Infinitely many solutions for nonlinear perturbed fractional boundary value problems

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Abstract. In this paper, we prove the existence of infinitely many solutions to nonlinear perturbed fractional boundary value problems. The approach is based on critical point theory and variational methods.

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1. Introduction

In this paper, we are interested in ensuring the existence of infinitely many solutions for the following perturbed fractional boundary value problem

\[
\frac{d}{dt} \left( \int_0^t \frac{\partial \psi}{\partial \tau} \right) - \int_T^\tau \frac{\partial \psi}{\partial \tau} \int_0^t \frac{\partial \psi}{\partial \tau} d\tau - \lambda f(u(t)) + \mu g(u(t)) = 0, \quad \text{a.e. } t \in [0, T],
\]

\[u(0) = u(T) = 0,
\]

where \( \alpha \in (1/2, 1] \), \( _0D_t^{\alpha-1} \) and \( _T^\alpha \) are the left and right Riemann-Liouville fractional integrals of order \( 1 - \alpha \) respectively, \( \frac{\partial \psi}{\partial \tau} \) and \( \frac{\partial \psi}{\partial \tau} \) are the left and right Caputo fractional derivatives of order \( 0 < \alpha \leq 1 \) respectively, \( \lambda \) is a positive real parameter, \( \mu \) is a non-negative real parameter and \( f, g : \mathbb{R} \rightarrow \mathbb{R} \) are continuous functions.

Because of its wide applicability in the modeling of many phenomena in various fields of physics, chemistry, biology, engineering and economics, the theory of fractional differential equations has recently been attracting increasing interest, see for instance the monographs of Miller and Ross [33], Samko et al [38], Podlubny [35], Hilfer [25], Kilbas et al [27] and the papers [2, 3, 6, 7, 8, 9, 28, 29, 40, 41, 42] and the references therein.

Critical point theory has been very useful in determining the existence of solution for integer order differential equations with some boundary conditions, for example [17, 29, 30, 32, 36, 39]. But until now, there are few results on the solution to fractional boundary value problems which were established by the critical point theory, since it is often very difficult to establish a suitable space and variational functional for fractional boundary value problems. Recently, Jiao and Zhou in [26] by using the critical point theory investigated the fractional boundary-value problem

\[
\frac{d}{dt} \left( \int_0^t \frac{\partial \psi}{\partial \tau} \right) - \int_T^\tau \frac{\partial \psi}{\partial \tau} \int_0^t \frac{\partial \psi}{\partial \tau} d\tau - \lambda f(u(t)) + \mu g(u(t)) = 0, \quad \text{a.e. } t \in [0, T],
\]

\[u(0) = u(T) = 0
\]

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where $0D_t^{-\beta}$ and $tD_T^{-\beta}$ are the left and right Riemann-Liouville fractional integrals of order $0 \leq \beta < 1$ respectively, $F : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ is a given function and $\nabla F(t, x)$ is the gradient of $F$ at $x$. Also, Chen and Tang in [16] studied the existence and multiplicity of solutions for the following fractional boundary value problem
\begin{equation*}
dt\left(\frac{1}{2}0D_t^{-\beta}(u'(t)) + \frac{1}{2}tD_T^{-\beta}(u'(t))\right) + \nabla F(t, u(t)) = 0, \quad \text{a.e. } t \in [0, T],
\end{equation*}
u(0) = u(T) = 0,
\end{equation*}
where $F(t, \cdot)$ are superquadratic, asymptotically quadratic, and subquadratic, respectively. In particular, Bai in [4], by using a local minimum theorem due to Bonanno ([10]), investigated the existence of at least one non-trivial solution to the problem (1).

In the present paper, motivated by [4], employing a smooth version of Theorem 2.1 of [13] which is a more precise version of Ricceri’s Variational Principle [37, Theorem 2.5] (see Theorem 2.6), requiring that the nonlinear term $f$ has a suitable oscillating behavior at infinity, in Theorem 3.1, we establish the existence of a precise interval of parameters $\Lambda$ such that, for each $\lambda \in \Lambda$ and every arbitrary continuous function $g$ which satisfies a certain growth at infinity, choosing $\mu$ sufficiently small, the perturbed problem (1) admits a sequence of solutions which are unbounded in the fractional derivative space $E_0^\alpha$. We also list some consequences of Theorem 3.1 and one example. Finally, we present an analogous result (see Theorem 3.6), in which we replace the oscillating behavior condition at infinity in Theorem 3.1, by a similar one at zero. In this setting, a sequence of pairwise distinct non-zero solutions which converges to zero is achieved.

A special case of our main result is the following theorem.

**Theorem 1.1.** Let $\frac{1}{2} < \alpha \leq 1$. Assume that
\begin{equation*}
\liminf_{\xi \to +\infty} \max_{|x| \leq \xi} \frac{F(x)}{\xi^2} = 0 \quad \text{and} \quad \limsup_{\xi \to +\infty} \frac{1}{\xi^2} \int_0^\Gamma F(x)dx = +\infty
\end{equation*}
where $F(x) = \int_0^x f(s)ds$ for every $x \in \mathbb{R}$ and $\Gamma$ is the gamma function. Then, the problem
\begin{equation*}
\frac{d}{dt}\left(0D_t^{\alpha-1}(0D_t^\alpha u(t)) - tD_T^{\alpha-1}(tD_T^\alpha u(t))\right) + f(u(t)) = 0, \quad \text{a.e. } t \in [0, T],
u(0) = u(T) = 0,
\end{equation*}
admits a sequence of pairwise distinct positive solutions.

For a discussion about the existence of infinitely many solutions for boundary value problems, using Ricceri’s Variational Principle [37] and its variants ([13, Theorem 2.1] and [31, Theorem 1.1]) we refer the reader to the papers [5, 11, 12, 14, 15, 18, 21, 22, 23].

For a through on the subject, we also refer the reader to [1, 19, 20, 24, 34].

2. Preliminaries

In this section, we will introduce some notations, definitions and preliminary facts which are used throughout this paper.
Definition 2.1 ([27]). Let \( f \) be a function defined on \([a, b]\) and \( \alpha > 0 \). The left and right Riemann-Liouville fractional integrals of order \( \alpha \) for the function \( f \) are defined by

\[
\begin{align*}
\frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha-1} f(s) \, ds, & \quad t \in [a, b], \\
\frac{1}{\Gamma(\alpha)} \int_t^b (s - t)^{\alpha-1} f(s) \, ds, & \quad t \in [a, b],
\end{align*}
\]

provided the right-hand sides are pointwise defined on \([a, b]\), where \( \Gamma(\alpha) \) is the gamma function.

Definition 2.2 ([27]). Let \( \gamma \geq 0 \) and \( n \in \mathbb{N} \).

(i) If \( \gamma \in (n - 1, n) \) and \( f \in AC^n([a, b], \mathbb{R}^N) \), then the left and right Caputo fractional derivatives of order \( \gamma \) for function \( f \) denoted by \( \frac{c}{a}D_t^\gamma f(t) \) and \( \frac{c}{b}D_t^\gamma f(t) \), respectively, exist almost everywhere on \([a, b]\), \( \frac{c}{a}D_t^\gamma f(t) \) and \( \frac{c}{b}D_t^\gamma f(t) \) are represented by

\[
\begin{align*}
\frac{c}{a}D_t^\gamma f(t) &= \frac{1}{\Gamma(n - \gamma)} \int_a^t (t - s)^{n-\gamma-1} f^{(n)}(s) \, ds, \quad t \in [a, b], \\
\frac{c}{b}D_t^\gamma f(t) &= \frac{(-1)^n}{\Gamma(n - \gamma)} \int_t^b (s - t)^{n-\gamma-1} f^{(n)}(s) \, ds, \quad t \in [a, b],
\end{align*}
\]

respectively.

(ii) If \( \gamma = n - 1 \) and \( f \in AC^{n-1}([a, b], \mathbb{R}^N) \), then \( \frac{c}{a}D_t^{n-1} f(t) \) and \( \frac{c}{b}D_t^{n-1} f(t) \) are represented by

\[
\frac{c}{a}D_t^{n-1} f(t) = f^{(n-1)}(t), \quad \text{and} \quad \frac{c}{b}D_t^{n-1} f(t) = (-1)^{(n-1)} f^{(n-1)}(t), \quad t \in [a, b].
\]

With these definitions, we have the rule for fractional integration by parts, and the composition of the Riemann-Liouville fractional integration operator with the Caputo fractional differentiation operator, which were proved in [27, 38].

Proposition 2.1 ([27, 38]). We have the following property of fractional integration

\[
\int_a^b [\frac{c}{a}D_t^\gamma f(t)]g(t) \, dt = \int_a^b [\frac{c}{b}D_t^\gamma g(t)]f(t) \, dt, \quad \gamma > 0,
\]

provided that \( f \in L^p([a, b], \mathbb{R}^N) \), \( g \in L^q([a, b], \mathbb{R}^N) \) and \( p \geq 1, q \geq 1, 1/p + 1/q = 1 + \gamma \) or \( p \neq 1, q \neq 1, 1/p + 1/q = 1 + \gamma \).

Proposition 2.2 ([27]). Let \( n \in \mathbb{N} \) and \( n - 1 < \gamma \leq n \). If \( f \in AC^n([a, b], \mathbb{R}^N) \) or \( f \in C^n([a, b], \mathbb{R}^N) \), then

\[
\begin{align*}
\frac{c}{a}D_t^{-\gamma} (\frac{c}{a}D_t^\gamma f(t)) &= f(t) - \sum_{j=0}^{n-1} \frac{f^{(j)}(a)}{j!} (t - a)^j, \\
\frac{c}{b}D_t^{-\gamma} (\frac{c}{b}D_t^\gamma f(t)) &= f(t) - \sum_{j=0}^{n-1} \frac{(-1)^j f^{(j)}(b)}{j!} (b - t)^j,
\end{align*}
\]

for \( t \in [a, b] \). In particular, if \( 0 < \gamma \leq 1 \) and \( f \in AC([a, b], \mathbb{R}^N) \) or \( f \in C^1([a, b], \mathbb{R}^N) \), then

\[
\begin{align*}
\frac{c}{a}D_t^{-\gamma} (\frac{c}{a}D_t^\gamma f(t)) &= f(t) - f(a), \quad \text{and} \quad \frac{c}{b}D_t^{-\gamma} (\frac{c}{b}D_t^\gamma f(t)) &= f(t) - f(b).
\end{align*}
\]
Remark 2.1. In view of (2) and Definition 2.2, it is obvious that $u \in AC([0, T])$ is a solution of (1) if and only if $u$ is a solution of the problem

$$
\frac{d}{dt} \left( \alpha D_t^{-\beta}(u'(t)) + \lambda f(u(t)) + \mu g(u(t)) \right) = 0, \quad \text{a.e. } t \in [0, T],
$$

where $\beta = 2(1 - \alpha) \in [0, 1)$.

To establish a variational structure for (1), it is necessary to construct appropriate function spaces.

Definition 2.3 ([26]). Let $0 < \alpha \leq 1$. The fractional derivative space $E_0^\alpha$ is defined by the closure of $C_0^\infty[0, T]$ with respect to the norm

$$
\|u\|_\alpha = \left( \int_0^T |\partial_0^\alpha D_t^\alpha u(t)|^2 dt + \int_0^T |u(t)|^2 dt \right)^{1/2}, \quad \forall u \in E_0^\alpha,
$$

where $C_0^\infty[0, T]$ denotes the set of all functions $u \in C^\infty[0, T]$ with $u(0) = u(T) = 0$. It is obvious that the fractional derivative space $E_0^\alpha$ is the space of functions $u \in L^2[0, T]$ having an $\alpha$-order Caputo fractional derivative $\partial_0^\alpha D_t^\alpha u \in L^2[0, T]$ and $u(0) = u(T) = 0$.

Proposition 2.3 ([26]). Let $0 < \alpha \leq 1$. The fractional derivative space $E_0^\alpha$ is reflexive and separable Banach space.

Proposition 2.4 ([26]). Let $0 < \alpha \leq 1$. For all $u \in E_0^\alpha$, we have

$$
\|u\|_{L^2} \leq \frac{T^\alpha}{\Gamma(\alpha + 1)} \|\partial_0^\alpha D_t^\alpha u\|_{L^2},
$$

$$
\|u\|_\infty \leq \frac{T^{\alpha-1/2}}{\Gamma(\alpha)(2(\alpha - 1) + 1)^{1/2}} \|\partial_0^\alpha D_t^\alpha u\|_{L^2}.
$$

According to (5), we can consider $E_0^\alpha$ with respect to the norm

$$
\|u\|_\alpha = \left( \int_0^T |\partial_0^\alpha D_t^\alpha u(t)|^2 dt \right)^{1/2} = \|\partial_0^\alpha D_t^\alpha u\|_{L^2}, \quad \forall u \in E_0^\alpha
$$

in the following analysis.

Proposition 2.5 ([26]). Let $1/2 < \alpha \leq 1$, then for all any $u \in E_0^\alpha$, we have

$$
|\cos(\pi\alpha)||u|_\alpha^2 \leq - \int_0^T \partial_0^\alpha D_t^\alpha u(t) \cdot \partial_0^\alpha D_t^\alpha u(t) dt \leq \frac{1}{|\cos(\pi\alpha)|} \|u\|_\alpha^2.
$$

By Proposition 2.4, when $\alpha > 1/2$, for each $u \in E_0^\alpha$ we have

$$
\|u\|_\infty \leq \Omega \left( \int_0^T |\partial_0^\alpha D_t^\alpha u(t)|^2 dt \right)^{1/2} = \Omega \|u\|_\alpha,
$$

where

$$
\Omega = \frac{T^{\alpha-1/2}}{\Gamma(\alpha)\sqrt{2(\alpha - 1) + 1}}
$$

Our main tool is the celebrated Ricceri’s Variational Principle [37, Theorem 2.5] that we now recall as given by Bonanno and Molica Bisci in [13].
Theorem 2.6. Let $X$ be a reflexive real Banach space, let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that $\Phi$ is sequentially weakly lower semicontinuous, strongly continuous, and coercive and $\Psi$ is sequentially weakly upper semicontinuous. For every $r > \inf X \Phi$, let us put

$$\varphi(r) := \inf_{u \in \Phi^{-1}([-\infty, r])} \sup_{v \in \Phi^{-1}([-\infty, r])} \frac{\Psi(v) - \Psi(u)}{r - \Phi(u)}$$

and

$$\gamma := \liminf_{r \to +\infty} \varphi(r), \quad \delta := \liminf_{r \to (\inf X \Phi)^+} \varphi(r).$$

Then, one has

(a) for every $r > \inf X \Phi$ and every $\lambda \in ]0, \frac{1}{\varphi(r)}[$, the restriction of the functional $I_\lambda = \Phi - \lambda \Psi$ to $\Phi^{-1}([-\infty, r])$ admits a global minimum, which is a critical point (local minimum) of $I_\lambda$ in $X$.

(b) If $\gamma < +\infty$ then, for each $\lambda \in ]0, \frac{1}{\gamma}[$, the following alternative holds:

either

(b1) $I_\lambda$ possesses a global minimum,

or

(b2) there is a sequence $\{u_n\}$ of critical points (local minima) of $I_\lambda$ such that

$$\lim_{n \to +\infty} \Phi(u_n) = +\infty.$$

(c) If $\delta < +\infty$ then, for each $\lambda \in ]0, \frac{1}{\delta}[$, the following alternative holds:

either

(c1) there is a global minimum of $\Phi$ which is a local minimum of $I_\lambda$,

or

(c2) there is a sequence of pairwise distinct critical points (local minima) of $I_\lambda$ which weakly converges to a global minimum of $\Phi$.

3. Main results

Put

$$\omega_\alpha := \frac{4\Gamma^2(2 - \alpha)}{\Gamma(4 - 2\alpha)} T^{1 - 2\alpha}(2^{2\alpha - 1} - 1)$$

and $F(x) = \int_0^x f(s)ds$ for every $x \in \mathbb{R}$.

We state our main result as follows:

Theorem 3.1. Let $\frac{1}{2} < \alpha \leq 1$. Assume that

(A1) $\liminf_{|\xi| \to +\infty} \frac{\max_{|x| \leq |\xi|} F(x)}{|\xi|^\alpha} < \frac{|\cos(\pi \alpha)|}{\Gamma(2 - \alpha) T^{1 - \alpha} \omega_\alpha}$

Then, for each $\lambda \in ]\lambda_1, \lambda_2[$ where

$$\lambda_1 := \frac{\Gamma(2 - \alpha)\omega_\alpha}{T \limsup_{|\xi| \to +\infty} \frac{\int_0^{T(2 - \alpha)\omega_\alpha |\xi|^\alpha F(x)dx}{|\xi|^\alpha}}$$

and

$$\lambda_2 := \frac{|\cos(\pi \alpha)|}{\Omega^2 T \liminf_{|\xi| \to +\infty} \frac{\max_{|x| \leq |\xi|} F(x)}{|\xi|^\alpha}},$$
for every arbitrary continuous function \( g : \mathbb{R} \to \mathbb{R} \) whose potential \( G(x) = \int_0^x g(s)ds \) for every \( x \in \mathbb{R} \), is a non-negative function satisfying the condition

\[
G_\infty := \lim_{\xi \to +\infty} \frac{\max_{|x| \leq \xi} G(x)}{\xi^2} < +\infty
\]

and for every \( \mu \in [0, \mu_{G,\lambda}] \) where

\[
\mu_{G,\lambda} := \frac{\left| \cos(\pi \alpha) \right|}{\Omega^2 TG_\infty} \left( 1 - \lambda T \frac{\Omega^2}{\left| \cos(\pi \alpha) \right|} \liminf_{\xi \to +\infty} \frac{\max_{|x| \leq \xi} F(t)}{\xi^2} \right),
\]

the problem (1) has an unbounded sequence of solutions in \( E_0^\alpha \).

**Proof.** In order to apply Theorem 2.6 to our problem, let \( X \) be the fractional derivative space \( E_0^\alpha \) equipped with the norm

\[
\|u\|_\alpha := \left( \int_0^T \|D_\tau^\alpha u(t)\|^2 dt \right)^{1/2},
\]

and we introduce the functionals \( \Phi, \Psi : X \to \mathbb{R} \) for each \( u \in X \) as follows:

\[
\Phi(u) := -\int_0^T \left( \frac{1}{\alpha} D_\tau^\alpha u(t) \cdot \partial_\tau D_\tau^\alpha u(t) dt, \quad \Psi(u) := \int_0^T (F(u(t)) + \frac{\pi}{\lambda} G(u(t))) dt.
\]

Clearly, \( \Phi \) and \( \Psi \) are Gâteaux differentiable functional whose Gâteaux derivative at the point \( u \in X \) are given by

\[
\Phi'(u)v = -\int_0^T \left( \frac{1}{\alpha} D_\tau^\alpha u(t) \cdot \partial_\tau D_\tau^\alpha v(t) + \frac{\pi}{\lambda} D_\tau^\alpha u(t) \cdot \partial_\tau D_\tau^\alpha v(t) dt,
\]

\[
\Psi'(u)v = \int_0^T (f(u(t))) v(t) dt = -\int_0^T \int_0^t f(u(s)) ds \cdot v'(t) dt + \frac{\pi}{\lambda} \int_0^T \int_0^t g(u(s)) ds \cdot v'(t) dt
\]

for every \( v \in X \). By Definition 2.2 and (3), we have

\[
\Phi'(u)v = \int_0^T \left( \frac{1}{\alpha} D_\tau^{\alpha-1} (\frac{1}{\alpha} D_\tau^\alpha u(t)) + \frac{\pi}{\lambda} D_\tau^{\alpha-1} (\frac{\pi}{\lambda} D_\tau^\alpha u(t)) \right) \cdot v'(t) dt.
\]

Put \( L_{\lambda} := \Phi - \lambda \Psi \). The solutions of the problem (1) are exactly the solutions of the equation \( L_{\lambda}(u) = 0 \) (see [4]). Fix \( \lambda \in ]\lambda_1, \lambda_2[ \) and let \( G \) be a non-negative function satisfies the condition (11). Since, \( \lambda < \lambda_2 \), one has

\[
\mu_{G,\lambda} := \frac{\left| \cos(\pi \alpha) \right|}{\Omega^2 TG_\infty} \left( 1 - \lambda T \frac{\Omega^2}{\left| \cos(\pi \alpha) \right|} \liminf_{\xi \to +\infty} \frac{\max_{|x| \leq \xi} F(t)}{\xi^2} \right) > 0.
\]

Fix \( \pi \in ]0, \mu_{G,\lambda}[ \) and set \( \nu_1 := \lambda_1 \) and \( \nu_2 := \frac{\pi^2}{\Omega^2 TG_\infty} \frac{\lambda_2}{1 + \frac{\lambda_2}{\pi^2 TG_\infty}} \). If \( G_\infty = 0 \), clearly, \( \nu_1 = \lambda_1, \nu_2 = \lambda_2 \) and \( \lambda \in ]\nu_1, \nu_2[ \). If \( G_\infty \neq 0 \), since \( \pi < \mu_{G,\lambda} \), we obtain

\[
\frac{\lambda}{\lambda_2} + \frac{\Omega^2}{\left| \cos(\pi \alpha) \right|} \frac{\pi T G_\infty}{\lambda_1} < 1,
\]

and so

\[
\frac{\lambda}{1 + \frac{\Omega^2}{\left| \cos(\pi \alpha) \right|} \frac{\pi T G_\infty}{\lambda_1}} > \lambda_2,
\]

namely, \( \lambda < \nu_2 \). Hence, bering in mind that \( \lambda > \lambda_1 = \nu_1 \), one has \( \lambda \in ]\nu_1, \nu_2[ \).

Now, let us show that

\[
\gamma < +\infty.
\]
Let \( \{\xi_n\} \) be a real sequence such that \( \xi_n \to +\infty \) as \( n \to \infty \) and
\[
\lim_{n \to \infty} \max_{|x| \leq \xi_n} F(x) + \frac{r}{\lambda} \max_{|x| \leq \xi_n} G(x) = \liminf_{\xi \to +\infty} \max_{|x| \leq \xi} F(x) + \frac{r}{\lambda} \max_{|x| \leq \xi} G(x).
\]
For every \( n \in \mathbb{N} \) let us consider \( r_n = \frac{1}{\max_{|x| \leq \xi_n}} \). Taking (8) into account, for all \( u \in X \) such that \( u \in \Phi^{-1}([-\infty, r_n]) \), we have
\[
|\cos(\pi \alpha)|\|u\|_2^2 \leq \Phi(u) \leq r_n,
\]
which implies
\[
\|u\|_2^2 \leq \frac{1}{|\cos(\pi \alpha)|} r_n. \tag{12}
\]
Thus, by (9) and (12) we obtain
\[
|u(t)| \leq \Omega\|u\|_2 \leq \Omega \sqrt{r} \frac{r}{|\cos(\pi \alpha)|} = \xi_n, \quad \forall t \in [0, T],
\]
which from the definition of \( \Psi \) follows
\[
\sup_{u \in \Phi^{-1}([-\infty, r_n])} \Psi(u) \leq T \max_{|x| \leq \xi_n} (F(x) + \frac{r}{\lambda} G(x)) \leq T(\max_{|x| \leq \xi_n} F(x) + \frac{r}{\lambda} \max_{|x| \leq \xi_n} G(x)).
\]
Therefore, since \( \Phi(0) = \Psi(0) = 0 \), for every \( n \) large enough, one has
\[
\varphi(r_n) = \inf_{u \in \Phi^{-1}([-\infty, r_n])} \frac{(\sup_{u \in \Phi^{-1}([-\infty, r_n])} \Psi(v)) - \Psi(u)}{r_n - \Phi(u)} \leq \frac{\sup_{u \in \Phi^{-1}([-\infty, r_n])} \Psi(v)}{r_n} \leq \varphi(r_n) \leq T \frac{\max_{|x| \leq \xi_n} F(x) + \frac{r}{\lambda} \max_{|x| \leq \xi_n} G(x)}{|\cos(\pi \alpha)| \xi_n^2}.
\]
Moreover, from Assumption (A1) and the condition (11) one has
\[
\lim_{n \to \infty} \max_{|x| \leq \xi_n} F(x) + \frac{r}{\lambda} \max_{|x| \leq \xi_n} G(x) < +\infty.
\]
So,
\[
\gamma \leq \liminf_{n \to +\infty} \varphi(r_n) \leq T \frac{\Omega^2}{|\cos(\pi \alpha)|} \lim_{n \to \infty} \max_{|x| \leq \xi_n} F(x) + \frac{r}{\lambda} \max_{|x| \leq \xi_n} G(x) < +\infty. \tag{13}
\]
Taking (11) into account, one has
\[
\liminf_{\xi \to +\infty} \max_{|x| \leq \xi} F(x) + \frac{r}{\lambda} \max_{|x| \leq \xi} G(x) \leq \liminf_{\xi \to +\infty} \max_{|x| \leq \xi} F(x) + \frac{r}{\lambda} G(x) < +\infty, \tag{14}
\]
which follows
\[
\exists \epsilon \in [\nu_1, \nu_2] \ni
\]
\[
\frac{\Gamma(2 - \alpha)}{\omega_1} \int_0^T \frac{F(x) + \frac{r}{\lambda} G(x)}{\xi^2} dx \leq \Omega^2 T \liminf_{\xi \to +\infty} \max_{|x| \leq \xi} (F(x) + \frac{r}{\lambda} G(x)) \frac{|\cos(\pi \alpha)|}{\xi^4}.
\]
Assumption (A1) in conjunction with (13), implies
\[
\left( \frac{1}{\Gamma(2-\alpha)\omega_\alpha} T \limsup_{\xi \to +\infty} \frac{1}{\xi^2} \int_0^T (F(x) + \frac{\omega}{\xi} G(x)) \, dx \right) \leq \left( \frac{\omega_\alpha}{\Gamma(2-\alpha)} \frac{|\cos(\pi \alpha)|}{\max_{1 \leq j \leq \xi} (F(x) + \frac{\omega}{\xi} G(x))} \right) \Omega^2 T \liminf_{\xi \to +\infty} \frac{1}{\xi^2}.
\]
For the fixed $\Xi$, the inequality (13) ensures that the condition (b) of Theorem 2.6 can be applied and either $I_\Xi$ has a global minimum or there exists a sequence $\{u_n\}$ of weak solutions of the problem (1) such that $\lim_{n \to \infty} \|u_n\| = +\infty$.

The other step is to show that for the fixed $\Xi$ the functional $I_\Xi$ has no global minimum. Let us verify that the functional $I_\Xi$ is unbounded from below. Since
\[
\frac{1}{\lambda} < \frac{T}{\Gamma(2-\alpha)\omega_\alpha} \limsup_{\xi \to +\infty} \frac{1}{\xi^2} \int_0^T (F(x) + \frac{\omega}{\xi} G(x)) \, dx
\]
we can consider a real sequence $\{d_n\}$ and a positive constant $\tau$ such that $d_n \to +\infty$ as $n \to \infty$ and
\[
\frac{1}{\lambda} < \tau < \frac{T}{\Gamma(2-\alpha)\omega_\alpha} \frac{1}{d_n} \int_0^T (F(x) + \frac{\omega}{\xi} G(x)) \, dx
\]
for each $n \in \mathbb{N}$ large enough. Let $\{w_n\}$ be a sequence in $X$ defined by putting
\[
w_n(t) = \begin{cases} \frac{2\Gamma(2-\alpha)d_n}{T} t, & t \in [0, T/2), \\ \frac{2\Gamma(2-\alpha)d_n}{T} (T - t), & t \in [T/2, T]. \end{cases}
\]
It is easy to check that $w_n(0) = w_n(T) = 0$ and $w_n \in L^2[0, T]$. The direct calculation shows that
\[
\delta D_{1}^\alpha w_n(t) = \begin{cases} \frac{2d_n}{T} t^{1-\alpha}, & t \in [0, T/2), \\ \frac{2d_n}{T} (t^{1-\alpha} - 2(t - \frac{T}{2})^{1-\alpha}), & t \in [T/2, T]. \end{cases}
\]
and
\[
\|w_n\|_\alpha^2 = \int_0^T \left( \delta D_{1}^\alpha w_n(t) \right)^2 \, dt = \int_0^{T/2} \left( \delta D_{1}^\alpha w_n(t) \right)^2 \, dt + \int_{T/2}^T \left( \delta D_{1}^\alpha w_n(t) \right)^2 \, dt
\]
\[
= 4d_n^2 \left[ \int_0^{T/2} t^{2(1-\alpha)} \, dt - \frac{T}{4} \int_{T/2}^T t^{1-\alpha} (t - \frac{T}{2})^{1-\alpha} \, dt + 4 \int_{T/2}^T (t - \frac{T}{2})^{2(1-\alpha)} \, dt \right]
\]
\[
= 4(1 + 2^{2\alpha-1})d_n^2 T^{1-2\alpha} - \frac{16d_n^2}{T^2} \int_{T/2}^T t^{1-\alpha} (t - \frac{T}{2})^{1-\alpha} \, dt < \infty.
\]
That is, $\delta D_{1}^\alpha w_n \in L^2[0, T]$. Thus, $w_n \in X$. Moreover, the direct calculation shows
\[
\delta D_{1}^\alpha w_n(t) = \begin{cases} \frac{2d_n}{T} ((T - t)^{1-\alpha} - 2(T - t)^{1-\alpha}), & t \in [0, T/2), \\ \frac{2d_n}{T} (T - t)^{1-\alpha}, & t \in [T/2, T]. \end{cases}
\]
Let \( \lambda T \) under the conditions \( \lambda \tau \in (2.6, \infty) \) we deduce that there is a sequence \( \{ \lambda_n \} \) such that it follows that

A direct calculation shows

Example 3.1. Theorem 3.1 ensures that for every \( \lambda > 0 \) and \( \alpha \in (0, 1] \), we achieve

\[ 0 \leq x \leq 96 \]

So, according to (15), (17) and (18) we achieve

\[ \Phi(w_n) = -\int_0^T \left[ \frac{\partial}{\partial t} I(t) \right] w_n(t) \, dt \]

\[ \Psi(w_n) = \int_0^T (F(w_n(t))) + \frac{T}{X} G(w_n(t)) \, dt = \frac{T}{\Gamma(2 - \alpha)} \int_0^\frac{T}{\Gamma(2 - \alpha)} (F(x) + \frac{T}{X} G(x)) \, dx. \]

\[ \lim_{n \to \infty} \| u_n \| = +\infty, \]

Remark 3.1. Under the conditions

\[ \lim_{\xi \to +\infty} \max_{|\xi| \leq \xi} \frac{F(x)}{\xi^2} = 0 \]

\[ \lim_{\xi \to +\infty} \frac{1}{|\xi|} \int_0^{\Gamma(2 - \alpha)|\xi|} F(x) \, dx = +\infty. \]

Theorem 3.1 ensures that for every \( \lambda > 0 \) and for each \( \mu \in (0, 1] \), the problem

(1) admits infinitely many solutions in \( E^\mu_\infty \). Moreover, if \( G_\infty = 0 \), the result holds for every \( \lambda > 0 \) and \( \mu \geq 0 \).

We now exhibit an example in which the hypotheses of Theorem 3.1 are satisfied.

Example 3.1. Let \( \frac{1}{2} < \alpha \leq 1 \). Let \( f : \mathbb{R} \to \mathbb{R} \) be the function defined by

\[ f(x) = \begin{cases} x \left( 2 - \cos(\ln(|x|)) - 2\sin(\ln(|x|)) \right) & \text{if } x \in (\mathbb{R} \setminus \{0\}), \\ 0 & \text{if } x = 0. \end{cases} \]

A direct calculation shows

\[ F(x) = \begin{cases} x^2 \left( 1 - \sin(\ln(|x|)) \right) & \text{if } x \in (\mathbb{R} \setminus \{0\}), \\ 0 & \text{if } x = 0. \end{cases} \]
Let \(C_1\) be the result of (A2), we obtain (A1), by choosing the same arguing as inside in Theorem 3.1, we obtain

\[
\lim_{\xi \to -\infty} \max_{|x| \leq \xi} \frac{F(x)dx}{\xi^2} = 0
\]

and

\[
\lim_{\xi \to -\infty} \sup_{\xi} \frac{1}{\xi^2} \int_{0}^{\infty} G(2-\alpha) F(x)dx = +\infty.
\]

Hence, using Theorem 3.1, the problem

\[
\frac{d}{dt} \left( a D_T^{\alpha-1} (0 D_T^a u(t)) - e D_T^{\alpha-1} (e D_T^a u(t)) \right) + \lambda f(u(t)) + \mu u(t) e^{-u(t)}(2 - u(t)) = 0,
\]

\(a.e. \ t \in [0, 1],
\)

\(u(0) = u(1) = 0,
\]

since \(G_{\infty} = 0\), for every \((\lambda, \mu) \in [0, +\infty] \times [0, +\infty]\) has an unbounded sequence of solutions in \(E_0^a\).

Now we want to present the following existence result which instead of Assumption (A1) in Theorem 3.1 a more general condition is assumed.

**Theorem 3.2.** Let \(\frac{1}{2} < \alpha \leq 1\). Assume that

(A2) there exist two sequence \(\{a_n\}\) and \(\{b_n\}\) with \(\omega_\alpha a_n^2 < \frac{|\cos (\pi \alpha)| b_n^2}{\Gamma(2-\alpha)}\) for every \(n \in \mathbb{N}\) and \(\lim_{n \to +\infty} b_n = +\infty\) such that

\[
\lim_{n \to +\infty} \max_{|x| \leq b_n} \frac{F(x) - \frac{1}{\omega_\alpha} \int_{0}^{\Gamma(2-\alpha)}[a_n] F(x)dx}{\frac{|\cos (\pi \alpha)| b_n^2}{\Gamma(2-\alpha)} - \omega_\alpha a_n^2} < \frac{1}{\omega_\alpha} \lim_{\xi \to -\infty} \sup_{\xi} \frac{1}{\xi^2} \int_{0}^{\infty} G(2-\alpha) F(x)dx.
\]

Then, for each

\[
\lambda' := \frac{\omega_\alpha}{T \lim_{\xi \to -\infty} \sup_{\xi} \frac{1}{\xi^2} \int_{0}^{\infty} G(2-\alpha) F(x)dx}
\]

\[
T \lim_{n \to +\infty} \max_{|x| \leq b_n} \frac{F(x) - \frac{1}{\omega_\alpha} \int_{0}^{\Gamma(2-\alpha)}[a_n] F(x)dx}{\frac{|\cos (\pi \alpha)| b_n^2}{\Gamma(2-\alpha)} - \omega_\alpha a_n^2}
\]

for every arbitrary continuous function \(g : \mathbb{R} \to \mathbb{R}\) whose potential \(G(x) = \int_{0}^{x} g(s)ds\) for every \(x \in \mathbb{R}\), is a non-negative function satisfying the condition (11) and for every \(\mu \in [0, \mu_{G, \lambda}]\) where

\[
\mu_{G, \lambda} := \frac{|\cos (\pi \alpha)|}{\Omega^2 G_{\infty}} \left( 1 - \lambda T \lim_{n \to +\infty} \max_{|x| \leq b_n} \frac{F(x) - \frac{1}{\omega_\alpha} \int_{0}^{\Gamma(2-\alpha)}[a_n] F(x)dx}{\frac{|\cos (\pi \alpha)| b_n^2}{\Gamma(2-\alpha)} - \omega_\alpha a_n^2} \right),
\]

the problem (1) has an unbounded sequence of solutions in \(E_0^a\).

**Proof.** Clearly, from (A2) we obtain (A1), by choosing \(a_n = 0\) for all \(n \in \mathbb{N}\). Moreover, if we assume (A2) instead of (A1) and set \(r_n = \frac{|\cos (\pi \alpha)| b_n^2}{\Gamma(2-\alpha)}\) for all \(n \in \mathbb{N}\), by the same arguing as in Theorem 3.1, we obtain
Assume that the assumptions in Theorem 3.1 hold. Then, for each $\lambda \in \Lambda := \left\{ \frac{\Gamma(2-\alpha)\omega_\alpha}{T\limsup_{t \to +\infty} \frac{1}{T} \int_0^T (F(x) + \frac{\alpha}{2} G(x)) \, dx} , \frac{\cos(\pi\alpha)}{\Omega^2 T \liminf_{t \to +\infty} \max_{|x| \leq b_n} F(x)} \right\}$, the problem (1), for $r = 0$, has an unbounded sequence of solutions in $E_0^n$.

Here we point out the following consequence of Theorem 3.3.

**Corollary 3.4.** Let $\frac{1}{2} < \alpha \leq 1$. Assume that

(B1) $\liminf_{t \to +\infty} \frac{\max_{|x| \leq b_n} F(x)}{\xi^2} < \frac{\cos(\pi\alpha)}{\Omega^2}$;

(B2) $\limsup_{t \to +\infty} \frac{\frac{1}{2}\int_0^T (F(x) + \frac{\alpha}{2} G(x)) \, dx}{\xi^2} > \Gamma(2-\alpha)\omega_\alpha$.

Then, for each $\lambda \in [\lambda_1, \lambda_2]$ where

$\lambda_1 := \frac{\Gamma(2-\alpha)\omega_\alpha}{T\limsup_{t \to +\infty} \frac{1}{T} \int_0^T (F(x) + \frac{\alpha}{2} G(x)) \, dx}$

and

$\lambda_2 := \frac{\cos(\pi\alpha)}{\Omega^2 T \liminf_{t \to +\infty} \max_{|x| \leq b_n} F(x)}$, we have the desired conclusion.
Theorem 1.1 in Introduction is an immediate consequence of Corollary 3.4.

Proof. has an unbounded sequence of solutions in \( E_0^\alpha \).

Remark 3.2. Theorem 1.1 in Introduction is an immediately consequence of Corollary 3.4.

We here give the following consequence of the main result:

**Corollary 3.5.** Let \( \frac{1}{2} < \alpha \leq 1 \). Let \( h_1 : \mathbb{R} \to \mathbb{R} \) be a non-negative continuous function, and denote that \( H_1(x) = \int_0^x h_1(s) \, ds \) for all \( x \in \mathbb{R} \). Assume that

1. \( \liminf_{\xi \to +\infty} \frac{H_1(\xi)}{\xi^2} < +\infty \);
2. \( \limsup_{\xi \to +\infty} \frac{1}{\xi^2} \int_0^\xi f^{(2-\alpha)(\xi)} H_1(x) \, dx = +\infty \).

Then, for every non-negative continuous \( h_i : \mathbb{R} \to \mathbb{R} \) for \( 2 \leq i \leq n \) satisfying

\[
\max \left\{ \sup_{\xi \in \mathbb{R}} H_i(\xi) : 2 \leq i \leq n \right\} \leq 0
\]

and

\[
\min \left\{ \liminf_{\xi \to +\infty} \frac{H_i(\xi)}{\xi^2} : 2 \leq i \leq n \right\} > -\infty
\]

where \( H_i(x) = \int_0^x h_i(s) \, ds \) for all \( x \in \mathbb{R} \) for \( 2 \leq i \leq n \), for each

\[
\lambda \in \left[ 0, \frac{\cos(\pi \alpha)}{\Omega^2 \liminf_{\xi \to +\infty} \frac{H_1(\xi)}{\xi^2}} \right],
\]

and for every arbitrary continuous function \( g : \mathbb{R} \to \mathbb{R} \) whose potential \( G(x) = \int_0^x g(s) \, ds \) for every \( x \in \mathbb{R} \), is a non-negative function satisfying the condition (11) and for every \( \mu \in [0, \mu G, \lambda] \) where

\[
\mu G, \lambda := \frac{\cos(\pi \alpha)}{\Omega^2 T G_\infty} \left( 1 - \lambda \frac{\Omega^2}{\cos(\pi \alpha)} \liminf_{\xi \to +\infty} \frac{H_1(\xi)}{\xi^2} \right),
\]

the problem

\[
\frac{d}{dt} \left( \alpha D_t^{\alpha-1} \left( \frac{\partial}{\partial t} D_t^\alpha u(t) \right) - \beta D_T^{\alpha-1} \left( \frac{\partial}{\partial t} D_T^\alpha u(t) \right) \right) + \lambda \sum_{i=1}^n h_i(u(t)) + \mu g(u(t)) = 0,
\]

a.e. \( t \in [0, T] \),

\[
u(0) = u(T) = 0,
\]

has an unbounded sequence of solutions in \( E_0^\alpha \).

**Proof.** Set \( f(x) = \sum_{i=1}^n h_i(x) \) for all \( x \in \mathbb{R} \). Assumption (C2) together with the condition

\[
\min \left\{ \liminf_{\xi \to +\infty} \frac{H_i(\xi)}{\xi^2} : 2 \leq i \leq n \right\} > -\infty
\]

ensures

\[
\limsup_{\xi \to +\infty} \frac{1}{\xi^2} \int_0^\xi F(x) \, dx = \limsup_{\xi \to +\infty} \frac{1}{\xi^2} \int_0^\xi \frac{f^{(2-\alpha)(\xi)}}{\xi} \sum_{i=1}^n H_i(x) \, dx = +\infty.
\]
Moreover, Assumption (C1) along with condition
\[
\max \left\{ \sup_{\xi \in \mathbb{R}} H_i(\xi) ; \ 2 \leq i \leq n \right\} \leq 0
\]
follows
\[
\liminf_{\xi \to +\infty} \max_{|x| \leq \xi} \frac{F(x)}{\xi^2} \leq \liminf_{\xi \to +\infty} \frac{H_1(\xi)}{\xi^2} < +\infty.
\]
Hence, from Corollary 3.1 the conclusion follows. \[\square\]

Finally, we observe that by similar reasonings as in the proof of Theorem 3.1 but using conclusion (c) of Theorem 2.6 instead of (b), the following result holds.

**Theorem 3.6.** Let \( \frac{1}{2} < \alpha \leq 1 \). Assume that

(A3) \( \liminf_{\xi \to 0^+} \frac{\max_{|x| \leq \xi} F(x)}{\xi^2} < \frac{\cos(\pi \alpha)}{\Gamma(2-\alpha) \omega} \limsup_{\xi \to 0^+} \frac{\int_0^{\Gamma(2-\alpha)(\xi)} F(x) dx}{\xi^2} \).

Then, for each \( \lambda \in ]\lambda_1, \lambda_2[ \) where
\[
\lambda_3 := \frac{\Gamma(2 - \alpha) \omega}{T \limsup_{\xi \to 0^+} \frac{\int_0^{\Gamma(2-\alpha)(\xi)} F(x) dx}{\xi^2}}
\]
and
\[
\lambda_4 := \frac{\cos(\pi \alpha)}{\Omega^2 T \liminf_{\xi \to 0^+} \frac{\max_{|x| \leq \xi} F(x)}{\xi^2}}
\]
for every arbitrary continuous function \( g : \mathbb{R} \to \mathbb{R} \) whose potential \( G(x) = \int_0^x g(s) ds \) for every \( x \in \mathbb{R} \), is a non-negative function satisfying the condition
\[
G_0 := \lim_{\xi \to 0^+} \frac{\max_{|x| \leq \xi} G(x)}{\xi^2} < +\infty \quad (19)
\]
and for every \( \mu \in [0, \mu_{G, x}] \) where
\[
\mu_{G, x} := \frac{\cos(\pi \alpha)}{\Omega^2 T G_0} \left( 1 - \frac{\Omega^2}{\cos(\pi \alpha)} \liminf_{\xi \to 0^+} \frac{\max_{|x| \leq \xi} F(x)}{\xi^2} \right)
\]
the problem (1) has a sequence of solutions, which strongly converges to 0 in \( E_0^\alpha \).

**Proof.** Fix \( \lambda \in ]\lambda_1, \lambda_2[ \) and let \( G \) be a function satisfies the condition (19). Since, \( \lambda < \lambda_2 \), one has
\[
\mu_{G, \lambda} := \frac{\cos(\pi \alpha)}{\Omega^2 T G_0} \left( 1 - \frac{\Omega^2}{\cos(\pi \alpha)} \liminf_{\xi \to 0^+} \frac{\max_{|x| \leq \xi} F(x)}{\xi^2} \right) > 0.
\]
Fix \( \mu \in [0, \mu_{G, \lambda}] \) and put \( \nu_1 := \lambda_3 \) and \( \nu_2 := \frac{\lambda_4}{1 + \frac{\Omega^2}{\cos(\pi \alpha)} \lambda_2 G_0} \). If \( G_0 = 0 \), clearly, \( \nu_1 = \lambda_3, \nu_2 = \lambda_4 \) and \( \lambda \in ]\nu_1, \nu_2[ \). If \( G_0 \neq 0 \), since \( \mu < \mu_{G, \lambda} \), we obtain
\[
\frac{\lambda_3}{\lambda_2} + \frac{\Omega^2}{\cos(\pi \alpha)} \mu T G_0 < 1,
\]
and so
\[
\frac{\lambda_3}{1 + \frac{\Omega^2}{\cos(\pi \alpha)} \lambda_2 T G_0} > \lambda,
\]
namely, $\lambda < \nu$. Hence, being in mind that $\lambda > \lambda_1 = \nu_1$, one has $\lambda \in [0, \nu_2)$. Taking (19) into account, one has

$$\liminf_{\xi \to 0^+} \max_{|x| \leq \xi} F(x) + \frac{\pi}{\xi^2} \max_{|x| \leq \xi} G(x) \leq \liminf_{\xi \to 0^+} \frac{\max_{|x| \leq \xi} F(x)}{\xi^2} + \frac{\pi}{\lambda} G_0.$$  \hfill (20)

Therefore, from (20), we observe

$$\lambda \in [\nu_1, \nu_2].$$

We take $X$, $\Phi$, $\Psi$ and $I_\lambda$ as in the proof of Theorem 3.1. Let $\{\xi_n\}$ be a sequence of positive numbers such that $\xi_n \to 0^+$ as $n \to +\infty$ and

$$\lim_{n \to \infty} \frac{\max_{|x| \leq \xi_n} F(x) + \frac{\pi}{\lambda} \max_{|x| \leq \xi_n} G(x)}{\xi_n^2} < +\infty. $$

Putting $r_n = \frac{\max_{|x| \leq \xi_n} F(x) + \frac{\pi}{\lambda} \max_{|x| \leq \xi_n} G(x)}{\xi_n^2}$ for every $n \in \mathbb{N}$ and working as in the proof of Theorem 3.1 it follows that $\delta < +\infty$. Let us show that the functional $I_\lambda$ has not a local minimum at zero. For this, let $\{d_n\}$ be a sequence of positive numbers and $\tau > 0$ such that $d_n \to 0^+$ as $n \to \infty$ and

$$\frac{1}{\lambda} < \tau < \frac{T}{\Gamma(2 - \alpha) \omega_\alpha} \frac{1}{d_n^2} \int_0^{\Gamma(2 - \alpha) |d_n|} (F(x) + \frac{\pi}{\lambda} G(x)) dx \quad \text{for each } n \in \mathbb{N} \text{ large enough.} \quad \text{(21)}$$

Putting together (17), (18) and (21) we achieve

$$I_\lambda(w_n) = \Phi(w_n) - \lambda \Psi(w_n) = \omega_\alpha d_n^2 - \lambda \int_0^{\Gamma(2 - \alpha) |d_n|} (F(x) + \frac{\pi}{\lambda} G(x)) dx < (1 - \lambda \tau) \omega_\alpha d_n^2 < 0$$

for every $n \in \mathbb{N}$ large enough. Since $I_\lambda(0) = 0$, this ensures that the functional $I_\lambda$ has not a local minimum at zero. Hence, recalling (8), the part (c) of Theorem 2.6 ensures that there exists a sequence $\{u_n\}$ in $X$ of critical points of $I_\lambda$ such that $||u_n|| \to 0$ as $n \to \infty$, and the proof is complete.

\[ \square \]

**Remark 3.3.** Applying Theorem 3.6, results similar to Theorems 1.1, 3.2 and 3.3 and Corollaries 3.4 and 3.5 can be obtained by replacing $\xi \to +\infty$ with $\xi \to 0^+$. 

**References**


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