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Infinitely many solutions for nonlinear perturbed fractional boundary value problems

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Abstract. In this paper, we prove the existence of infinitely many solutions to nonlinear perturbed fractional boundary value problems. The approach is based on critical point theory and variational methods.

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1. Introduction

In this paper, we are interested in ensuring the existence of infinitely many solutions for the following perturbed fractional boundary value problem

$$\begin{gathered}\frac{d}{dt}\Big({}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(t)) - {}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(t))\Big) + \lambda f(u(t)) + \mu g(u(t)) = 0, \quad \text{a.e. } t \in [0,T],\\ u(0) = u(T) = 0,\end{gathered} \tag{1}$$

where $\alpha \in (1/2, 1]$, ${}_0D_t^{\alpha-1}$ and ${}_tD_T^{\alpha-1}$ are the left and right Riemann-Liouville fractional integrals of order $1-\alpha$ respectively, ${}_0^cD_t^{\alpha}$ and ${}_t^cD_T^{\alpha}$ are the left and right Caputo fractional derivatives of order $0 < \alpha \le 1$ respectively, λ is a positive real parameter, μ is a non-negative real parameter and $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions.

Because of its wide applicability in the modeling of many phenomena in various fields of physic, chemistry, biology, engineering and economics, the theory of fractional differential equations has recently been attracting increasing interest, see for instance the monographs of Miller and Ross [33], Samko et al [38], Podlubny [35], Hilfer [25], Kilbas et al [27] and the papers [2, 3, 6, 7, 8, 9, 28, 29, 40, 41, 42] and the references therein.

Critical point theory has been very useful in determining the existence of solution for integer order differential equations with some boundary conditions, for example [17, 29, 30, 32, 36, 39]. But until now, there are few results on the solution to fractional boundary value problems which were established by the critical point theory, since it is often very difficult to establish a suitable space and variational functional for fractional boundary value problems. Recently, Jiao and Zhou in [26] by using the critical point theory investigated the fractional boundary-value problem

$$\begin{gathered}\frac{d}{dt}\Big(\frac{1}{2}{}_0D_t^{-\beta}(u'(t)) + \frac{1}{2}{}_tD_T^{-\beta}(u'(t))\Big) + \nabla F(t, u(t)) = 0, \quad \text{a.e. } t \in [0,T],\\ u(0) = u(T) = 0\end{gathered}$$

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where ${}_0D_t^{-\beta}$ and ${}_tD_T^{-\beta}$ are the left and right Riemann-Liouville fractional integrals of order $0 \leq \beta < 1$ respectively, $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ is a given function and $\nabla F(t,x)$ is the gradient of F at x. Also, Chen and Tang in [16] studied the existence and multiplicity of solutions for the following fractional boundary value problem

$$\frac{d}{dt}\Big(\frac{1}{2}{}_0D_t^{-\beta}(u'(t)) + \frac{1}{2}{}_tD_T^{-\beta}(u'(t))\Big) + \nabla F(t,u(t)) = 0, \quad \text{a.e. } t \in [0,T],$$
$$u(0) = u(T) = 0,$$

where $F(t,\cdot)$ are superquadratic, asymptotically quadratic, and subquadratic, respectively. In particular, Bai in [4], by using a local minimum theorem due to Bonanno ([10]), investigated the existence of at least one non-trivial solution to the problem (1).

In the present paper, motivated by [4], employing a smooth version of Theorem 2.1 of [13] which is a more precise version of Ricceri's Variational Principle [37, Theorem 2.5] (see Theorem 2.6), requiring that the nonlinear term f has a suitable oscillating behavior at infinity, in Theorem 3.1, we establish the existence of a precise interval of parameters Λ such that, for each $\lambda \in \Lambda$ and every arbitrary continuous function g which satisfies a certain growth at infinity, choosing μ sufficiently small, the perturbed problem (1) admits a sequence of solutions which are unbounded in the fractional derivative space E_0^α. We also list some consequences of Theorem 3.1 and one example. Finally, we present an analogous result (see Theorem 3.6), in which we replace the oscillating behavior condition at infinity in Theorem 3.1, by a similar one at zero. In this setting, a sequence of pairwise distinct non-zero solutions which converges to zero is achieved.

A special case of our main result is the following theorem.

Theorem 1.1. *Let $\frac{1}{2} < \alpha \leq 1$. Assume that*

$$\liminf_{\xi \to +\infty} \frac{\max_{|x| \leq \xi} F(x)}{\xi^2} = 0 \quad \text{and} \quad \limsup_{\xi \to +\infty} \frac{\frac{1}{|\xi|}\int_0^{\Gamma(2-\alpha)|\xi|} F(x)dx}{\xi^2} = +\infty$$

where $F(x) = \int_0^x f(s)ds$ for every $x \in \mathbb{R}$ and Γ is the gamma function. Then, the problem

$$\frac{d}{dt}\Big({}_0D_t^{\alpha-1}({}_0^cD_t^\alpha u(t)) - {}_tD_T^{\alpha-1}({}_t^cD_T^\alpha u(t))\Big) + f(u(t)) = 0, \quad \text{a.e. } t \in [0,T],$$
$$u(0) = u(T) = 0,$$

admits a sequence of pairwise distinct positive solutions.

For a discussion about the existence of infinitely many solutions for boundary value problems, using Ricceri's Variational Principle [37] and its variants ([13, Theorem 2.1] and [31, Theorem 1.1]) we refer the reader to the papers [5, 11, 12, 14, 15, 18, 21, 22, 23].

For a through on the subject, we also refer the reader to [1, 19, 20, 24, 34].

2. Preliminaries

In this section, we will introduce some notations, definitions and preliminary facts which are used throughout this paper.

Definition 2.1 ([27])**.** Let f be a function defined on $[a,b]$ and $\alpha > 0$. The left and right Riemann-Liouville fractional integrals of order α for the function f are defined by

$$
\begin{aligned}
{}_aD_t^{-\alpha} f(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds, \quad t \in [a,b], \\
{}_tD_b^{-\alpha} f(t) &= \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s)ds, \quad t \in [a,b],
\end{aligned}
$$

provided the right-hand sides are pointwise defined on $[a,b]$, where $\Gamma(\alpha)$ is the gamma function.

Definition 2.2 ([27])**.** Let $\gamma \geq 0$ and $n \in \mathbb{N}$.
(i) If $\gamma \in (n-1,n)$ and $f \in AC^n([a,b],\mathbb{R}^N)$, then the left and right Caputo fractional derivatives of order γ for function f denoted by ${}_a^cD_t^\gamma f(t)$ and ${}_t^cD_b^\gamma f(t)$, respectively, exist almost everywhere on $[a,b]$, ${}_a^cD_t^\gamma f(t)$ and ${}_t^cD_b^\gamma f(t)$ are represented by

$$
\begin{aligned}
{}_a^cD_t^\gamma f(t) &= \frac{1}{\Gamma(n-\gamma)} \int_a^t (t-s)^{n-\gamma-1} f^{(n)}(s)ds, \quad t \in [a,b], \\
{}_t^cD_b^\gamma f(t) &= \frac{(-1)^n}{\Gamma(n-\gamma)} \int_t^b (s-t)^{n-\gamma-1} f^{(n)}(s)ds, \quad t \in [a,b],
\end{aligned}
$$

respectively.
(ii) If $\gamma = n-1$ and $f \in AC^{n-1}([a,b],\mathbb{R}^N)$, then ${}_a^cD_t^{n-1} f(t)$ and ${}_t^cD_b^{n-1} f(t)$ are represented by

$$
{}_a^cD_t^{n-1} f(t) = f^{(n-1)}(t), \quad \text{and} \quad {}_t^cD_b^{n-1} f(t) = (-1)^{(n-1)} f^{(n-1)}(t), \quad t \in [a,b].
$$

With these definitions, we have the rule for fractional integration by parts, and the composition of the Riemann-Liouville fractional integration operator with the Caputo fractional differentiation operator, which were proved in [27, 38].

Proposition 2.1 ([27, 38])**.** *We have the following property of fractional integration*

$$
\int_a^b [{}_aD_t^{-\gamma} f(t)] g(t)dt = \int_a^b [{}_tD_b^{-\gamma} g(t)] f(t)dt, \quad \gamma > 0, \tag{2}
$$

provided that $f \in L^p([a,b],\mathbb{R}^N)$, $g \in L^q([a,b],\mathbb{R}^N)$ *and* $p \geq 1$, $q \geq 1$, $1/p + 1/q \leq 1+\gamma$ *or* $p \neq 1$, $q \neq 1$, $1/p+1/q = 1+\gamma$.

Proposition 2.2 ([27])**.** *Let* $n \in \mathbb{N}$ *and* $n-1 < \gamma \leq n$. *If* $f \in AC^n([a,b],\mathbb{R}^N)$ *or* $f \in C^n([a,b],\mathbb{R}^N)$, *then*

$$
\begin{aligned}
{}_aD_t^{-\gamma}({}_a^cD_t^\gamma f(t)) &= f(t) - \sum_{j=0}^{n-1} \frac{f^{(j)}(a)}{j!}(t-a)^j, \\
{}_tD_b^{-\gamma}({}_t^cD_b^\gamma f(t)) &= f(t) - \sum_{j=0}^{n-1} \frac{(-1)^j f^{(j)}(b)}{j!}(b-t)^j,
\end{aligned}
$$

for $t \in [a,b]$. *In particular, if* $0 < \gamma \leq 1$ *and* $f \in AC([a,b],\mathbb{R}^N)$ *or* $f \in C^1([a,b],\mathbb{R}^N)$, *then*

$$
{}_aD_t^{-\gamma}({}_a^cD_t^\gamma f(t)) = f(t) - f(a), \quad \text{and} \quad {}_tD_b^{-\gamma}({}_t^cD_b^\gamma f(t)) = f(t) - f(b). \tag{3}
$$

Remark 2.1. In view of (2) and Definition 2.2, it is obvious that $u \in AC([0,T])$ is a solution of (1) if and only if u is a solution of the problem

$$\frac{d}{dt}\Big({}_0D_t^{-\beta}(u'(t)) + {}_tD_T^{-\beta}(u'(t))\Big) + \lambda f(u(t)) + \mu g(u(t)) = 0, \quad \text{a.e. } t \in [0,T], \qquad u(0) = u(T) = 0, \tag{4}$$

where $\beta = 2(1-\alpha) \in [0,1)$.

To establish a variational structure for (1), it is necessary to construct appropriate function spaces.

Definition 2.3 ([26])**.** Let $0 < \alpha \le 1$. The fractional derivative space E_0^α is defined by the closure of $C_0^\infty[0,T]$ with respect to the norm

$$\|u\|_\alpha = \Big(\int_0^T |{}_0^cD_t^\alpha u(t)|^2 dt + \int_0^T |u(t)|^2 dt\Big)^{1/2}, \quad \forall u \in E^\alpha,$$

where $C_0^\infty[0,T]$ denotes the set of all functions $u \in C^\infty[0,T]$ with $u(0) = u(T) = 0$. It is obvious that the fractional derivative space E_0^α is the space of functions $u \in L^2[0,T]$ having an α-order Caputo fractional derivative ${}_0^cD_t^\alpha u \in L^2[0,T]$ and $u(0) = u(T) = 0$.

Proposition 2.3 ([26])**.** *Let $0 < \alpha \le 1$. The fractional derivative space E_0^α is reflexive and separable Banach space.*

Proposition 2.4 ([26])**.** *Let $0 < \alpha \le 1$. For all $u \in E_0^\alpha$, we have*

$$\|u\|_{L^2} \le \frac{T^\alpha}{\Gamma(\alpha+1)} \|{}_0^cD_t^\alpha u\|_{L^2}, \tag{5}$$

$$\|u\|_\infty \le \frac{T^{\alpha-1/2}}{\Gamma(\alpha)(2(\alpha-1)+1)^{1/2}} \|{}_0^cD_t^\alpha u\|_{L^2}. \tag{6}$$

According to (5), we can consider E_0^α with respect to the norm

$$\|u\|_\alpha = \Big(\int_0^T |{}_0^cD_t^\alpha u(t)|^2 dt\Big)^{1/2} = \|{}_0^cD_t^\alpha u\|_{L^2}, \quad \forall u \in E_0^\alpha \tag{7}$$

in the following analysis.

Proposition 2.5 ([26])**.** *Let $1/2 < \alpha \le 1$, then for all any $u \in E_0^\alpha$, we have*

$$|\cos(\pi\alpha)| \|u\|_\alpha^2 \le -\int_0^T {}_0^cD_t^\alpha u(t) \cdot {}_t^cD_T^\alpha u(t) dt \le \frac{1}{|\cos(\pi\alpha)|}\|u\|_\alpha^2. \tag{8}$$

By Proposition 2.4, when $\alpha > 1/2$, for each $u \in E_0^\alpha$ we have

$$\|u\|_\infty \le \Omega\Big(\int_0^T |{}_0^cD_t^\alpha u(t)|^2 dt\Big)^{1/2} = \Omega\|u\|_\alpha, \tag{9}$$

where

$$\Omega = \frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)\sqrt{2(\alpha-1)+1}}. \tag{10}$$

Our main tool is the celebrated Ricceri's Variational Principle [37, Theorem 2.5] that we now recall as given by Bonanno and Molica Bisci in [13].

Theorem 2.6. *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \longrightarrow \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous, and coercive and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, let us put*

$$\varphi(r) := \inf_{u\in\Phi^{-1}(]-\infty,r[)} \frac{\sup_{v\in\Phi^{-1}(]-\infty,r])} \Psi(v) - \Psi(u)}{r - \Phi(u)}$$

and

$$\gamma := \liminf_{r\to+\infty} \varphi(r), \quad \delta := \liminf_{r\to(\inf_X \Phi)^+} \varphi(r).$$

Then, one has
(a) for every $r > \inf_X \Phi$ and every $\lambda \in]0, \frac{1}{\varphi(r)}[$, the restriction of the functional $I_\lambda = \Phi - \lambda\Psi$ to $\Phi^{-1}(]-\infty, r[)$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.
(b) If $\gamma < +\infty$ then, for each $\lambda \in]0, \frac{1}{\gamma}[$, the following alternative holds:
either
(b_1) I_λ possesses a global minimum,
or
(b_2) there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that

$$\lim_{n\to+\infty} \Phi(u_n) = +\infty.$$

(c) If $\delta < +\infty$ then, for each $\lambda \in]0, \frac{1}{\delta}[$, the following alternative holds:
either
(c_1) there is a global minimum of Φ which is a local minimum of I_λ,
or
(c_2) there is a sequence of pairwise distinct critical points (local minima) of I_λ which weakly converges to a global minimum of Φ.

3. Main results

Put

$$\omega_\alpha := \frac{4\Gamma^2(2-\alpha)}{\Gamma(4-2\alpha)} T^{1-2\alpha}(2^{2\alpha-1} - 1)$$

and $F(x) = \int_0^x f(s)ds$ for every $x \in \mathbb{R}$.

We state our main result as follows:

Theorem 3.1. *Let $\frac{1}{2} < \alpha \leq 1$. Assume that*
(A1) $\liminf_{\xi\to+\infty} \frac{\max_{|x|\leq\xi} F(x)}{\xi^2} < \frac{|\cos(\pi\alpha)|}{\Gamma(2-\alpha)\omega_\alpha\Omega^2} \limsup_{\xi\to+\infty} \frac{\frac{1}{|\xi|}\int_0^{\Gamma(2-\alpha)|\xi|} F(x)dx}{\xi^2}$.

Then, for each $\lambda \in]\lambda_1, \lambda_2[$ where

$$\lambda_1 := \frac{\Gamma(2-\alpha)\omega_\alpha}{T \limsup_{\xi\to+\infty} \frac{\frac{1}{|\xi|}\int_0^{\Gamma(2-\alpha)|\xi|} F(x)dx}{\xi^2}}$$

and

$$\lambda_2 := \frac{|\cos(\pi\alpha)|}{\Omega^2 T \liminf_{\xi\to+\infty} \frac{\max_{|x|\leq\xi} F(x)}{\xi^2}},$$

for every arbitrary continuous function $g : \mathbb{R} \to \mathbb{R}$ *whose potential* $G(x) = \int_0^x g(s)ds$ *for every* $x \in \mathbb{R}$*, is a non-negative function satisfying the condition*

$$G_\infty := lim_{\xi\to+\infty} \frac{\max_{|x|\le\xi} G(x)}{\xi^2} < +\infty \tag{11}$$

and for every $\mu \in [0, \mu_{G,\lambda}[$ *where*

$$\mu_{G,\lambda} := \frac{|\cos(\pi\alpha)|}{\Omega^2 T G_\infty}\left(1 - \lambda T \frac{\Omega^2}{|\cos(\pi\alpha)|} \liminf_{\xi\to+\infty} \frac{\max_{|x|\le\xi} F(x)}{\xi^2}\right),$$

the problem (1) has an unbounded sequence of solutions in E_0^α*.*

Proof. In order to apply Theorem 2.6 to our problem, let X be the fractional derivative space E_0^α equipped with the norm

$$\|u\|_\alpha = \Big(\int_0^T |{}_0^cD_t^\alpha u(t)|^2 dt\Big)^{1/2},$$

and we introduce the functionals $\Phi,\ \Psi : X \to \mathbb{R}$ for each $u \in X$ as follows:

$$\Phi(u) := -\int_0^T {}_0^cD_t^\alpha u(t) \cdot {}_t^cD_T^\alpha u(t)dt, \quad \Psi(u) := \int_0^T (F(u(t)) + \frac{\overline{\mu}}{\overline{\lambda}} G(u(t)))dt.$$

Clearly, Φ and Ψ are Gâteaux differentiable functional whose Gâteaux derivative at the point $u \in X$ are given by

$$\begin{aligned}\Phi'(u)v &= -\int_0^T ({}_0^cD_t^\alpha u(t) \cdot {}_t^cD_T^\alpha v(t) + {}_t^cD_T^\alpha u(t) \cdot {}_0^cD_t^\alpha v(t))dt,\\ \Psi'(u)v &= \int_0^T (f(u(t)) + \frac{\overline{\mu}}{\overline{\lambda}} g(u(t)))v(t)dt\\ &= -\int_0^T \int_0^t f(u(s))ds \cdot v'(t)dt - \frac{\overline{\mu}}{\overline{\lambda}} \int_0^T \int_0^t g(u(s))ds \cdot v'(t)dt\end{aligned}$$

for every $v \in X$. By Definition 2.2 and (3), we have

$$\Phi'(u)v = \int_0^T ({}_0D_t^{\alpha-1}({}_0^cD_t^\alpha u(t)) - {}_tD_T^{\alpha-1}({}_t^cD_T^\alpha u(t))) \cdot v'(t)dt.$$

Put $I_{\overline{\lambda}} := \Phi - \overline{\lambda}\Psi$. The solutions of the problem (1) are exactly the solutions of the equation $I'_{\overline{\lambda}}(u) = 0$ (see [4]). Fix $\overline{\lambda} \in]\lambda_1, \lambda_2[$ and let G be a non-negative function satisfies the condition (11). Since, $\overline{\lambda} < \lambda_2$, one has

$$\mu_{G,\overline{\lambda}} := \frac{|\cos(\pi\alpha)|}{\Omega^2 T G_\infty}\left(1 - \overline{\lambda} T \frac{\Omega^2}{|\cos(\pi\alpha)|} \liminf_{\xi\to+\infty} \frac{\max_{|t|\le\xi} F(t)}{\xi^2}\right) > 0.$$

Fix $\overline{\mu} \in]0, \mu_{G,\overline{\lambda}}[$ and set $\nu_1 := \lambda_1$ and $\nu_2 := \frac{\lambda_2}{1+\frac{\Omega^2}{|\cos(\pi\alpha)|}\frac{\overline{\mu}}{\overline{\lambda}}\lambda_2 T G_\infty}$. If $G_\infty = 0$, clearly, $\nu_1 = \lambda_1$, $\nu_2 = \lambda_2$ and $\lambda \in]\nu_1, \nu_2[$. If $G_\infty \neq 0$, since $\overline{\mu} < \mu_{G,\overline{\lambda}}$, we obtain

$$\frac{\overline{\lambda}}{\lambda_2} + \frac{\Omega^2}{|\cos(\pi\alpha)|} \overline{\mu} T G_\infty < 1,$$

and so

$$\frac{\lambda_2}{1 + \frac{\Omega^2}{|\cos(\pi\alpha)|}\frac{\overline{\mu}}{\overline{\lambda}}\lambda_2 T G_\infty} > \overline{\lambda},$$

namely, $\overline{\lambda} < \nu_2$. Hence, bering in mind that $\overline{\lambda} > \lambda_1 = \nu_1$, one has $\overline{\lambda} \in]\nu_1, \nu_2[$.
Now, let us show that

$$\gamma < +\infty.$$

Let $\{\xi_n\}$ be a real sequence such that $\xi_n \to +\infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{\max_{|x|\le\xi_n} F(x) + \frac{\overline{\mu}}{\lambda}\max_{|x|\le\xi_n} G(x)}{\xi_n^2} = \liminf_{\xi\to+\infty} \frac{\max_{|x|\le\xi} F(x) + \frac{\overline{\mu}}{\lambda}\max_{|x|\le\xi} G(x)}{\xi^2}.$$

For every $n \in \mathbb{N}$ let us consider $r_n = \frac{|\cos(\pi\alpha)|}{\Omega^2}\xi_n^2$. Taking (8) into account, for all $u \in X$ such that $u \in \Phi^{-1}(]-\infty, r_n])$, we have

$$|\cos(\pi\alpha)|\|u\|_\alpha^2 \le \Phi(u) \le r_n,$$

which implies

$$\|u\|_\alpha^2 \le \frac{1}{|\cos(\pi\alpha)|} r_n. \tag{12}$$

Thus, by (9) and (12) we obtain

$$|u(t)| < \Omega\|u\|_\alpha \le \Omega\sqrt{\frac{r}{|\cos(\pi\alpha)|}} = \xi_n, \quad \forall t \in [0,T],$$

which from the definition of Ψ follows

$$\sup_{u\in\Phi^{-1}(]-\infty,r_n])} \Psi(u) \le T \max_{|x|\le\xi_n}(F(x) + \frac{\overline{\mu}}{\overline{\lambda}}G(x)) \le T(\max_{|x|\le\xi_n} F(x) + \frac{\overline{\mu}}{\overline{\lambda}}\max_{|x|\le\xi_n} G(x)).$$

Therefore, since $\Phi(0) = \Psi(0) = 0$, for every n large enough, one has

$$\begin{aligned}
\varphi(r_n) &= \inf_{u\in\Phi^{-1}(]-\infty,r_n[)} \frac{(\sup_{v\in\Phi^{-1}(]-\infty,r_n])}\Psi(v)) - \Psi(u)}{r_n - \Phi(u)} \\
&\le \frac{\sup_{v\in\Phi^{-1}(]-\infty,r_n])}\Psi(v)}{r_n} \\
&\le T\frac{\max_{|x|\le\xi_n} F(x) + \frac{\overline{\mu}}{\lambda}\max_{|x|\le\xi_n} G(x)}{\frac{|\cos(\pi\alpha)|}{\Omega^2}\xi_n^2}.
\end{aligned}$$

Moreover, from Assumption (A1) and the condition (11) one has

$$\lim_{n\to\infty} \frac{\max_{|x|\le\xi_n} F(x) + \frac{\overline{\mu}}{\lambda}\max_{|x|\le\xi_n} G(x)}{\xi_n^2} < +\infty.$$

So,

$$\gamma \le \liminf_{n\to+\infty} \varphi(r_n) \le T\frac{\Omega^2}{|\cos(\pi\alpha)|}\lim_{n\to\infty} \frac{\max_{|x|\le\xi_n} F(x) + \frac{\overline{\mu}}{\lambda}\max_{|x|\le\xi_n} G(x)}{\xi_n^2} < +\infty. \tag{13}$$

Taking (11) into account, one has

$$\liminf_{\xi\to+\infty} \frac{\max_{|x|\le\xi} F(x) + \frac{\overline{\mu}}{\lambda}\max_{|x|\le\xi} G(x)}{\xi^2} \le \liminf_{\xi\to+\infty} \frac{\max_{|x|\le\xi} F(x)}{\xi^2} + \frac{\overline{\mu}}{\overline{\lambda}}G_\infty, \tag{14}$$

which follows

$$\overline{\lambda} \in]\nu_1, \nu_2[\subseteq$$

$$\left]\frac{\Gamma(2-\alpha)\omega_\alpha}{T\limsup_{\xi\to+\infty}\frac{\frac{1}{|\xi|}\int_0^{\Gamma(2-\alpha)|\xi|}(F(x)+\frac{\overline{\mu}}{\lambda}G(x))dx}{\xi^2}}, \frac{|\cos(\pi\alpha)|}{\Omega^2 T\liminf_{\xi\to+\infty}\frac{\max_{|x|\le\xi}(F(x)+\frac{\overline{\mu}}{\lambda}G(x))}{\xi^2}}\right[.$$

Assumption (A1) in conjunction with (13), implies

$$\left]\frac{\Gamma(2-\alpha)\omega_\alpha}{T\limsup_{\xi\to+\infty}\frac{\frac{1}{|\xi|}\int_0^{\Gamma(2-\alpha)|\xi|}(F(x)+\frac{\overline{\mu}}{\overline{\lambda}}G(x))dx}{\xi^2}}, \frac{|\cos(\pi\alpha)|}{\Omega^2 T\liminf_{\xi\to+\infty}\frac{\max_{|x|\leq\xi}(F(x)+\frac{\overline{\mu}}{\overline{\lambda}}G(x))}{\xi^2}}\right[$$

$$\subseteq\left]0,\frac{1}{\gamma}\right[.$$

For the fixed $\overline{\lambda}$, the inequality (13) ensures that the condition (b) of Theorem 2.6 can be applied and either $I_{\overline{\lambda}}$ has a global minimum or there exists a sequence $\{u_n\}$ of weak solutions of the problem (1) such that $\lim_{n\to\infty}\|u_n\|=+\infty$.
The other step is to show that for the fixed $\overline{\lambda}$ the functional $I_{\overline{\lambda}}$ has no global minimum. Let us verify that the functional $I_{\overline{\lambda}}$ is unbounded from below. Since

$$\begin{aligned}\frac{1}{\overline{\lambda}}&<\frac{T}{\Gamma(2-\alpha)\omega_\alpha}\limsup_{\xi\to+\infty}\frac{\frac{1}{|\xi|}\int_0^{\Gamma(2-\alpha)|\xi|}F(x)dx}{\xi^2}\\&\leq\frac{T}{\Gamma(2-\alpha)\omega_\alpha}\limsup_{\xi\to+\infty}\frac{\frac{1}{|\xi|}\int_0^{\Gamma(2-\alpha)|\xi|}(F(x)+\frac{\overline{\mu}}{\overline{\lambda}}G(x))dx}{\xi^2},\end{aligned}$$

we can consider a real sequence $\{d_n\}$ and a positive constant τ such that $d_n\to+\infty$ as $n\to\infty$ and

$$\frac{1}{\overline{\lambda}}<\tau<\frac{T}{\Gamma(2-\alpha)\omega_\alpha}\frac{\frac{1}{|d_n|}\int_0^{\Gamma(2-\alpha)|d_n|}(F(x)+\frac{\overline{\mu}}{\overline{\lambda}}G(x))dx}{d_n^2} \tag{15}$$

for each $n\in\mathbb{N}$ large enough. Let $\{w_n\}$ be a sequence in X defined by putting

$$w_n(t)=\begin{cases}\frac{2\Gamma(2-\alpha)d_n}{T}t, & t\in[0,T/2),\\ \frac{2\Gamma(2-\alpha)d_n}{T}(T-t), & t\in[T/2,T]\end{cases} \tag{16}$$

It is easy to check that $w_n(0)=w_n(T)=0$ and $w_n\in L^2[0,T]$. The direct calculation shows that

$${}_0^cD_t^\alpha w_n(t)=\begin{cases}\frac{2d_n}{T}t^{1-\alpha}, & t\in[0,T/2),\\ \frac{2d_n}{T}(t^{1-\alpha}-2(t-\frac{T}{2})^{1-\alpha}), & t\in[T/2,T]\end{cases}$$

and

$$\begin{aligned}\|w_n\|_\alpha^2&=\int_0^T({}_0^cD_t^\alpha w_n(t))^2dt=\int_0^{\frac{T}{2}}+\int_{T/2}^T({}_0^cD_t^\alpha w_n(t))^2dt\\&=\frac{4d_n^2}{T^2}\Big[\int_0^T t^{2(1-\alpha)}dt-4\int_{T/2}^T t^{1-\alpha}(t-\frac{T}{2})^{1-\alpha}dt+4\int_{T/2}^T(t-\frac{T}{2})^{2(1-\alpha)}dt\Big]\\&=\frac{4(1+2^{2\alpha-1})d_n^2}{3-2\alpha}T^{1-2\alpha}-\frac{16d_n^2}{T^2}\int_{T/2}^T t^{1-\alpha}(t-\frac{T}{2})^{1-\alpha}dt<\infty.\end{aligned}$$

That is, ${}_0^cD_t^\alpha w_n\in L^2[0,T]$. Thus, $w_n\in X$. Moreover, the direct calculation shows

$${}_t^cD_T^\alpha w_n(t)=\begin{cases}\frac{2d_n}{T}((T-t)^{1-\alpha}-2(\frac{T}{2}-t)^{1-\alpha}), & t\in[0,T/2),\\ \frac{2d_n}{T}(T-t)^{1-\alpha}, & t\in[T/2,T]\end{cases}$$

and

$$\begin{aligned}
\Phi(w_n) &= -\int_0^T {}_0^cD_t^\alpha w_n(t)\cdot {}_t^cD_T^\alpha w_n(t)dt\\
&= -(\frac{2d_n}{T})^2\Big[\int_0^{\frac{T}{2}} t^{1-\alpha}\Big((T-t)^{1-\alpha}-2\big(\frac{T}{2}-t\big)^{1-\alpha}\Big)dt\\
&\quad+\int_{T/2}^T (T-t)^{1-\alpha}\cdot\Big(t^{1-\alpha}-2(t-\frac{T}{2})^{1-\alpha}\Big)dt\Big]\\
&= -(\frac{2d_n}{T})^2\Big[\int_0^T t^{1-\alpha}(T-t)^{1-\alpha}dt - 4\int_0^{\frac{T}{2}} t^{1-\alpha}\big(\frac{T}{2}-t\big)^{1-\alpha}dt\Big]\\
&= -(\frac{2d_n}{T})^2\Big[\frac{\Gamma^2(2-\alpha)}{\Gamma(4-2\alpha)}T^{3-2\alpha} - 4\frac{\Gamma^2(2-\alpha)}{\Gamma(4-2\alpha)}(\frac{T}{2})^{3-2\alpha}\Big]\\
&= \frac{4\Gamma^2(2-\alpha)}{\Gamma(4-2\alpha)}T^{1-2\alpha}(2^{2\alpha-1}-1)d_n^2 = \omega_\alpha d_n^2 \qquad (17)
\end{aligned}$$

and

$$\Psi(w_n) = \int_0^T (F(w_n(t))+\frac{\overline{\mu}}{\overline{\lambda}}G(w_n(t))dt = \frac{T}{\Gamma(2-\alpha)|d_n|}\int_0^{\Gamma(2-\alpha)|d_n|}(F(x)+\frac{\overline{\mu}}{\overline{\lambda}}G(x))dx. \qquad (18)$$

So, according to (15), (17) and (18) we achieve

$$I_{\overline{\lambda}}(w_n) = \omega_\alpha d_n^2 - \overline{\lambda}\frac{T}{\Gamma(2-\alpha)|d_n|}\int_0^{\Gamma(2-\alpha)|d_n|}(F(x)+\frac{\overline{\mu}}{\overline{\lambda}}G(x))dx < (1-\overline{\lambda}\tau)\omega_\alpha d_n^2$$

for every $n\in\mathbb{N}$ large enough. Hence, the functional $I_{\overline{\lambda}}$ is unbounded from below, and it follows that $I_{\overline{\lambda}}$ has no global minimum. Therefore, recalling (8), applying Theorem 2.6 we deduce that there is a sequence $\{u_n\}\subset X$ of critical points of $I_{\overline{\lambda}}$ such that $\lim_{n\to\infty}\|u_n\| = +\infty$, and the proof is complete. □

Remark 3.1. Under the conditions

$$\liminf_{\xi\to+\infty}\frac{\max_{|x|\le\xi}F(x)}{\xi^2} = 0$$

and

$$\limsup_{\xi\to+\infty}\frac{\frac{1}{|\xi|}\int_0^{\Gamma(2-\alpha)|\xi|}F(x)dx}{\xi^2} = +\infty,$$

Theorem 3.1 ensures that for every $\lambda>0$ and for each $\mu\in[0,\frac{|\cos(\pi\alpha)|}{\Omega^2 TG_\infty}[$ the problem (1) admits infinitely many solutions in E_0^α. Moreover, if $G_\infty=0$, the result holds for every $\lambda>0$ and $\mu\ge 0$.

We now exhibit an example in which the hypotheses of Theorem 3.1 are satisfied.

Example 3.1. Let $\frac{1}{2}<\alpha\le 1$. Let $f:\mathbb{R}\to\mathbb{R}$ be the function defined by

$$f(x)=\begin{cases} x\Big(2-\cos(\ln(|x|))-2\sin(\ln(|x|))\Big) & \text{if } x\in(\mathbb{R}-\{0\}),\\ 0 & \text{if } x=0.\end{cases}$$

A direct calculation shows

$$F(x)=\begin{cases} x^2\Big(1-\sin(\ln(|x|))\Big) & \text{if } x\in(\mathbb{R}-\{0\}),\\ 0 & \text{if } x=0.\end{cases}$$

So,

$$\liminf_{\xi\to+\infty}\frac{\max_{|x|\le\xi}F(x)dx}{\xi^2}=0$$

and

$$\limsup_{\xi\to+\infty}\frac{\frac{1}{|\xi|}\int_0^{\Gamma(2-\alpha)|\xi|}F(x)dx}{\xi^2}=+\infty.$$

Hence, using Theorem 3.1, the problem

$$\begin{gathered}\frac{d}{dt}\Big({}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(t))-{}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(t))\Big)+\lambda f(u(t))+\mu u(t)e^{-u(t)}(2-u(t))=0,\\ \text{a.e. } t\in[0,1],\\ u(0)=u(1)=0,\end{gathered}$$

since $G_\infty=0$, for every $(\lambda,\mu)\in]0,+\infty[\times[0,+\infty[$ has an unbounded sequence of solutions in E_0^α.

Now we want to present the following existence result which instead of Assumption (A1) in Theorem 3.1 a more general condition is assumed.

Theorem 3.2. *Let $\frac{1}{2}<\alpha\le 1$. Assume that*
(A2) there exist two sequence $\{a_n\}$ and $\{b_n\}$ with $\omega_\alpha a_n^2<\frac{|\cos(\pi\alpha)|}{\Omega^2}b_n^2$ for every $n\in\mathbb{N}$ and $\lim_{n\to+\infty}b_n=+\infty$ such that

$$\lim_{n\to+\infty}\frac{\max_{|x|\le b_n}F(x)-\frac{1}{|a_n|}\int_0^{\Gamma(2-\alpha)|a_n|}F(x)dx}{\frac{|\cos(\pi\alpha)|}{\Omega^2}b_n^2-\omega_\alpha a_n^2}<\frac{1}{\omega_\alpha}\limsup_{\xi\to+\infty}\frac{\frac{1}{|\xi|}\int_0^{\Gamma(2-\alpha)|\xi|}F(x)dx}{\xi^2}.$$

Then, for each

$$\Lambda':=\Bigg]\frac{\omega_\alpha}{T\limsup_{\xi\to+\infty}\frac{\frac{1}{|\xi|}\int_0^{\Gamma(2-\alpha)|\xi|}F(x)dx}{\xi^2}},$$

$$\frac{\frac{|\cos(\pi\alpha)|}{\Omega^2}b_n^2}{T\lim_{n\to+\infty}\frac{\max_{|x|\le b_n}F(x)-\frac{1}{|a_n|}\int_0^{\Gamma(2-\alpha)|a_n|}F(x)dx}{\frac{|\cos(\pi\alpha)|}{\Omega^2}b_n^2-\omega_\alpha a_n^2}}\Bigg[,$$

for every arbitrary continuous function $g:\mathbb{R}\to\mathbb{R}$ whose potential $G(x)=\int_0^x g(s)ds$ for every $x\in\mathbb{R}$, is a non-negative function satisfying the condition (11) *and for every $\mu\in[0,\mu_{G,\lambda}[$ where*

$$\mu_{G,\lambda}:=\frac{|\cos(\pi\alpha)|}{\Omega^2TG_\infty}\left(1-\lambda T\lim_{n\to+\infty}\frac{\max_{|x|\le b_n}F(x)-\frac{1}{|a_n|}\int_0^{\Gamma(2-\alpha)|a_n|}F(x)dx}{\frac{|\cos(\pi\alpha)|}{\Omega^2}b_n^2-\omega_\alpha a_n^2}\right),$$

the problem (1) has an unbounded sequence of solutions in E_0^α.

Proof. Clearly, from (A2) we obtain (A1), by choosing $a_n=0$ for all $n\in\mathbb{N}$. Moreover, if we assume (A2) instead of (A1) and set $r_n=\frac{|\cos(\pi\alpha)|}{\Omega^2}b_n^2$ for all $n\in\mathbb{N}$, by the same arguing as inside in Theorem 3.1, we obtain

$$\begin{aligned}
\varphi(r_n) &= \inf_{u\in\Phi^{-1}(]-\infty,r_n[)} \frac{(\sup_{v\in\Phi^{-1}(]-\infty,r_n])}\Psi(v))-\Psi(u)}{r_n-\Phi(u)}\\
&\le \frac{\sup_{v\in\Phi^{-1}(]-\infty,r_n])}\Psi(v)-\int_0^T(F(w_n(t))+\frac{\mu}{\lambda}G(w_n(t))dt}{r_n+\int_0^T {}_0^cD_t^\alpha w_n(t)\cdot {}_t^cD_T^\alpha w_n(t)dt}\\
&\le \frac{\max_{|x|\le b_n}F(x)+\frac{\mu}{\lambda}\max_{|x|\le b_n}G(x)-\frac{1}{|a_n|}\int_0^{\Gamma(2-\alpha)|a_n|}(F(x)+\frac{\mu}{\lambda}G(x))dx}{\frac{|\cos(\pi\alpha)|}{\Omega^2}b_n^2-\omega_\alpha a_n^2}
\end{aligned}$$

where

$$w_n(t)=\begin{cases}\frac{2\Gamma(2-\alpha)a_n}{T}t, & t\in[0,T/2),\\ \frac{2\Gamma(2-\alpha)a_n}{T}(T-t), & t\in[T/2,T].\end{cases}$$

Moreover, from Assumption (A2) and the condition (11) one has

$$\lim_{n\to\infty}\frac{\max_{|x|\le b_n}F(x)+\frac{\mu}{\lambda}\max_{|x|\le b_n}G(x)-\frac{1}{|a_n|}\int_0^{\Gamma(2-\alpha)|a_n|}(F(x)+\frac{\mu}{\lambda}G(x))dx}{\frac{|\cos(\pi\alpha)|}{\Omega^2}b_n^2-\omega_\alpha a_n^2}<+\infty.$$

Therefore,

$$\begin{aligned}
\gamma &\le \liminf_{n\to+\infty}\varphi(r_n)\\
&\le T\lim_{n\to\infty}\frac{\max_{|x|\le b_n}F(x)+\frac{\mu}{\lambda}\max_{|x|\le b_n}G(x)-\frac{1}{|a_n|}\int_0^{\Gamma(2-\alpha)|a_n|}(F(x)+\frac{\mu}{\lambda}G(x))dx}{\frac{|\cos(\pi\alpha)|}{\Omega^2}b_n^2-\omega_\alpha a_n^2}\\
&< +\infty.
\end{aligned}$$

So, we have the desired conclusion. □

The following result is a special case of Theorem 3.1 with $\mu=0$.

Theorem 3.3. *Assume that the assumptions in Theorem 3.1 hold. Then, for each*

$$\lambda\in\Lambda:=\left]\frac{\Gamma(2-\alpha)\omega_\alpha}{T\limsup_{\xi\to+\infty}\frac{\frac{1}{|\xi|}\int_0^{\Gamma(2-\alpha)|\xi|}F(x)dx}{\xi^2}},\ \frac{|\cos(\pi\alpha)|}{\Omega^2T\liminf_{\xi\to+\infty}\frac{\max_{|x|\le\xi}F(x)}{\xi^2}}\right[$$

the problem (1), for $\mu=0$, has an unbounded sequence of solutions in E_0^α.

Here we point out the following consequence of Theorem 3.3.

Corollary 3.4. *Let $\frac{1}{2}<\alpha\le 1$. Assume that*

(B1) $\liminf_{\xi\to+\infty}\frac{\max_{|x|\le\xi}F(x)}{\xi^2}<\frac{|\cos(\pi\alpha)|}{\Omega^2}.$;

(B2) $\limsup_{\xi\to+\infty}\frac{\frac{1}{|\xi|}\int_0^{\Gamma(2-\alpha)|\xi|}F(x)dx}{\xi^2}>\Gamma(2-\alpha)\omega_\alpha.$

Then, for each $\lambda\in]\lambda_1,\lambda_2[$ where

$$\lambda_1:=\frac{\Gamma(2-\alpha)\omega_\alpha}{T\limsup_{\xi\to+\infty}\frac{\frac{1}{|\xi|}\int_0^{\Gamma(2-\alpha)|\xi|}F(x)dx}{\xi^2}}$$

and

$$\lambda_2:=\frac{|\cos(\pi\alpha)|}{\Omega^2T\liminf_{\xi\to+\infty}\frac{\max_{|x|\le\xi}F(x)}{\xi^2}},$$

Then, the problem

$$\frac{d}{dt}\Big({}_0D_t^{\alpha-1}({}_0^cD_t^\alpha u(t)) - {}_tD_T^{\alpha-1}({}_t^cD_T^\alpha u(t))\Big) + f(u(t)) = 0, \quad a.e.\ t\in[0,T],$$
$$u(0)=u(T)=0,$$

has an unbounded sequence of solutions in E_0^α.

Remark 3.2. Theorem 1.1 in Introduction is an immediately consequence of Corollary 3.4.

We here give the following consequence of the main result:

Corollary 3.5. *Let* $\frac{1}{2}<\alpha\leq 1$. *Let* $h_1:\mathbb{R}\to\mathbb{R}$ *be a non-negative continuous function, and denote that* $H_1(x)=\int_0^x h_1(s)ds$ *for all* $x\in\mathbb{R}$. *Assume that*

(C1) $\liminf_{\xi\to+\infty}\frac{H_1(\xi)}{\xi^2}<+\infty$;

(C2) $\limsup_{\xi\to+\infty}\frac{\frac{1}{|\xi|}\int_0^{\Gamma(2-\alpha)|\xi|}H_1(x)dx}{\xi^2}=+\infty.$

Then, for every non-negative continuous $h_i:\mathbb{R}\to\mathbb{R}$ *for* $2\leq i\leq n$ *satisfying*

$$\max\Big\{\sup_{\xi\in\mathbb{R}}H_i(\xi);\ 2\leq i\leq n\Big\}\leq 0$$

and

$$\min\Big\{\liminf_{\xi\to+\infty}\frac{H_i(\xi)}{\xi^2};\ 2\leq i\leq n\Big\}>-\infty$$

where $H_i(x)=\int_0^x h_i(s)ds$ *for all* $x\in\mathbb{R}$ *for* $2\leq i\leq n$, *for each*

$$\lambda\in\left]0,\ \frac{|\cos(\pi\alpha)|}{\Omega^2T\liminf_{\xi\to+\infty}\frac{H_1(\xi)}{\xi^2}}\right[,$$

and for every arbitrary continuous function $g:\mathbb{R}\to\mathbb{R}$ *whose potential* $G(x)=\int_0^x g(s)ds$ *for every* $x\in\mathbb{R}$, *is a non-negative function satisfying the condition* (11) *and for every* $\mu\in[0,\mu_{G,\lambda}[$ *where*

$$\mu_{G,\lambda}:=\frac{|\cos(\pi\alpha)|}{\Omega^2TG_\infty}\left(1-\lambda T\frac{\Omega^2}{|\cos(\pi\alpha)|}\liminf_{\xi\to+\infty}\frac{H_1(\xi)}{\xi^2}\right),$$

the problem

$$\frac{d}{dt}\Big({}_0D_t^{\alpha-1}({}_0^cD_t^\alpha u(t)) - {}_tD_T^{\alpha-1}({}_t^cD_T^\alpha u(t))\Big) + \lambda\sum_{i=1}^n h_i(u(t)) + \mu g(u(t)) = 0,$$
$$a.e.\ t\in[0,T],$$
$$u(0)=u(T)=0,$$

has an unbounded sequence of solutions in E_0^α.

Proof. Set $f(x)=\sum_{i=1}^n h_i(x)$ for all $x\in\mathbb{R}$. Assumption (C2) together with the condition

$$\min\Big\{\liminf_{\xi\to+\infty}\frac{H_i(\xi)}{\xi^2};\ 2\leq i\leq n\Big\}>-\infty$$

ensures

$$\limsup_{\xi\to+\infty}\frac{\frac{1}{|\xi|}\int_0^{\Gamma(2-\alpha)|\xi|}F(x)dx}{\xi^2}=\limsup_{\xi\to+\infty}\frac{\frac{1}{|\xi|}\int_0^{\Gamma(2-\alpha)|\xi|}\sum_{i=1}^n H_i(x)dx}{\xi^2}=+\infty.$$

Moreover, Assumption (C1) along with condition

$$\max\left\{\sup_{\xi\in\mathbb{R}} H_i(\xi);\ 2\le i\le n\right\}\le 0$$

follows

$$\liminf_{\xi\to+\infty}\frac{\max_{|x|\le\xi}F(x)}{\xi^2}\le\liminf_{\xi\to+\infty}\frac{H_1(\xi)}{\xi^2}<+\infty.$$

Hence, from Corollary 3.1 the conclusion follows. □

Finally, we observe that by similar reasonings as in the proof of Theorem 3.1 but using conclusion (c) of Theorem 2.6 instead of (b), the following result holds.

Theorem 3.6. *Let $\frac{1}{2}<\alpha\le 1$. Assume that*

(A3) $\liminf_{\xi\to 0^+}\frac{\max_{|x|\le\xi}F(x)}{\xi^2}<\frac{|\cos(\pi\alpha)|}{\Gamma(2-\alpha)\omega_\alpha\Omega^2}\limsup_{\xi\to 0^+}\frac{\frac{1}{|\xi|}\int_0^{\Gamma(2-\alpha)|\xi|}F(x)dx}{\xi^2}$.

Then, for each $\lambda\in]\lambda_1,\lambda_2[$ where

$$\lambda_3:=\frac{\Gamma(2-\alpha)\omega_\alpha}{T\limsup_{\xi\to 0^+}\frac{\frac{1}{|\xi|}\int_0^{\Gamma(2-\alpha)|\xi|}F(x)dx}{\xi^2}}$$

and

$$\lambda_4:=\frac{|\cos(\pi\alpha)|}{\Omega^2T\liminf_{\xi\to 0^+}\frac{\max_{|x|\le\xi}F(x)}{\xi^2}},$$

for every arbitrary continuous function $g:\mathbb{R}\to\mathbb{R}$ whose potential $G(x)=\int_0^x g(s)ds$ for every $x\in\mathbb{R}$, is a non-negative function satisfying the condition

$$G_0:=lim_{\xi\to 0^+}\frac{\max_{|x|\le\xi}G(x)}{\xi^2}<+\infty \tag{19}$$

and for every $\mu\in[0,\mu_{G,\lambda}[$ where

$$\mu_{G,\lambda}:=\frac{|\cos(\pi\alpha)|}{\Omega^2TG_0}\left(1-\lambda T\frac{\Omega^2}{|\cos(\pi\alpha)|}\liminf_{\xi\to 0^+}\frac{\max_{|x|\le\xi}F(x)}{\xi^2}\right),$$

the problem (1) has a sequence of solutions, which strongly converges to 0 in E_0^α.

Proof. Fix $\overline{\lambda}\in]\lambda_3,\lambda_4[$ and let G be a function satisfies the condition (19). Since, $\overline{\lambda}<\lambda_2$, one has

$$\mu_{G,\overline{\lambda}}:=\frac{|\cos(\pi\alpha)|}{\Omega^2TG_0}\left(1-\overline{\lambda}T\frac{\Omega^2}{|\cos(\pi\alpha)|}\liminf_{\xi\to 0^+}\frac{\max_{|x|\le\xi}F(x)}{\xi^2}\right)>0.$$

Fix $\overline{\mu}\in]0,\mu_{G,\overline{\lambda}}[$ and put $\nu_1:=\lambda_3$ and $\nu_2:=\frac{\lambda_4}{1+\frac{\Omega^2}{|\cos(\pi\alpha)|}\frac{\overline{\mu}}{\overline{\lambda}}\lambda_2G_0}$. If $G_0=0$, clearly, $\nu_1=\lambda_3$, $\nu_2=\lambda_4$ and $\lambda\in]\nu_1,\nu_2[$. If $G_0\ne 0$, since $\overline{\mu}<\mu_{G,\overline{\lambda}}$, we obtain

$$\frac{\overline{\lambda}}{\lambda_2}+\frac{\Omega^2}{|\cos(\pi\alpha)|}\overline{\mu}TG_0<1,$$

and so

$$\frac{\lambda_2}{1+\frac{\Omega^2}{|\cos(\pi\alpha)|}\frac{\overline{\mu}}{\overline{\lambda}}\lambda_2TG_0}>\overline{\lambda},$$

namely, $\overline{\lambda} < \nu_2$. Hence, bering in mind that $\overline{\lambda} > \lambda_3 = \nu_1$, one has $\overline{\lambda} \in]0\nu_1, \nu_2[$. Taking (19) into account, one has

$$\liminf_{\xi\to 0^+} \frac{\max_{|x|\le\xi} F(x) + \frac{\overline{\mu}}{\overline{\lambda}} \max_{|x|\le\xi} G(x)}{\xi^2} \le \liminf_{\xi\to 0^+} \frac{\max_{|x|\le\xi} F(x)}{\xi^2} + \frac{\overline{\mu}}{\overline{\lambda}} G_0. \tag{20}$$

Therefore, from (20), we observe

$$\overline{\lambda} \in]\nu_1, \nu_2[\subseteq$$

$$\left] \frac{\Gamma(2-\alpha)\omega_\alpha}{T \limsup_{\xi\to 0^+} \frac{\frac{1}{|\xi|}\int_0^{\Gamma(2-\alpha)|\xi|}(F(x)+\frac{\overline{\mu}}{\overline{\lambda}}G(x))dx}{\xi^2}}, \frac{|\cos(\pi\alpha)|}{\Omega^2 T \liminf_{\xi\to 0^+} \frac{\max_{|x|\le\xi} F(x)+\frac{\overline{\mu}}{\overline{\lambda}}\max_{|x|\le\xi} G(x)}{\xi^2}} \right[.$$

We take X, Φ, Ψ and $I_{\overline{\lambda}}$ as in the proof of Theorem 3.1. Let $\{\xi_n\}$ be a sequence of positive numbers such that $\xi_n \to 0^+$ as $n \to +\infty$ and

$$\lim_{n\to\infty} \frac{\max_{|x|\le\xi_n} F(x) + \frac{\overline{\mu}}{\overline{\lambda}} \max_{|x|\le\xi_n} G(x)}{\xi_n^2} < +\infty.$$

Putting $r_n = \frac{|\cos(\pi\alpha)|}{\Omega^2}\xi_n^2$ for every $n \in \mathbb{N}$ and working as in the proof of Theorem 3.1 it follows that $\delta < +\infty$. Let us show that the functional $I_{\overline{\lambda}}$ has not a local minimum at zero. For this, let $\{d_n\}$ be a sequence of positive numbers and $\tau > 0$ such that $d_n \to 0^+$ as $n \to \infty$ and

$$\frac{1}{\overline{\lambda}} < \tau < \frac{T}{\Gamma(2-\alpha)\omega_\alpha} \frac{\frac{1}{|d_n|}\int_0^{\Gamma(2-\alpha)|d_n|}(F(x) + \frac{\overline{\mu}}{\overline{\lambda}}G(x))dx}{d_n^2} \tag{21}$$

for each $n \in \mathbb{N}$ large enough. Let $\{w_n\}$ be a sequence in X defined by setting w_n as given in (16). Putting together (17), (18) and (21) we achieve

$$\begin{aligned} I_{\overline{\lambda}}(w_n) &= \Phi(w_n) - \overline{\lambda}\Psi(w_n) \\ &= \omega_\alpha d_n^2 - \overline{\lambda}\frac{T}{\Gamma(2-\alpha)|d_n|}\int_0^{\Gamma(2-\alpha)|d_n|}(F(x) + \frac{\overline{\mu}}{\overline{\lambda}}G(x))dx \\ &< (1-\overline{\lambda}\tau)\omega_\alpha d_n^2 < 0 \end{aligned}$$

for every $n \in \mathbb{N}$ large enough. Since $I_{\overline{\lambda}}(0) = 0$, this ensures that the functional I_λ has not a local minimum at zero. Hence, recalling (8), the part (c) of Theorem 2.6 ensures that there exists a sequence $\{u_n\}$ in X of critical points of $I_{\overline{\lambda}}$ such that $||u_n|| \to 0$ as $n \to \infty$, and the proof is complete. □

Remark 3.3. Applying Theorem 3.6, results similar to Theorems 1.1, 3.2 and 3.3 and Corollaries 3.4 and 3.5 can be obtained by replacing $\xi \to +\infty$ with $\xi \to 0^+$.

References

[1] G.A. Afrouzi, A. Hadjian and G. Molica Bisci, Some results for one dimensional fractional problems, preprint.

[2] R.P. Agarwal, M. Benchohra and S. Hamani, A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions, *Acta Appl. Math.* **109** (2010), 973–1033.

[3] B. Ahmad and J.J. Nieto, Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions, *Comput. Math. Appl.* **58** (2009), 1838–1843.

[4] C. Bai, Existence of solutions for a nonlinear fractional boundary value problem via a local minimum theorem, *Electronic Journal of Differential Equations* Vol. **2012** (2012), no. 176, 1–9.

[5] C. Bai, Infinitely many solutions for a perturbed nonlinear fractional boundary-value problem, *Electronic Journal of Differential Equations* Vol. **2013** (2013), no. 136, 1–12.

[6] C. Bai, Impulsive periodic boundary value problems for fractional differential equation involving Riemann-Liouville sequential fractional derivative, *J. Math. Anal. Appl.* **384** (2011), 211–231.

[7] C. Bai, Solvability of multi-point boundary value problem of nonlinear impulsive fractional differential equation at resonance, *Electron. J. Qual. Theory Differ. Equ.* **2011** (2011), no. 89, 1–19.

[8] Z. Bai and H. Lu, Positive solutions for boundary value problem of nonlinear fractional differential equation, *J. Math. Anal. Appl.* **311** (2005), 495–505.

[9] M. Benchohra, S. Hamani anf S.K. Ntouyas, Boundary value problems for differential equations with fractional order and nonlocal conditions, *Nonlinear Anal. TMA* **71** (2009), 2391–2396.

[10] G. Bonanno, A critical point theorem via the Ekeland variational principle, *Nonlinear Anal. TMA* **75** (2012), 2992–3007.

[11] G. Bonanno and G. D'Aguì, On the Neumann problem for elliptic equations involving the p-Laplacian, *J. Math. Anal. Appl.* **358** (2009), 223–228.

[12] G. Bonanno and B. Di Bella, Infinitely many solutions for a fourth-order elastic beam equation, *Nonlinear Differential Equations and Applications NoDEA* **18** (2011), 357–368.

[13] G. Bonanno and G. Molica Bisci, Infinitely many solutions for a boundary value problem with discontinuous nonlinearities, *Bound. Value Probl.* **2009** (2009), 1–20.

[14] G. Bonanno and G. Molica Bisci, Infinitely many solutions for a Dirichlet problem involving the p-Laplacian, *Proceedings of the Royal Society of Edinburgh A* **140** (2010), no. 04, 737–752.

[15] G. Bonanno, G. Molica Bisci and D. O'Regan, Infinitely many weak solutions for a class of quasilinear elliptic systems, *Math. Comput. Modelling* **52** (2010), 152–160.

[16] J. Chen andd X.H. Tang, Existence and multiplicity of solutions for some fractional boundary value problem via critical point theory, *Abstract and Applied Analysis* **2012** (2012), 1–21.

[17] J.-N. Corvellec, V.V. Motreanu and C. Saccon, Doubly resonant semilinear elliptic problems via nonsmooth critical point theory, *J. Differential Equations* **248** (2010), 2064–2091.

[18] G. D'Aguì, Infinitely many solutions for a double Sturm-Liouville problem, *J. Global Optim.* **54** (2012), 619–625.

[19] M. Ferrara and G. Molica Bisci, Remarks for one-dimensional fractional equations, *Opuscula Mathematica*, to appear.

[20] M. Galewski and G. Molica Bisci, Existence results for one-dimensional fractional equations, arXiv:1402.1529v1 [math.CA] 6 Feb 2014.

[21] J.R. Graef, S. Heidarkhani and L. Kong, Infinitely many solutions for systems of multi-point boundary value problems using variational methods, *Topological Methods in Nonlinear Analysis* **42** (2013), no. 1, 105–118.

[22] J.R. Graef, S. Heidarkhani and L. Kong, Infinitely many solutions for systems of Sturm-Liouville boundary value problems, *Results in Mathematics*, to appear.

[23] S. Heidarkhani, Infinitely many solutions for systems of n two-point boundary value Kirchhoff-type problems, *Annales Polonici Mathematici* **107** (2013), no 2, 133–152.

[24] S. Heidarkhani, Multiple solutions for a nonlinear perturbed fractional boundary value problem, *Dynamic Systems and Applications*, to appear.

[25] R. Hilferm, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[26] F. Jiao and Y. Zhou, Existence of solutions for a class of fractional boundary value problems via critical point theory, *Comput. Math. Appl.* **62** (2011), 1181–1199.

[27] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[28] N. Kosmatov, Integral equations and initial value problems for nonlinear differential equations of fractional order, *Nonlinear Anal. TMA* **70** (2009), 2521–2529.

[29] V. Lakshmikantham and A.S. Vatsala, Basic theory of fractional differential equations, *Nonlinear Anal. TMA* **69** (2008), 2677–2682.

[30] F. Li, Z. Liang and Q. Zhang, Existence of solutions to a class of nonlinear second order two-point boundary value problems, *J. Math. Anal. Appl.* **312** (2005), 357–373.

[31] S.A. Marano and D. Motreanu, Infinitely many critical points of non-differentiable functions and applications to a Neumann type problem involving the p-Laplacian, *J. Differential Equations* **182** (2002), 108–120.

[32] J. Mawhin and M. Willem, *Critical Point Theorey and Hamiltonian Systems*, Springer, New York, 1989.

[33] K.S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, 1993.
[34] G. Molica Bisci, Fractional equations with bounded primitive, *Applied Mathematics Letters* **27** (2014), 53–58.
[35] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.
[36] P.H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, in: CBMS, vol. 65, American Mathematical Society, 1986.
[37] B. Ricceri, A general variational principle and some of its applications, *J. Comput. Appl. Math.* **113** (2000), 401–410.
[38] S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional Integral and Derivatives: Theory and Applications*, Gordon and Breach, Longhorne, PA, 1993.
[39] C. Tang and X. Wu, Some critical point theorems and their applications to periodic solution for second order Hamiltonian systems, *J. Differential Equations* **248** (2010), 660–692.
[40] J. Wang and Y. Zhou, A class of fractional evolution equations and optimal controls, *Nonlinear Anal. RWA* **12** (2011), 262–272.
[41] Z. Wei, W. Dong and J. Che, Periodic boundary value problems for fractional differential equations involving a Riemann-Liouville fractional derivative, *Nonlinear Anal. TMA* **73** (2010), 3232–3238.
[42] S. Zhang, Positive solutions to singular boundary value problem for nonlinear fractional differential equation, *Comput. Math. Appl.* **59** (2010), 1300–1309.

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