \Omega. \tag{5.4}$$

Remark 5.1. Equation (5.3) is formally obtained through pointwise multiplication of (1.1) by $S'(u)$. All the terms in (5.3) have a meaning in $\mathcal{D}'(Q)$.

Now we announce the main result of this section.

Theorem 5.1. *Let $f \in L^1(Q)$ and $b(u_0) \in L^1(\Omega)$. Assume that* (**A1**)-(**A3**) *hold true. Then, there exists at least one renormalized solution u of problem* (1.1)

The proof of this theorem is divided into 4 steps.

Step 1. A priori estimates in Generalized Lebesgue spaces. For $n > 0$, let us define the following approximation of b, H, f and u_0:

$$b_n(r) = b(T_n(r)) + \frac{1}{n}r \quad \text{for } n > 0, \tag{5.5}$$

$$H_n(x,t,s,\xi) = \frac{H(x,t,s,\xi)}{1 + \frac{1}{n}|H(x,t,s,\xi)|}.$$

$$f_n \in V^* \quad \text{and} \quad f_n \to f \quad \text{a.e. in } Q \text{ and strongly in } L^1(Q) \text{ as } n \to +\infty, \tag{5.6}$$

$$u_{0n} \in \mathcal{D}(\Omega), \quad \|b_n(u_{0n})\|_{L^1} \le \|b(u_0)\|_{L^1}, \tag{5.7}$$

$$b_n(u_{0n}) \to b(u_0) \text{ a.e. in } \Omega \text{ and strongly in } L^1(\Omega). \tag{5.8}$$

Then we consider the approximate problem:

$$\begin{aligned} &\frac{\partial b_n(u_n)}{\partial t} - \operatorname{div}(a(x,t,u_n,\nabla u_n)) + H_n(x,t,u_n,\nabla u_n) = f_n \quad \text{in } \mathcal{D}'(Q),\\ &u_n = 0 \quad \text{in } (0,T)\times\partial\Omega,\\ &b_n(u_n(t=0)) = b_n(u_{0n}). \end{aligned} \tag{5.9}$$

Note that $H_n(x,t,s,\xi)$ satisfies the following conditions

$$|H_n(x,t,s,\xi)| \le H(x,t,s,\xi) \quad \text{and} \quad |H_n(x,t,s,\xi)| \le n.$$

Moreover, since $f_n \in V^*$, proving existence of a weak solution $u_n \in V$ of (5.9) is an easy task (see [17]).

Let $\varphi \in V \cap L^\infty(Q)$ with $\varphi > 0$, choosing $v = \exp(G(u_n))\varphi$ as a test function in (5.9) with $G(s) = \int_0^s \frac{g(r)}{\alpha} dr$ (the function g appears in (3.5)). We have

$$\begin{aligned} &\int_Q \frac{\partial b_n(u_n)}{\partial t} \exp(G(u_n))\varphi\,dx\,dt + \int_Q a(x,t,u_n,\nabla u_n)\nabla(\exp(G(u_n))\varphi)\,dx\,dt\\ &\quad = \int_Q H_n(x,t,u_n,\nabla u_n)\exp(G(u_n))\varphi\,dx\,dt + \int_Q f_n \exp(G(u_n))\varphi\,dx\,dt. \end{aligned}$$

According to (3.5), we obtain

$$\begin{aligned} &\int_Q \frac{\partial b_n(u_n)}{\partial t} \exp(G(u_n))\varphi\,dx\,dt + \int_Q a(x,t,u_n,\nabla u_n)\nabla u_n \frac{g(u_n)}{\alpha}\exp(G(u_n))\varphi\,dx\,dt\\ &\quad + \int_Q a(x,t,u_n,\nabla u_n)\exp(G(u_n))\nabla\varphi\,dx\,dt\\ &\quad \le \int_Q \gamma(x,t)\exp(G(u_n))\varphi\,dx\,dt + \int_Q g(u_n)\sum_{i=1}^N \Big|\frac{\partial u_n}{\partial x_i}\Big| \exp(G(u_n))\varphi dx dt\\ &\quad + \int_Q f_n \exp(G(u_n))\varphi\,dx\,dt. \end{aligned}$$

From (3.4), we obtain

$$\begin{aligned}&\int_Q \frac{\partial b_n(u_n)}{\partial t}\exp(G(u_n))\varphi\,dx\,dt+\int_Q a(x,t,u_n,\nabla u_n)\exp(G(u_n))\nabla\varphi\,dx\,dt\\&\quad\le\int_Q\gamma(x,t)\exp(G(u_n))\varphi\,dx\,dt+\int_Q f_n\exp(G(u_n))\varphi\,dx\,dt,\end{aligned}\tag{5.10}$$

for all $\varphi\in V\cap L^\infty(Q),\varphi>0$. On the other hand, taking $v=\exp(-G(u_n))\varphi$ as a test function in (5.9), we deduce, as in (5.10), that

$$\begin{aligned}&\int_Q \frac{\partial b_n(u_n)}{\partial t}\exp(-G(u_n))\varphi\,dx\,dt+\int_Q a(x,t,u_n,\nabla u_n)\exp(-G(u_n))\nabla\varphi\,dx\,dt\\&\qquad+\int_Q\gamma(x,t)\exp(-G(u_n))\varphi\,dx\,dt\\&\quad\ge\int_Q f_n\exp(-G(u_n))\varphi\,dx\,dt,\end{aligned}\tag{5.11}$$

for all $\varphi\in V\cap L^\infty(Q),\varphi>0$.
For every $\tau\in[0,T]$, let $\varphi=T_k(u_n)^+\chi_{(0,\tau)}$, in (5.11) we have,

$$\begin{aligned}&\int_\Omega B_k^n(u_n(\tau))\exp(G(u_n))dx+\int_{Q_\tau}a(x,t,u_n,\nabla u_n)\exp(G(u_n))\nabla T_k(u_n)^+\,dx\,dt\\&\quad\le\int_{Q_\tau}\gamma(x,t)\exp(G(u_n))T_k(u_n)^+\,dx\,dt+\int_{Q_\tau}f_n\exp(G(u_n))T_k(u_n)^+\,dx\,dt\\&\qquad+\int_\Omega B_k^n(u_{0n})dx,\end{aligned}\tag{5.12}$$

where $B_k^n(r)=\int_0^r T_k(s)^+\frac{\partial b_n(s)}{\partial s}ds$. Due to this definition, we have

$$0\le\int_\Omega B_k^n(u_{0n})dx\le k\int_\Omega|b_n(u_{0n})|dx\le k\|b(u_0)\|_{L^1(\Omega)}.\tag{5.13}$$

Using the above result, $B_k^n(u_n)\ge 0$ and $G(u_n)\le\frac{\|g\|_{L^1(I\!R)}}{\alpha}$, we get

$$\begin{aligned}&\int_{Q_\tau}a(x,t,u_n,\nabla T_k(u_n)^+)\nabla T_k(u_n)^+\exp(G(u_n))\,dx\,dt\\&\quad\le k\exp\Big(\frac{\|g\|_{L^1(I\!R)}}{\alpha}\Big)\Big(\|f_n\|_{L^1(Q)}+\|\gamma\|_{L^1(Q)}+\|b_n(u_{0n})\|_{L^1(\Omega)}\Big)\\&\quad\le c_1k.\end{aligned}$$

Thanks to (3.4), we conclude that

$$\alpha\int_{Q_\tau}\sum_{i=1}^N\Big|\frac{\partial T_k(u_n)^+}{\partial x_i}\Big|^{p(x)}\exp(G(u_n))\,dx\,dt\le c_1k.\tag{5.14}$$

Hence

$$\alpha\int_Q\sum_{i=1}^N\Big|\frac{\partial T_k(u_n)^+}{\partial x_i}\Big|^{p(x)}\,dx\,dt\le c_1k.\tag{5.15}$$

Similarly to (5.15), choosing $\varphi=T_k(u_n)^-\chi_{(0,\tau)}$ as a test function in (5.11) leads to

$$\alpha\int_Q\sum_{i=1}^N\Big|\frac{\partial T_k(u_n)^-}{\partial x_i}\Big|^{p(x)}\,dx\,dt\le c_2k\tag{5.16}$$

where c_2 is a positive constant.
Combining (5.15) and (5.16), we conclude that

$$\|T_k(u_n)\|_V^\gamma \leq ck, \tag{5.17}$$

where

$$\gamma = \begin{cases} \frac{1}{p^-} & \text{if } \|\nabla T_k(u_n)\|_{L^{p(x)}(Q)} > 1, \\ \frac{1}{p^+} & \text{if } \|\nabla T_k(u_n)\|_{L^{p(x)}(Q)} \leq 1. \end{cases} \tag{5.18}$$

The above inequality together with (5.12) and (5.13) make it possible to obtain

$$\int_\Omega B_k^n(u_n)dx \leq k(\|f\|_{L^1(Q)} + \|b(u_0)\|_{L^1(\Omega)}) \equiv Ck. \tag{5.19}$$

Since $T_k(u_n)$ is bounded in V, there exists some v_k such that

$$T_k(u_n) \rightharpoonup v_k \text{ in the space } V,$$

and by the compact embedding (see Lemma (4.1)), we have

$$T_k(u_n) \to v_k \quad \text{strongly in } L^{p(x)}(Q) \text{ and a.e. in } Q.$$

Let $k > 0$ be large enough. Combining the generalized Hölder's inequality and Poincaré inequality, one has

$$\begin{aligned} k\,\text{meas}(\{|u_n| > k\} \times [0,T]) &= \int_0^T \int_{\{|u_n|>k\}} |T_k(u_n)|\,dx\,dt \\ &\leq \int_0^T \int_\Omega |T_k(u_n)|\,dx\,dt \\ &\leq (\frac{1}{p^-} + \frac{1}{p'^-})\|T_k(u_n)\|_{L^{p(x)}(Q)}\|1\|_{L^{p'(x)}(Q)} \\ &\leq c\|\nabla T_k(u_n)\|_{L^{p(x)}(Q)} \\ &\leq ck^\gamma, \end{aligned}$$

which yields,

$$\text{meas}(\{|u_n| > k\} \times [0,T]) \leq \frac{c_1}{k^{1-\gamma}}, \quad \forall k \geq 1.$$

Moreover, we have

$$\lim_{k\to+\infty} (\text{meas}(\{|u_n| > k\} \times [0,T])) = 0.$$

Now we turn to prove the almost every convergence of u_n and $b_n(u_n)$. Consider now a non decreasing function $g_k \in \mathcal{C}^2(I\!R)$ such that $g_k(s) = s$ for $|s| \leq \frac{k}{2}$ and $g_k(s) = k$ for $|s| \geq k$. Multiplying the approximate equation by $g'_k(b_n(u_n))$, we get

$$\begin{aligned} &\frac{\partial g_k(b_n(u_n))}{\partial t} - \text{div}(a(x,t,u_n,\nabla u_n)g'_k(b_n(u_n))) \\ &\quad + a(x,t,u_n,\nabla u_n)g''_k(b_n(u_n))b'_n(u_n)\nabla u_n + H_n(x,t,u_n,\nabla u_n)g'_k(b_n(u_n)) \\ &= f_n g'_k(b_n(u_n)) \end{aligned} \tag{5.20}$$

in the sense of distributions, which implies that

$$g_k(b_n(u_n)) \text{ is bounded in } V \tag{5.21}$$

and

$$\frac{\partial g_k(b_n(u_n))}{\partial t} \text{ is bounded in } V^* + L^1(Q), \tag{5.22}$$

independently of n as soon as $k < n$.
Due to Definition (3.1) and (5.5) of b_n, it is clear that

$$\{|b_n(u_n)| \le k\} \subset \{|u_n| \le k^*\}$$

as soon as $k < n$ and k^* is a constant independent of n.
As a first consequence we have

$$\nabla g_k(b_n(u_n)) = g'_k(b_n(u_n))b'_n(T_{k^*}(u_n))\nabla T_{k^*}(u_n) \quad \text{a.e. in } Q \quad \text{as long as } k < n. \tag{5.23}$$

Secondly, the following estimate holds true

$$\|g'_k(b_n(u_n))b'_n(T_{k^*}(u_n))\|_{L^\infty(Q)} \le \|g'_k\|_{L^\infty(Q)}(\max_{|r|\le k^*}(b'(r)) + 1).$$

As a consequence of (5.17) and (5.23), we then obtain (5.21).
To show that (5.22) holds, we use (5.20) to obtain

$$\begin{aligned}\frac{\partial g_k(b_n(u_n))}{\partial t} &= \operatorname{div}(a(x,t,u_n,\nabla u_n)g'_k(b_n(u_n))) - a(x,t,u_n,\nabla u_n)g''_k(b_n(u_n))b'_n(u_n)\nabla u_n \\ &\quad - H_n(x,t,u_n,\nabla u_n)g'_k(b_n(u_n)) + f_n g'_k(b_n(u_n)).\end{aligned} \tag{5.24}$$

Each term in the right hand side of (5.24) is bounded either in V^* or in $L^1(Q)$. Actually, since supp g'_k and supp g''_k are both included in $[-k,k]$, u_n may be replaced by $T_{k^*}(u_n)$ in each of these terms. As a consequence, Lemma 4.3 allows us to conclude that $g_k(b_n(u_n))$ is compact in $L^{p(x)}_{\text{loc}}(Q)$.
Thus, for a subsequence, it also converges in measure and almost every where in Q, due to the choice of g_k, we conclude that for each k, the sequence $T_k(b_n(u_n))$ converges almost everywhere in Q (since we have, for every $\lambda > 0$)

$$\begin{aligned}&\text{meas}(\{|b_n(u_n) - b_m(u_m)| > \lambda\} \times [0,T]) \le \text{meas}(\{|b_n(u_n)| > k\} \times [0,T]) \\ &\quad + \text{meas}(\{|b_m(u_m)| > k\} \times [0,T]) + \text{meas}(\{|g_k(b_n(u_n)) - g_k(b_m(u_m))| > \lambda\}).\end{aligned}$$

Let $\varepsilon > 0$, then there exist $k(\varepsilon) > 0$ such that

$$\text{meas}(\{|b_n(u_n) - b_m(u_m)| > \lambda\} \times [0,T]) \le \varepsilon$$

for all $n, m \ge n_0(k(\varepsilon), \lambda)$. This proves that $(b_n(u_n))$ is a Cauchy sequence in measure in $\Omega \times [0,T]$, thus converges almost everywhere to some measurable function v. Then for a subsequence denoted again u_n,

$$u_n \to u \quad \text{a.e. in } Q, \tag{5.25}$$

$$b_n(u_n) \to b(u) \quad \text{a.e. in } Q. \tag{5.26}$$

We can deduce from (5.17) that

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in } V \tag{5.27}$$

and then, the compact embedding (4.1) gives

$$T_k(u_n) \to T_k(u) \quad \text{strongly in } L^{p(x)}(Q) \text{ and a.e. in } Q.$$

Which implies, by using (3.2), for all $k > 0$ that there exists a function $h_k \in \prod_{i=1}^N L^{p'(x)}(Q)$, such that

$$a(x,t,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup h_k \quad \text{weakly in } \prod_{i=1}^N L^{p'(x)}(Q). \tag{5.28}$$

We now establish that $b(u)$ belongs to $L^\infty(0,T;L^1(\Omega))$. Using (5.25) and passing to the limit-inf in (5.19) as n tends to $+\infty$, we obtain that

$$\frac{1}{k}\int_\Omega B_k(u)(\tau)dx \leq [\|f\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)}] \equiv C,$$

for almost any τ in $(0,T)$. Due to the definition of $B_k(s)$ and the fact that $\frac{1}{k}B_k(u)$ converges pointwise to $b(u)$, as k tends to $+\infty$, shows that $b(u)$ belong to $L^\infty(0,T;L^1(\Omega))$.

Lemma 5.1. *Let u_n be a solution of the approximate problem* (5.9). *Then*

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{m\leq|u_n|\leq m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt = 0. \tag{5.29}$$

Proof. Considering the function $\varphi = T_1(u_n - T_m(u_n))^- := \alpha_m(u_n)$ in (5.11), this function is admissible since $\varphi \in V$ and $\varphi \geq 0$. Then, we have

$$\begin{aligned}&\int_Q \frac{\partial b_n(x,u_n)}{\partial t}\alpha_m(u_n)\,dx\,dt + \int_{\{-(m+1)\leq u_n\leq -m\}} a(x,t,u_n,\nabla u_n)\nabla u_n\alpha'_m(u_n)\,dx\,dt\\ &\quad + \int_Q f_n\exp(-G(u_n))\alpha_m(u_n)\,dx\,dt\\ &\quad \leq \int_Q \gamma(x,t)\exp(-G(u_n))\alpha_m(u_n)\,dx\,dt.\end{aligned}$$

Setting $B_n^m(x,r) = \displaystyle\int_0^r \frac{\partial b_n(s)}{\partial s}\alpha_m(s)ds$, permit us to write

$$\begin{aligned}&\int_\Omega B_n^m(u_n)(T)dx + \int_{\{-(m+1)\leq u_n\leq -m\}} a(x,t,u_n,\nabla u_n)\nabla u_n\alpha'_m(u_n)\,dx\,dt\\ &\quad + \int_Q f_n\exp(-G(u_n))\alpha_m(u_n)\,dx\,dt\\ &\quad \leq \int_Q \gamma(x,t)\exp(-G(u_n))\alpha_m(u_n)\,dx\,dt + \int_\Omega B_n^m(u_{0n})dx.\end{aligned}$$

Since $B_n^m(u_n)(T) \geq 0$ and by Lebesgue's theorem, we have

$$\lim_{m\to\infty}\lim_{n\to\infty}\int_Q f_n\exp(-G(u_n))\alpha_m(u_n)\,dx\,dt = 0. \tag{5.30}$$

Similarly, since $\gamma \in L^1(\Omega)$, we obtain

$$\lim_{m\to\infty}\lim_{n\to\infty}\int_Q \gamma\exp(-G(u_n))\alpha_m(u_n)\,dx\,dt = 0. \tag{5.31}$$

Therefore,

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{-(m+1)\leq u_n\leq -m\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt = 0. \tag{5.32}$$

On the other hand, let $\varphi = T_1(u_n - T_m(u_n))^+$ as a test function in (5.10) and reasoning as in the proof of (5.32) we deduce that

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{m)\leq u_n\leq m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt = 0. \tag{5.33}$$

Thus (5.29) follows from (5.32) and (5.33). $\square$

Step 2. Almost everywhere convergence of the gradients. This step is devoted to introduce for $k \geq 0$ fixed, a time regularization of the function $T_k(u)$, in order to perform the monotonicity method.
Let $\psi_i \in \mathcal{D}(\Omega)$ be a sequence which converge strongly to u_0 in $L^1(\Omega)$. Set $w^i_\mu = (T_k(u))_\mu + e^{-\mu t} T_k(\psi_i)$ where $(T_k(u))_\mu$ is the mollification with respect to time of $T_k(u)$. Note that w^i_μ is a smooth function having the following properties:

$$\frac{\partial w^i_\mu}{\partial t} = \mu(T_k(u) - w^i_\mu), \quad w^i_\mu(0) = T_k(\psi_i) \quad \text{and } \left|w^i_\mu\right| \leq k, \tag{5.34}$$

$$w^i_\mu \to T_k(u) \quad \text{in } V \text{ as } \mu \to \infty. \tag{5.35}$$

We introduce the following function of one real

$$h_m(s) = \begin{cases} 1 & \text{if } |s| \leq m \\ 0 & \text{if } |s| \geq m+1 \\ m+1-s & \text{if } m \leq s \leq m+1 \\ m+1+s & \text{if } -(m+1) \leq s \leq -m \end{cases}$$

where $m > k$.
Let $\varphi = (T_k(u_n) - w^i_\mu)^+ h_m(u_n) \in V \cap L^\infty(Q)$ and $\varphi \geq 0$, then we take this function in (5.10), to write

$$\begin{aligned} &\int_{\{T_k(u_n) - w^i_\mu \geq 0\}} \frac{\partial b_n(x,u_n)}{\partial t} \exp(G(u_n))(T_k(u_n) - w^i_\mu) h_m(u_n)\,dx\,dt \\ &\quad + \int_{\{T_k(u_n) - w^i_\mu \geq 0\}} a(x,t,u_n,\nabla u_n)\nabla(T_k(u_n) - w^i_\mu) h_m(u_n)\,dx\,dt \\ &\quad - \int_{\{m \leq u_n \leq m+1\}} \exp(G(u_n)) a(x,t,u_n,\nabla u_n)\nabla u_n (T_k(u_n) - w^i_\mu)^+\,dx\,dt \\ &\leq \int_Q \gamma(x,t)\exp(G(u_n))(T_k(u_n) - w^i_\mu)^+ h_m(u_n)\,dx\,dt \\ &\quad + \int_Q f_n \exp(G(u_n))(T_k(u_n) - w^i_\mu)^+ h_m(u_n)\,dx\,dt. \end{aligned} \tag{5.36}$$

Observe that

$$\begin{aligned} &\int_{\{m \leq u_n \leq m+1\}} \exp(G(u_n)) a(x,t,u_n,\nabla u_n)\nabla u_n (T_k(u_n) - w^i_\mu)^+\,dx\,dt \\ &\quad \leq 2k \int_{\{m \leq u_n \leq m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt. \end{aligned}$$

Thanks to (5.29) the third integral tends to zero as n and m go to infinity, and by Lebesgue's theorem, we deduce that the right hand side converges to zero as n, m and μ go to infinity. Since

$$(T_k(u_n) - w^i_\mu)^+ h_m(u_n) \rightharpoonup (T_k(u) - w^i_\mu)^+ h_m(u) \quad \text{weakly–* in } L^\infty(Q) \text{ as } n \to \infty$$
$$\text{and } (T_k(u) - w^i_\mu)^+ h_m(u) \rightharpoonup 0 \quad \text{weakly–* in } L^\infty(Q) \text{ as } \mu \to \infty.$$

Let $\varepsilon_l(n,m,\mu,i)$, $l = 1,\ldots,n$ various functions which converge to zero as n, m, i and μ tend to infinity.
The definition of the sequence w^i_μ makes it possible to establish the following Lemma.

Lemma 5.2. *For $k \geq 0$, we have*

$$\int_{\{T_k(u_n)-w_\mu^i\geq 0\}} \frac{\partial b_n(u_n)}{\partial t} \exp(G(u_n))(T_k(u_n) - w_\mu^i)h_m(u_n)\,dx\,dt \geq \varepsilon(n,m,\mu,i). \tag{5.37}$$

Proof. The proof of this Lemma is a slight modification of the analogues one of [19]. □

On the other hand, the second term of left hand side of (5.36) reads as follows

$$\begin{aligned}
&\int_{\{T_k(u_n)-w_\mu^i\geq 0\}} a(x,t,u_n,\nabla u_n)\nabla(T_k(u_n) - w_\mu^i)h_m(u_n)\,dx\,dt \\
&\quad= \int_{\{T_k(u_n)-w_\mu^i\geq 0,|u_n|\leq k\}} a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla(T_k(u_n) - w_\mu^i)h_m(u_n)\,dx\,dt \\
&\quad\quad- \int_{\{T_k(u_n)-w_\mu^i\geq 0,|u_n|\geq k\}} a(x,t,u_n,\nabla u_n)\nabla w_\mu^i h_m(u_n)\,dx\,dt.
\end{aligned}$$

Since $m > k$, $h_m(u_n) = 0$ on $\{|u_n| \geq m+1\}$, one has

$$\begin{aligned}
&\int_{\{T_k(u_n)-w_\mu^i\geq 0\}} a(x,t,u_n,\nabla u_n)\nabla(T_k(u_n) - w_\mu^i)h_m(u_n)\,dx\,dt \\
&\quad= \int_{\{T_k(u_n)-w_\mu^i\geq 0\}} a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla(T_k(u_n) - w_\mu^i)h_m(u_n)\,dx\,dt \\
&\quad\quad- \int_{\{T_k(u_n)-w_\mu^i\geq 0,|u_n|\geq k\}} a(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\nabla w_\mu^i h_m(u_n)\,dx\,dt \\
&\quad= J_1 + J_2.
\end{aligned} \tag{5.38}$$

In the following we pass to the limit in (5.38): letting first n goes to $+\infty$, then μ and finally m tend toward $+\infty$. Since $a(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n))$ is bounded in $\prod_{i=1}^N L^{p'(x)}(Q)$, we have that

$$a(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n))h_m(u_n)\chi_{\{|u_n|>k\}} \to h_m h_m(u)\chi_{\{|u|>k\}}$$

strongly in $\prod_{i=1}^N L^{p'(x)}(Q)$ as n tends to infinity, it follows that

$$\begin{aligned}
J_2 &= \int_{\{T_k(u_n)-w_\mu^i\geq 0\}} h_m \nabla w_\mu^i h_m(u)\chi_{\{|u|>k\}}\,dx\,dt + \varepsilon(n) \\
&= \int_{\{T_k(u_n)-w_\mu^i\geq 0\}} h_m(\nabla T_k(u)_\mu - e^{-\mu t}\nabla T_k(\psi_i))h_m(u)\chi_{\{|u|>k\}}\,dx\,dt + \varepsilon(n).
\end{aligned}$$

By letting $\mu \to +\infty$, we obtain

$$J_2 = \int_{\{T_k(u_n)-w_\mu^i\geq 0\}} h_m \nabla T_k(u)\,dx\,dt + \varepsilon(n,\mu).$$

Using now the term J_1 of (5.38), one can easily show that

$$\begin{aligned}
&\int_{\{T_k(u_n)-w_\mu^i\geq 0\}} a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla(T_k(u_n)-w_\mu^i)h_m(u_n)\,dx\,dt\\
&= \int_{\{T_k(u_n)-w_\mu^i\geq 0\}} [a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u))]\\
&\quad\times[\nabla T_k(u_n)-\nabla T_k(u)]\,h_m(u_n)\,dx\,dt\\
&\quad+\int_{\{T_k(u_n)-w_\mu^i\geq 0\}} a(x,t,T_k(u_n),\nabla T_k(u))(\nabla T_k(u_n)-\nabla T_k(u))h_m(u_n)\,dx\,dt\\
&\quad+\int_{\{T_k(u_n)-w_\mu^i\geq 0\}} a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u)h_m(u_n)\,dx\,dt\\
&\quad-\int_{\{T_k(u_n)-w_\mu^i\geq 0\}} a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla w_\mu^i h_m(u_n)\,dx\,dt\\
&= K_1+K_2+K_3+K_4.
\end{aligned} \tag{5.39}$$

We shall go to the limit as n and $\mu\to+\infty$ in the three integrals of the right-hand side. Starting with K_2, we have by letting $n\to+\infty$,

$$K_2=\varepsilon(n). \tag{5.40}$$

Concerning K_3, can be see letting $n\to+\infty$ and using (5.28),

$$K_3=\int_{\{T_k(u_n)-w_\mu^i\geq 0\}} h_k\nabla T_k(u)h_m(u)\chi_{\{|u|>k\}}\,dx\,dt+\varepsilon(n)$$

By letting $\mu\to+\infty$, we get

$$K_3=\int_{\{T_k(u_n)-w_\mu^i\geq 0\}} h_k\nabla T_k(u)\,dx\,dt+\varepsilon(n,\mu). \tag{5.41}$$

For K_4 we can write

$$K_4=-\int_{\{T_k(u_n)-w_\mu^i\geq 0\}} h_k\nabla w_\mu^i h_m(u)\,dx\,dt+\varepsilon(n),$$

by letting $\mu\to+\infty$,

$$K_4=-\int_{\{T_k(u_n)-w_\mu^i\geq 0\}} h_k\nabla T_k(u)\,dx\,dt+\varepsilon(n,\mu). \tag{5.42}$$

Then, we conclude that

$$\begin{aligned}
&\int_{\{T_k(u_n)-w_\mu^i\geq 0\}} a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla(T_k(u_n)-w_\mu^i)h_m(u_n)\,dx\,dt\\
&= \int_{\{T_k(u_n)-w_\mu^i\geq 0\}} [a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u))]\\
&\quad\times[\nabla T_k(u_n)-\nabla T_k(u)]\,h_m(u_n)\,dx\,dt+\varepsilon(n,\mu).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\int_{\{T_k(u_n)-w_\mu^i\geq 0\}}[a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(u))]\\
&\quad\times[\nabla T_k(u_n)-\nabla T_k(u)]\,dx\,dt\\
&\quad=\int_{\{T_k(u_n)-w_\mu^i\geq 0\}}[a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(u))]\\
&\quad\times[\nabla T_k(u_n)-\nabla T_k(u)]\,h_m(u_n)\,dx\,dt\\
&\quad+\int_{\{T_k(u_n)-w_\mu^i\geq 0\}}a(x,t,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla T_k(u))\\
&\quad\times(1-h_m(u_n))\,dx\,dt\\
&\quad-\int_{\{T_k(u_n)-w_\mu^i\geq 0\}}a(x,t,T_k(u_n),\nabla T_k(u))(\nabla T_k(u_n)-\nabla T_k(u))\\
&\quad\times(1-h_m(u_n))\,dx\,dt.
\end{aligned}\tag{5.43}$$

Since $h_m(u_n)=1$ in $\{|u_n|\leq m\}$ and $\{|u_n|\leq k\}\subset\{|u_n|\leq m\}$ for m large enough, we deduce from (5.43) that

$$\begin{aligned}
&\int_{\{T_k(u_n)-w_\mu^i\geq 0\}}[a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(u))]\\
&\quad\times[\nabla T_k(u_n)-\nabla T_k(u)]\,dx\,dt\\
&\quad=\int_{\{T_k(u_n)-w_\mu^i\geq 0\}}[a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(u))]\\
&\quad\times[\nabla T_k(u_n)-\nabla T_k(u)]\,h_m(u_n)\,dx\,dt\\
&\quad+\int_{\{T_k(u_n)-w_\mu^i\geq 0,|u_n|>k\}}a(x,t,T_k(u_n),\nabla T_k(u))\nabla T_k(u)(1-h_m(u_n))\,dx\,dt.
\end{aligned}$$

It is easy to see that the last terms of the last equality tend to zero as $n\to+\infty$, which implies

$$\begin{aligned}
&\int_{\{T_k(u_n)-w_\mu^i\geq 0\}}[a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(u))]\\
&\quad\times[\nabla T_k(u_n)-\nabla T_k(u)]\,dx\,dt\\
&\quad=\int_{\{T_k(u_n)-w_\mu^i\geq 0\}}[a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(u))]\\
&\quad\times[\nabla T_k(u_n)-\nabla T_k(u)]\,h_m(u_n)\,dx\,dt+\varepsilon(n).
\end{aligned}$$

Combining (5.37), (5.39), (5.40), (5.41), (5.42) and (5.43), follows

$$\begin{aligned}
&\int_{\{T_k(u_n)-w_\mu^i\geq 0\}}[a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(u))]\\
&\quad\times[\nabla T_k(u_n)-\nabla T_k(u)]\,dx\,dt\leq\varepsilon(n,\mu,m).
\end{aligned}\tag{5.44}$$

Passing to the limit in (5.44) as n and m tend to infinity, we obtain

$$\begin{aligned}
&\lim_{n\to\infty}\int_{\{T_k(u_n)-w_\mu^i\geq 0\}}[a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(u))]\\
&\quad\times[\nabla T_k(u_n)-\nabla T_k(u)]\,dx\,dt=0.
\end{aligned}\tag{5.45}$$

On the other hand, taking $\varphi = (T_k(u_n) - w^i_\mu)^- h_m(u_n)$ in (5.11), we may adopt the same procedure in (5.45) to obtain

$$\lim_{n\to\infty} \int_{\{T_k(u_n)-w^i_\mu \le 0\}} [a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u))] \times [\nabla T_k(u_n) - \nabla T_k(u)]\, dx\, dt = 0. \tag{5.46}$$

Furthermore, combining (5.45) and (5.46), we conclude

$$\lim_{n\to\infty} \int_Q [a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u))] \times [\nabla T_k(u_n) - \nabla T_k(u)]\, dx\, dt = 0. \tag{5.47}$$

Which, from Lemma (4.4), it follows that

$$T_k(u_n) \to T_k(u) \quad \text{strongly in } V \text{ for all } k. \tag{5.48}$$

Now, observe that for every $\sigma > 0$,

$$\begin{aligned} \text{meas}\{(x,t) \in \Omega\times[0,T] : |\nabla u_n - \nabla u| > \sigma\} &\le \text{meas}\{(x,t) \in \Omega\times[0,T] : |\nabla u_n| > k\} \\ &+ \text{meas}\{(x,t) \in \Omega\times[0,T] : |u| > k\} \\ &+ \text{meas}\{(x,t) \in \Omega\times[0,T] : \big|\nabla T_k(u_n) - \nabla T_k(u)\big| > \sigma\}, \end{aligned}$$

then as a consequence of (5.48), it follows that ∇u_n converges to ∇u in measure and therefore, always reasoning for a subsequence,

$$\nabla u_n \to \nabla u \quad \text{a.e. in } Q. \tag{5.49}$$

Which yields,

$$a(x,t,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup a(x,t,T_k(u),\nabla T_k(u)) \quad \text{in } \prod_{i=1}^{N} L^{p'(x)}(Q). \tag{5.50}$$

Step 3. Compactness of the nonlinearities. In order to pass to the limit in the approximated equation, we now show that

$$H_n(x,t,u_n,\nabla u_n) \to H(x,t,u,\nabla u) \;\text{ strongly in } L^1(Q),$$

by using Vitali's theorem. Since $H_n(x,t,u_n,\nabla u_n) \to H(x,t,u,\nabla u)$ a.e. in Q, consider a function $\rho_h(s) = \int_0^s g(\nu)\chi_{\{\nu>h\}} d\nu$, we take $\varphi = \rho_h(u_n) = \int_0^{u_n} g(s)\chi_{\{s>h\}} ds$ as a test function in (5.10), to obtain

$$\begin{aligned} &\Big[\int_\Omega B^n_h(x,u_n)dx\Big]_0^T + \int_Q a(x,t,u_n,\nabla u_n)\nabla u_n g(u_n)\chi_{\{u_n>h\}}\, dx\, dt \\ &\le \Big(\int_h^\infty g(s)\chi_{\{s>h\}} ds\Big) \exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big(\|\gamma\|_{L^1(Q)} + \|f_n\|_{L^1(Q)}\Big), \end{aligned}$$

where $B^n_h(x,r) = \int_0^r \frac{\partial b_n(x,s)}{\partial s}\rho_h(s)ds$.
Which implies that (since $B^n_h(x,r) \ge 0$),

$$\begin{aligned} &\int_Q a(x,t,u_n,\nabla u_n)\nabla u_n g(u_n)\chi_{\{u_n>h\}}\, dx\, dt \\ &\le \Big(\int_h^\infty g(s) ds\Big) \exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\big(\|\gamma\|_{L^1(Q)} + \|f_n\|_{L^1(Q)}\big) + \int_\Omega B^n_h(x,u_{0n})dx. \end{aligned}$$

Now, using (3.4), we get

$$\int_{\{u_n>h\}} g(u_n)\sum_{i=1}^{N}\Big|\frac{\partial u_n}{\partial x_i}\Big|^{p(x)}\,dx\,dt \le C\int_h^{\infty} g(s)\,ds.$$

Since $g\in L^1(I\!R)$, we have

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{u_n>h\}} g(u_n)\sum_{i=1}^{N}\Big|\frac{\partial u_n}{\partial x_i}\Big|^{p(x)}\,dx\,dt = 0.$$

By the same procedure as above, choose $\varphi=\displaystyle\int_{u_n}^{0} g(s)\chi_{\{s<-h\}}ds$ as a test function in (5.11), we conclude that

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{u_n<-h\}} g(u_n)\sum_{i=1}^{N}\Big|\frac{\partial u_n}{\partial x_i}\Big|^{p(x)}\,dx\,dt = 0.$$

Consequently,

$$\lim_{h\to+\infty}\sup_{n\in\mathbb{N}}\int_{\{|u_n|>h\}} g(u_n)\sum_{i=1}^{N}\Big|\frac{\partial u_n}{\partial x_i}\Big|^{p(x)}\,dx\,dt = 0,$$

we may choose h large enough, such that

$$\begin{aligned}\int_Q g(u_n)\sum_{i=1}^{N}\Big|\frac{\partial u_n}{\partial x_i}\Big|^{p(x)}\,dx\,dt &\le \int_{\{|u_n|<h\}} g(u_n)\sum_{i=1}^{N}\Big|\frac{\partial u_n}{\partial x_i}\Big|^{p(x)}\,dx\,dt+1\\ &\le \int_Q g(T_k(u_n))\sum_{i=1}^{N}\Big|\frac{\partial T_k(u_n)}{\partial x_i}\Big|^{p(x)}\,dx\,dt+1.\end{aligned}$$

Then, by (5.48) and Vitali's theorem, we can deduce that $g(u_n)\displaystyle\sum_{i=1}^{N}\Big|\frac{\partial u_n}{\partial x_i}\Big|^{p(x)}$ converges to $g(u)\displaystyle\sum_{i=1}^{N}\Big|\frac{\partial u}{\partial x_i}\Big|^{p(x)}$ strongly in $L^1(Q)$.

Finally, (3.5) gives

$$H_n(x,t,u_n,\nabla u_n)\to H(x,t,u,\nabla u)\quad \text{strongly in } L^1(Q). \tag{5.51}$$

Step 4. In this step we prove that u satisfies (5.2), (5.3) and (5.4).

Lemma 5.3. *The limit u of the approximate solution u_n of* (5.9) *satisfies*

$$\lim_{m\to+\infty}\int_{\{m\le|u|\le m+1\}} a(x,t,u,\nabla u)\,\nabla u\,dx\,dt = 0.$$

Proof. To this end, remark that for any fixed $m\ge 0$ one has

$$\begin{aligned}\int_{\{m\le|u_n|\le m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n &= \int_Q a(x,t,u_n,\nabla u_n)(\nabla T_{m+1}(u_n)-\nabla T_m(u_n))\\ = \int_Q a(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\nabla T_{m+1}(u_n) &- \int_Q a(x,t,T_m(u_n),\nabla T_m(u_n))\nabla T_m(u_n).\end{aligned}$$

According to (5.50) and (5.48), one is at liberty to pass to the limit as $n \to +\infty$ for fixed $m \geq 0$ and to obtain

$$\begin{aligned}
&\lim_{n\to+\infty} \int_{\{m\leq|u_n|\leq m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n \, dx\, dt \\
&= \int_Q a(x,t,T_{m+1}(u),\nabla T_{m+1}(u))\nabla T_{m+1}(u)\, dx\, dt \\
&- \int_Q a(x,t,T_m(u),\nabla T_m(u))\nabla T_m(u_n)\, dx\, dt. \\
&= \int_{\{m\leq|u_n|\leq m+1\}} a(x,t,u,\nabla u)\nabla u \, dx\, dt.
\end{aligned} \tag{5.52}$$

Taking the limit as m tends ∞ in (5.52) and using the estimate (5.29) show that u satisfies (5.3). The proof is then complete. □

Now, let us show that u satisfies (5.3) and (5.4).
Let S be a function in $W^{1,\infty}(I\!R)$ such that S' has a compact support. Let M be a positive real number such that supp $S' \subset [-M, M]$.
Pointwise multiplication of the approximate equation (5.9) by $S'(u_n)$ leads to

$$\begin{aligned}
&\frac{\partial B_S^n(x,u_n)}{\partial t} - \mathrm{div}[S'(u_n)a(u_n,\nabla u_n)] + S''(u_n)a(u_n,\nabla u_n)\nabla u_n + S'(u_n)H_n(u_n,\nabla u_n) \\
&\quad = fS'(u_n) \quad \text{in } \mathcal{D}'(Q).
\end{aligned} \tag{5.53}$$

Passing to the limit, as n tends to $+\infty$, we have

• Since S is bounded and continuous, then the fact that $u_n \to u$ a.e. in Q implies that $B_S^n(u_n)$ converges to $B_S(u)$ a.e. in Q and L^∞ weak–*. Consequently,

$$\frac{\partial B_S^n(u_n)}{\partial t} \quad \text{converges to} \quad \frac{\partial B_S(u)}{\partial t}$$

in $\mathcal{D}'(Q)$ as n tends to $+\infty$.

• Since supp $S' \subset [-M, M]$, we have for $n \geq M$,

$$S'(u_n)a_n(u_n,\nabla u_n) = S'(u_n)a(T_M(u_n),\nabla T_M(u_n)) \quad \text{a.e. in } Q.$$

The pointwise convergence of u_n to u and (5.50) as n tends to $+\infty$ and the bounded character of S' permit us to conclude that

$$S'(u_n)a_n(u_n,\nabla u_n) \rightharpoonup S'(u)a(T_M(u),\nabla T_M(u)) \quad \text{in } \prod_{i=1}^N L^{p'(x)}(Q), \tag{5.54}$$

as n tends to $+\infty$. $S'(u)a(T_M(u),\nabla T_M(u))$ has been denoted by $S'(u)a(u,\nabla u)$ in equation (5.3).

• Regarding the 'energy' term, we have

$$S''(u_n)a(u_n,\nabla u_n)\nabla u_n = S''(u_n)a(T_M(u_n),\nabla T_M(u_n))\nabla T_M(u_n) \quad \text{a.e. in } Q.$$

The pointwise convergence of $S'(u_n) \to S'(u)$ and (5.50) as n tends to $+\infty$ and the bounded character of S'' permit us to conclude that

$$S''(u_n)a_n(u_n,\nabla u_n)\nabla u_n \rightharpoonup S''(u)a(T_M(u),\nabla T_M(u))\nabla T_M(u) \quad \text{weakly in } L^1(Q). \tag{5.55}$$

Recall that

$$S''(u)a(T_M(u),\nabla T_M(u))\nabla T_M(u) = S''(u)a(u,\nabla u)\nabla u \quad \text{a.e. in } Q.$$

- Since $\operatorname{supp} S' \subset [-M, M]$ and from (5.51), we have

$$S'(u_n)H_n(x,t,u_n,\nabla u_n) \to S'(u)H(x,t,u,\nabla u) \quad \text{strongly in } L^1(Q). \tag{5.56}$$

- Due to (5.6) and the fact that $u_n \to u$ a.e. in Q, we have

$$S'(u_n)f_n \to S'(u)f \quad \text{strongly in } L^1(Q).$$

As a consequence of the above convergence results, we are in a position to pass to the limit as n tends to $+\infty$ in equation (5.53) and to conclude that u satisfies (5.3).

It remains to show that $B_S(u)$ satisfies the initial condition (5.4). To this end, firstly remark that, S being bounded, $B_S^n(u_n)$ is bounded in $L^\infty(Q)$. Secondly, (5.53) and the above considerations on the behavior of the terms of this equation show that $\frac{\partial B_S^n(u_n)}{\partial t}$ is bounded in $L^1(Q) + V^*$. As a consequence, an Aubin's type lemma (see, e.g, [22]) implies that $B_S^n(u_n)$ lies in a compact set of $\mathcal{C}^0([0,T], L^1(\Omega))$. It follows that, on the one hand, $B_S^n(u_n)(t=0) = B_S^n(u_0^n)$ converges to $B_S(u)(t=0)$ strongly in $L^1(\Omega)$. On the other hand, the smoothness of S imply that

$$B_S(u)(t=0) = B_S(u_0) \quad \text{in } \Omega.$$

As a conclusion of Step 1, Step 2, Step 3 and Step 4 the proof of Theorem 5.1 is complete.

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(Elhoussine Azroul) Laboratoire d'Analyse Mathématique et Applications (LAMA), Faculté des Sciences Dhar El Mahraz, Université Sidi Mohamed Ben Abdellah, BP 1796 Atlas Fès, Maroc
E-mail address: azroul_elhoussine@yahoo.fr

(Mohamed Badr Benboubker) Ecole National des Sciences Appliquées (ENSA), Université Abdelmalek Essaadi, BP 2222 M'hannech Tétouan, Maroc
E-mail address: simo.ben@hotmail.com

(Hicham Redwane) Faculté des Sciences Juridiques, Economiques et Sociales Université Hassan 1, BP 784, Settat, Maroc
E-mail address: redwane_hicham@yahoo.fr

(Chihab Yazough) Faculté des Sciences et Techniques, Université Sultan Moulay Slimane, BP 523 Béni Mellal, Maroc
E-mail address: chihabyazough@gmail.com