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Towards a local definition of *body* in Continuum Mechanics

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Abstract. The purpose of this paper is to introduce the concept of *pyramidal manifold*, and to demonstrate that it is useful as a model definition of three-dimensional body. Pyramidal manifolds generalize three-dimensional manifolds with corners and represent an approach to the definition of body from the point of view of Differential Geometry, which facilitates the development of the mathematical theory of Continuum Mechanics. Two maps defined on a pyramidal manifold, the *degree* and the *index*, are introduced. Both of them are invariant under deformations and allow taking a first step towards a classification of bodies. The Stokes theorem for bodies is also discussed, and a proof thereof is provided by using differential forms on pyramidal manifolds.

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1. Introduction

The notion of (deformable) body is fundamental in Continuum Mechanics (hereinafter, CM), since bodies are the material objects that are affected by forces and experience deformations and motions. Nevertheless, an ultimate definition of body has not yet been established. To reinforce this assertion, we reproduce the following words of Sheldon R. Smith in [17, footnote 44]:

> "I should note here that the question of the structure of bodies is not a finished business decided once and for all but is, rather, an on-going area of research. [...]"

In fact, in recent years other authors have made research on this topic. As far as we know, the last scholarly article entirely devoted to this subject is the paper of Del Piero [3], where the author sharpens the class of fit regions defined by himself in [2]. In both papers, Del Piero follows the original lines established by Noll and Virga in [16] and, in particular, he takes from them the term *fit region* with the meaning of "sets fit to be occupied by continuous bodies and their subbodies".

We follow another approach, which combines the spirit of Gurtin's book [6] and the language of Differential Geometry; specifically, we employ the language of manifolds with corners to define a larger class that we call *pyramidal manifolds*.

There exist some references intersecting this one in the sense that they understand bodies as manifolds; perhaps the most remarkable one is the book by Marsden and Hughes [15]. In the same line, but much more recent, are [1] and [13]. Among all those three references, it is in the paper [13] where one finds the highest degree of proximity with our standpoint; there, the authors define a body as a manifold with corners, which shows their intention of understanding the concept of body in the way we do.

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The purpose of this paper is to introduce the concept of *pyramidal manifold*, and to demonstrate that it is useful as a model definition of three-dimensional (from now on, 3D) body. A definition of body expressed in the language of Differential Geometry has the nice property of being framed within a strong branch of modern Mathematics, the theorems of which can consequently be employed to CM concepts involving bodies, like for instance deformations, motions, and theorems of integration.[1]

To give an example, when speaking of "deformed bodies" it is understood that the deformation of a body (the image of a body under a deformation) is, in turn, a body. From a dynamical point of view, the same situation occurs when talking about motions, as a motion is a one-parameter family of deformations. *Inter alia*, the methodology employed in the present paper allows performing an easy proof of the statement that "the deformation of a body is again a body". Even when this statement might seem to be trivial, its proof is obviously related to the way in which the concept of body is defined, and typically a complex proof is required when a complicated definition is given.

This work assumes that, given a definition of 3D body $\mathscr{B}$, the following assertions should be true:

(1) It is easy to discern whether a given subset of $\mathbb{R}^3$ is a 3D body or not.
(2) $\mathscr{B}^\circ$ is connected.
(3) $\mathscr{B} = \overline{\mathscr{B}^\circ}$.
(4) $\mathscr{B}$ possesses some local structure of 3D manifold, in such a way that:
   (a) The set of 3D bodies includes as particular cases the 3D manifolds with corners, but it is strictly larger. (A cube is a manifold with corners, but a pyramid with four or more faces converging in its vertex is not, whereas it should be a body.)
   (b) The set of 3D bodies is invariant under deformations. In other words, the deformation of a 3D body is again a 3D body.
   (c) The Stokes theorem holds for bounded 3D bodies.

The contents of this paper are based upon the unpublished work [12], carried out by the first author, when he was an undergraduate senior student, under the supervision of the second and third ones.

After 3D, the symbols 0D, 1D and 2D will be employed with their obvious meaning. Throughout all of this paper, except for the last section, the word "body" shall mean "3D body".

2. Origins

The scientific origins of this work can be found in Gurtin's book [6], where it is said that "a body $\mathscr{B}$ is a (possibly unbounded) regular region in $\mathscr{E}$". The symbol $\mathscr{E}$ stands for $\mathbb{R}^3$, and the expression *regular region* is used "in the sense of Kellogg", which in turn means *in the sense of Kellogg's book* [8]. But if one wants to deepen into details and goes to [8], it soon becomes apparent that that definition is not entirely satisfactory, due to the following reasons:

A. Firstly, one notice that Kellogg's regular regions are necessarily bounded, while Gurtin wants to consider unbounded bodies in CM (an unbounded body may be a good approximation for the problem under consideration when one direction is much larger than the others).

[1]One cannot forget that the Stokes theorem is a matter of utmost concern in CM.

B. As a second point of disagreement, the example of two cones sharing the same vertex (and only that point) shows that a regular region in Kellogg's sense may fail to be the closure of a domain,[2] a property which is mandatory for Gurtin's bodies (as it is for ours; see Conditions 2 and 3 above). Other examples are furnished by thinking of two balls with one point of tangency or of two cubes with only one edge in common.
C. Lastly, Kellogg's definition is given in such a way that makes it difficult to check whether the deformation of a body is a body, as it should be, or not.

This paper is focused on giving a definition of body according to Gurtin's spirit from the viewpoint of the Differential Geometry.

It is by no means our intention to censure the definition of Kellogg, which was published in 1929, when the concept of manifold with corners did not even exist, and which possesses the good property of agreeing very well with what a body should be. It must be pointed out, moreover, that the class of Kellogg's bodies which satisfy Conditions 2 and 3 above is larger than the class of bounded bodies as defined in here, where we shall limit ourselves to study bodies which are locally deformable into a convex set which has Lipschitz boundary and is equal to the closure of its interior; precisely speaking, for every $\boldsymbol{x}$ of the body $\mathscr{B}$ there exists a neighborhood of $\boldsymbol{x}$, let us say U, in $\mathbb{R}^3$ such that $U \cap \mathscr{B}$ can be deformed, via an orientation-preserving $\mathscr{C}^1$-diffeomorphism, into a convex subset of $\mathbb{R}^3$ which has Lipschitz boundary and is equal to the closure of its interior. These statements set out the scope of the present study, which we consider not definitive as the word "towards" in the title tries to reflect.

3. Pyramidal manifolds

3.1. Pyramids. We will start by fixing what we mean by a differentiable map defined on an arbitrary subset of $\mathbb{R}^p$. We follow the definition given by Kobayashi and Nomizu [9, Appendix 3]; that is, we say that the map $\varphi\colon A \subset \mathbb{R}^p \longrightarrow \mathbb{R}^q$ is $\mathscr{C}^r$-differentiable (where p and q are natural numbers and $r \in \mathbb{N} \cup \{\infty\}$) at $\boldsymbol{x} \in A$ whenever there exists a $\mathscr{C}^r$-map $\varphi_{\boldsymbol{x}}$, defined on a open neighborhood of $\boldsymbol{x}$ in $\mathbb{R}^p$, which agrees with φ in the intersection of their domains. Every time that we use the term $\mathscr{C}^r$-differentiable we mean $r \geqslant 1$ and never $r = 0$ which will be mentioned as *continuous*.

We recall here that Gurtin uses in [6] the term smooth for $\mathscr{C}^1$-maps. We do not follow this convention. We say that a map is a $\mathscr{C}^r$-diffeomorphism when it is a $\mathscr{C}^r$-differentiable bijection with $\mathscr{C}^r$-differentiable inverse.

Following [5], we are going to work with convex polytopes in an Euclidean space: the intersection of a finite family of closed half-spaces. (We understand by closed half-space a subspace of the form

$$\{(x_1, \dots, x_p) \in \mathbb{R}^p\colon\ x_1 \geqslant c\}$$

for a given basis of $\mathbb{R}^p$ and a given real constant c.) Convex polytopes in $\mathbb{R}^3$ present an interesting property: they are locally generated by pyramids. Our purpose is to define the concept of body by means of local diffeomorphisms to convex polytopes. It is thus enough to take charts only to pyramids.

Definition 3.1 (Pyramid of n edges in $\mathbb{R}^3$, where $n \in \mathbb{N} \smallsetminus \{1, 2\}$)**.** Let us consider the family $\{\pi_k^n\}_{k=0}^{n-1}$ of planes in $\mathbb{R}^3$ containing the origin and two consecutive nth

[2]Here, a *domain* is an open and connected subset of $\mathbb{R}^3$.

roots of unity on the plane $x_3 = 1$. We define $\mathbf{P}_n$ to be the closure of the connected component of $\mathbb{R}^3 \smallsetminus \bigcup_{k=0}^{n-1} \pi_k^n$ containing $(0, 0, 1)$. We call this subspace the *pyramid of n edges in $\mathbb{R}^3$*.

This definition would fail for smaller n. But we would like to have an equivalent idea of convex topological cone with 2, 1 and 0 edges.

Definition 3.2 (Pyramid of n edges in $\mathbb{R}^3$, where $n \in \{0, 1, 2\}$)**.** We define for $n \in \{0, 1, 2\}$ the *pyramid of n edges in* $\mathbb{R}^3$ as the subspace of $\mathbb{R}^3$ given by

$$\begin{aligned}
\mathbf{P}_0 &:= \{(x_1, x_2, x_3) \in \mathbb{R}^3 \colon \quad x_2^2 \leqslant (x_3 + x_1)\ (x_3 - x_1)\ ,\ x_3 \geqslant 0\},\\
\mathbf{P}_1 &:= \{(x_1, x_2, x_3) \in \mathbb{R}^3 \colon \quad x_2^2 \leqslant (x_3 + x_1)\ (x_3 - x_1)^2,\ x_3 \geqslant |x_1|\},\\
\mathbf{P}_2 &:= \{(x_1, x_2, x_3) \in \mathbb{R}^3 \colon \quad x_2^2 \leqslant (x_3 + x_1)^2 (x_3 - x_1)^2,\ x_3 \geqslant |x_1|\}.
\end{aligned}$$

These pyramids are cones over some figures of the plane $x_3 = 1$ as in the previous cases. Evaluating $x_3 = 1$ in the defining equations of the pyramids we find that the generating figures are homeomorphic to closed discs of the plane with n corners, located in the nth roots of unity (we understand here that there are no 0th roots of unity). In fact, this might be done for any $n \in \mathbb{N}$.

Remark 3.1. It is possible to give a name to the regular surfaces bounding $\mathbf{P}_n$ for $n \in \{0, 1, 2\}$ as we have done in Definition 3.1. Those are given by taking the cones over the closed curves bounding the disc of the plane $x_3 = 1$ without the corners and removing the vertex $(0, 0, 0)$. The number of those surfaces is equal to n and will be denoted by π_k^n as before (except for $n = 0$, in which case there exists one surface, π_0^0).

3.2. Pyramidal manifolds. We might remark at this point that $\mathbf{P}_n = \overline{(\mathbf{P}_n)^\circ}$, a property which is desired for bodies. This appreciation leads us to the definition of *pyramidal manifold*. Avoiding formalisms, a pyramidal manifold is nothing but a subspace of $\mathbb{R}^3$ which looks locally like a pyramid.

Definition 3.3 (Pyramidal manifold in $\mathbb{R}^3$)**.** We say that a subset M of $\mathbb{R}^3$ is a *$\mathscr{C}^r$-differentiable pyramidal manifold*, where $r \in \mathbb{N} \cup \{\infty\}$, when for each $\boldsymbol{x}$ in M there exist

- a $\mathscr{C}^r$-diffeomorphism $\varphi_{\boldsymbol{x}} \colon U_{\boldsymbol{x}} \longrightarrow V_{\boldsymbol{x}}$ of an open neighborhood of $\boldsymbol{x}$ in $\mathbb{R}^3$ onto an open subset of $\mathbb{R}^3$, and
- a non-negative integer number $n_{\boldsymbol{x}}$

satisfying that $\varphi_{\boldsymbol{x}}(U_{\boldsymbol{x}} \cap M) = V_{\boldsymbol{x}} \cap \mathbf{P}_{n_{\boldsymbol{x}}}$.

In Figure 1 two examples of pyramidal manifolds are shown. Note that both sets are conceivable bodies, but they are not manifolds with corners.

Every pyramidal manifold of class $\mathscr{C}^r$ ($r \in \mathbb{N} \cup \{\infty\}$) is in particular of class $\mathscr{C}^1$. We will often use *pyramidal manifold* as shorthand for a $\mathscr{C}^r$-differentiable pyramidal manifold in $\mathbb{R}^3$ for some $r \in \mathbb{N} \cup \{\infty\}$.

We will call the triplet $(U_{\boldsymbol{x}}, \varphi_{\boldsymbol{x}}, n_{\boldsymbol{x}})$ a *pyramidal local chart* omitting the subindex $\boldsymbol{x}$ whenever possible.

Definition 3.3 might be confusing because $n_{\boldsymbol{x}}$ might change for the same points and might be fixed for some others. Let us imagine that we want to show that $\mathbf{P}_0$ is a pyramidal manifold. We might think of choosing for each point the pyramidal local chart $(\mathbb{R}^3, \mathrm{id}, 0)$ which gives indeed the structure of a pyramidal manifold to $\mathbf{P}_0$. But we could have chosen for any interior point an open ball U fully contained in the interior of $\mathbf{P}_0$ centered at that point and V an open ball in the interior of any pyramid $\mathbf{P}_n$ centered at $(0, 0, 1)$. It is possible to find an affine φ mapping U onto V. This time (U, φ, n) is again a pyramidal local chart for a point in the interior of

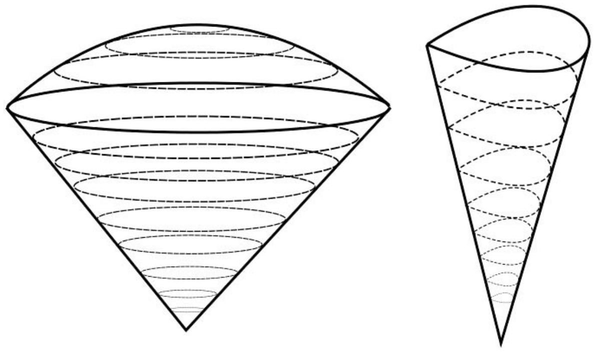

FIGURE 1. A spinning top (left) is modeled in $\mathbf{P}_0$. Many blades (clip point blades) and wedges (right) are modeled in $\mathbf{P}_1$. None of these sets is a manifold with corners (see Section 5).

$\mathbf{P}_0$. Nevertheless, the only choice for $n_{(0,0,0)}$ at each pyramid $\mathbf{P}_n$ is $n_{(0,0,0)} = n$ as we study in the following section.

In fact, the points away from the origin are locally like the intersection of at most two half-spaces. We use the notion of Λ-*quadrant of order* n *in* $\mathbb{R}^p$ [14, Definition 1.1.1]:

$$\Lambda_p^n = \{(x_1, \dots, x_p) \in \mathbb{R}^p : \quad x_i \geqslant 0;\ p-n < i \leqslant p\} = \mathbb{R}^{p-n} \times [0,\infty)^n.$$

Remark 3.2. The linear map $\Phi\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ sending in order the 3rd roots of unity of the plane $x_3 = 1$ to the usual basis of $\mathbb{R}^3$ shows that $\mathbf{P}_3$ is $\mathscr{C}^\infty$-diffeomorphic to its image Λ_3^3 under Φ.

Our aim is to take local charts around $\boldsymbol{x}$ to different objects such that $(0,0,0)$ is the image of $\boldsymbol{x}$. This is achieved by taking local charts to certain Λ-quadrants. We rename some of those spaces to better fit in our language:

$$\mathbf{P}_n := \Lambda_3^{n+3} \text{ for } n \in \{-3,-2,-1\}.$$

Remark 3.3. Although we denote those spaces by the same letter as that used for the pyramids, and in fact we will also call these Λ-quadrants pyramids, we want to highlight the complete different nature of those subspaces which are not cones over a topological disc with n corners.

We give the names π_0^{-1}, π_1^{-1} (respectively, π_0^{-2}) to the faces bounding $\mathbf{P}_{-1}$ (respectively, $\mathbf{P}_{-2}$) as we have done in Definition 3.1. In this case we need to use another point (for instance $(1,1,1)$) to identify the connected component whose closure is $\mathbf{P}_n$.

It is easy to check that the point $(0,0,0)$ is in each $\mathbf{P}_n$, $n \geqslant -3$. This fact will play a fundamental role in the theory.

Theorem 3.1. *A subset M of $\mathbb{R}^3$ is a $\mathscr{C}^r$-differentiable pyramidal manifold if, and only if, for each $\boldsymbol{x}$ in M there exist*

- *a $\mathscr{C}^r$-diffeomorphism $\varphi_{\boldsymbol{x}}\colon U_{\boldsymbol{x}} \longrightarrow V_{\boldsymbol{x}}$ of an open neighborhood of $\boldsymbol{x}$ in $\mathbb{R}^3$ onto an open neighborhood of $(0,0,0)$ in $\mathbb{R}^3$, and*
- *an integer number $n_{\boldsymbol{x}} \geqslant -3$*

satisfying that $\varphi_{\boldsymbol{x}}(\boldsymbol{x}) = (0,0,0)$ and $\varphi_{\boldsymbol{x}}(U_{\boldsymbol{x}} \cap M) = V_{\boldsymbol{x}} \cap \mathbf{P}_{n_{\boldsymbol{x}}}$.

We call the triplet $(U_{\boldsymbol{x}}, \varphi_{\boldsymbol{x}}, n_{\boldsymbol{x}})$ in this new sense an *adapted local chart* centered at $\boldsymbol{x}$, or even an adapted local chart.

Proof.

(1) Let M be a subset of $\mathbb{R}^3$ satisfying the properties required in the theorem. Given any point $\boldsymbol{x} \in M$ with an adapted local chart such that $n_{\boldsymbol{x}} \geqslant 0$ it is also a pyramidal local chart. We recall that $\mathbf{P}_3$ is diffeomorphic to Λ_3^3 through the diffeomorphism Φ defined in Remark 3.2. What is more, given any open neighborhood V of the origin in Λ_3^i ($i \in \{0, 1, 2\}$), by restricting to a small enough open ball W centered at the origin contained in V, there exists a translation ψ such that $\psi(W \cap \Lambda_3^i) = \psi(W) \cap \Lambda_3^3$, post-composing with Φ^{-1} we have built a pyramidal local chart centered at any point $\boldsymbol{x} \in M$.

(2) Let M be a pyramidal manifold. We fix a point $\boldsymbol{x} \in M$ and a pyramidal local chart $U_{\boldsymbol{x}}, \varphi_{\boldsymbol{x}}, n_{\boldsymbol{x}})$ around it. If $\varphi_{\boldsymbol{x}}(\boldsymbol{x}) = (0, 0, 0)$ the same chart is also an adapted local chart. We assume now that $\varphi_{\boldsymbol{x}}(\boldsymbol{x}) \neq (0, 0, 0)$. If $\varphi_{\boldsymbol{x}}(\boldsymbol{x})$ is in the interior of $\mathbf{P}_{n_{\boldsymbol{x}}}$, we take an open neighborhood fully contained in $V_{\boldsymbol{x}} \cap \mathbf{P}_{n_{\boldsymbol{x}}}$ and a translation by $-\varphi_{\boldsymbol{x}}(\boldsymbol{x})$ to get the new desired adapted chart. If $\varphi_{\boldsymbol{x}}(\boldsymbol{x})$ is on the boundary of $\mathbf{P}_{n_{\boldsymbol{x}}}$ and $n_{\boldsymbol{x}} \geqslant 3$, then $\varphi_{\boldsymbol{x}}(\boldsymbol{x})$ belongs to at most two planes $\pi_k^{n_{\boldsymbol{x}}}$. In this case we can take a small enough neighborhood of the point meeting the same amount of planes $\pi_k^{n_{\boldsymbol{x}}}$ and take an affine map sending $\varphi_{\boldsymbol{x}}(\boldsymbol{x})$ to $(0, 0, 0)$ and the plane or planes $\pi_k^{n_{\boldsymbol{x}}}$ to the plane or planes bounding Λ_3^1 or Λ_3^2 respectively.
For the remaining cases ($\varphi_{\boldsymbol{x}}(\boldsymbol{x}) \neq (0, 0, 0)$, $\varphi_{\boldsymbol{x}}(\boldsymbol{x})$ on the boundary of $\mathbf{P}_{n_{\boldsymbol{x}}}$, and $n_{\boldsymbol{x}} \in \{0, 1, 2\}$) we can approximate the intersection $\mathbf{P}_n \cap \{(y_1, y_2, y_3) \in \mathbb{R}^3 : \quad y_3 = x_3\}$ close to $\varphi_{\boldsymbol{x}}(\boldsymbol{x})$ by a line or by two intersecting half-lines (the generated by the lateral derivatives). In any case we can apply the previous procedure to get an adapted local chart. □

4. Index and degree

Theorem 3.1 shows in particular that every pyramid as in Definitions 3.1 and 3.2 is a $\mathscr{C}^\infty$-pyramidal manifold. It was already said that the use of the adapted charts stresses out the importance of the third component of a local chart.

Proposition 4.1 (Index and Degree)**.** *Let M be a $\mathscr{C}^r$-differentiable pyramidal manifold. Let $(U, \varphi, n_{\boldsymbol{x}})$ be an adapted local chart centered at a point $\boldsymbol{x}$ of M. Then $n_{\boldsymbol{x}}$ is independent of the adapted local chart chosen for $\boldsymbol{x}$ and thus the following maps, called index* (ind) *and degree* (deg)*, are well defined:*

$$\begin{aligned} &\mathrm{ind}\colon M \longrightarrow \{0, 1, 2, 3\}, \\ &\mathrm{deg}\colon \mathrm{ind}^{-1}(3) \longrightarrow \mathbb{N} \cup \{0\}, \end{aligned}$$

where the index assigns to $\boldsymbol{x} \in M$ the number $n_{\boldsymbol{x}} + 3$ if $n_{\boldsymbol{x}} < 0$ and 3 *otherwise, and the degree assigns to $\boldsymbol{x} \in \mathrm{ind}^{-1}(3)$ the number $n_{\boldsymbol{x}}$.*

Proof. Let $\boldsymbol{x}$ be an arbitrary point in M and (U, φ, m) and (V, ψ, n) two adapted local charts centered at $\boldsymbol{x}$. The map

$$h = \psi \circ \varphi^{-1} \colon \varphi(U \cap V) \longrightarrow \psi(U \cap V),$$

which is called *transition function*, is a $\mathscr{C}^r$-diffeomorphism, hence sending the boundary of $\varphi(U \cap V) \cap \mathbf{P}_m$ onto the boundary of $\psi(U \cap V) \cap \mathbf{P}_n$. What is more, since h is a diffeomorphism, it preserves regular submanifolds.

The proof of this proposition is done by studying the number of maximal dimensional $\mathscr{C}^1$-submanifolds of the boundary. We find that for $n \geqslant 1$ the number of $\mathscr{C}^1$-surfaces and curves is equal to n (the open sections of the planes π_k^n and the intersections between them), and the origin is not contained in any of them, so we have a

0D submanifold. If $n = 0$ there is no $\mathscr{C}^1$-curve but one surface and one remaining 0D submanifold. For $n < 0$ there is no 0D remaining submanifold, and there are exactly $n + 3$ regular $\mathscr{C}^1$-surfaces. The same consideration might be done for m.

Let n' and m' be, respectively, the number of maximal regular surfaces of the boundary of $\mathbf{P}_n$ and $\mathbf{P}_m$. Since h sends surfaces into surfaces, given $i \in \{0, \ldots, m'-1\}$ there exists a unique $j \in \{0, \ldots, n'-1\}$ such that

$$h(\varphi(U \cap V) \cap \pi_i^m) = \psi(U \cap V) \cap \pi_j^n.$$

It is now clear that $n' = m'$. The existence of a 0D remaining regular submanifold and of a $\mathscr{C}^1$-curve classifies the different cases that might appear for $n' = m' \leqslant 2$, which proves the proposition. □

We distinguish points of a pyramidal manifold in two ways (index and degree) because of several reasons. One of them is agreement with other authors (term Λ-index for [14]), another one is to differentiate between the points that look locally like the vertices of cones over topological discs (those where the degree is defined) and those which do not (index strictly smaller than 3). The differences between these two concepts will be developed in the following sections.

The same argument can be used to prove the following corollary. From now on we will just use (U, φ) for an adapted local chart centered at $\boldsymbol{x}$ (the third component will be referred to as the degree or the index of the point depending on the situation).

Corollary 4.2. *Let $f\colon M \longrightarrow N$ be a $\mathscr{C}^r$-diffeomorphism between two pyramidal manifolds M and N. Then*

$$\begin{aligned} &\operatorname{ind}(\boldsymbol{x}) = \operatorname{ind}(f(\boldsymbol{x}))\ \forall \boldsymbol{x} \in M \text{ and} \\ &\deg(\boldsymbol{x}) = \deg(f(\boldsymbol{x}))\ \forall \boldsymbol{x} \in \operatorname{ind}^{-1}(3). \end{aligned}$$

Proof. We fix a point $\boldsymbol{x} \in M$ and two adapted local charts (U, φ, n) centered at $\boldsymbol{x}$ and (V, ψ, m) centered at $f(\boldsymbol{x})$. The map

$$h = \psi \circ f \circ \varphi^{-1} \colon \varphi(U \cap f^{-1}(V)) \longrightarrow \psi(f(U) \cap V)$$

is a $\mathscr{C}^r$-diffeomorphism and we can apply the same procedure of the previous proof to get the result. (Note that a local extension of f on an open neighborhood of $\boldsymbol{x}$ and a local extension of f^{-1} on an open neighborhood of $f(\boldsymbol{x})$ need not be inverse of each other, but we can get smaller open neighborhoods so that this property holds.) □

Remark 4.1. Given a pyramid $\mathbf{P}_n$ we observe that there are points of, at most, four different indices: the interior points (index 0), those belonging to a single face (index 1), those belonging to a single edge (index 2) and, in case $(0, 0, 0)$ is not of one of the previous kinds, it is of index 3 and degree n. This assertion follows from Theorem 3.1.

This gives us more information about the index and degree on a pyramid. First of all, there are no points of index or degree greater than the index or degree of $(0, 0, 0)$ in any pyramid. Besides, in every open neighborhood of $(0, 0, 0)$ on $\mathbf{P}_n$ there are points of index $i \in \{0, \ldots, n + 3\}$ if $n < 0$, of all indices (0, 1, 2 and 3) for $n > 0$ and of index $i \in \{0, 1, 3\}$ for $n = 0$. What is more, there is only one point where the index is defined in any $\mathbf{P}_n$ with $n \geqslant 0$: the origin, whose degree is equal to n.

Definition 4.1 (ith stratum and ith skeleton)**.** Let M be a pyramidal manifold. Then, for each $i \in \{0, 1, 2, 3\}$, the sets

$$\begin{aligned} \partial_i M &= \{\boldsymbol{x} \in M\colon\ \operatorname{ind}(\boldsymbol{x}) = i\} \text{ and} \\ \partial^i M &= \{\boldsymbol{x} \in M\colon\ \operatorname{ind}(\boldsymbol{x}) \geqslant i\} \end{aligned}$$

are called, respectively, the *ith stratum* and the *ith skeleton* of M.

The vocabulary of strata and skeletons is taken from simplicial homotopy theory. Extensions of the meaning of these terms are common in algebraic topology (see [10], for example). The concept is also a generalization of Λ-index [14, Definition (1.2.6)] for small dimensions.

Both strata and skeletons have important properties, as will be shown in the following sections. As a first observation $\partial_3 M$ is the subset of M where the degree is defined. We begin with a result which also generalizes [7, Proposition 2.4] for dimension 3. The symbol $\sqcup$ is used to denote disjoint union.

Proposition 4.3. *Let M be a pyramidal manifold. Then:*
(1) $M = \bigsqcup_{i=0}^{3} \partial_i M$.
(2) $\partial^i M = \bigsqcup_{j=i}^{3} \partial_j M$.
(3) $M = \partial^0 M$.
(4) $\partial^3 M = \partial_3 M$ *is a set of isolated points, and so are* $\deg^{-1}(n) \subset \partial_3 M$ *for all* $n \in \mathbb{N} \cup \{0\}$.
(5) $\overline{\partial_i M} \cap M = \partial^i M$ *for all* $i \in \{0,1,3\}$. *If M is closed in* $\mathbb{R}^3$, $\overline{\partial_i M} = \partial^i M$.
(6) $\overline{\partial_2 M} \cap M \subset \partial^2 M$, *and when M is closed in* $\mathbb{R}^3$, $\overline{\partial_2 M} \subset \partial^2 M$. *Both equalities hold if, and only if,* $\deg^{-1}(0) = \emptyset$.

Proof.
1, 2 and 3. Assertions in these items follow immediately from Definition 4.1 and Proposition 4.1.
4. Given $\boldsymbol{x} \in M$ such that $\operatorname{ind}(\boldsymbol{x}) = 3$, there exists a neighborhood of $\boldsymbol{x}$ which is diffeomorphic to a neighborhood of $(0,0,0)$ in $\mathbf{P}_{\deg(\boldsymbol{x})}$ where $\deg(\boldsymbol{x}) \geqslant 0$. It follows from the definition of pyramid and Corollary 4.2 that the only point with index 3 in the open neighborhood is $\boldsymbol{x}$ itself.
5 and 6. Set $i \in \{0,1,2,3\}$. Fix a point $\boldsymbol{x} \in \partial(\partial_i M) \cap M$ and apply on $\boldsymbol{x}$ the index map. If the index is smaller than i, we have seen in Remark 4.1 that there is no point in a small enough neighborhood of $\boldsymbol{x}$ whose index is larger than $\operatorname{ind}(\boldsymbol{x})$, in particular no point of index i, which is a contradiction to the fact that $\boldsymbol{x} \in \partial(\partial_i M)$. That proves the inclusion

$$\overline{\partial_i M} \cap M = \partial_i M \cup (\partial(\partial_i M) \cap M) \subset \partial^i M.$$

In order to finish this proof we may apply again the Remark 4.1. If i is in $\{0,1,3\}$ and $\boldsymbol{x}$ in $\partial^i M$, then in any open neighborhood of $\boldsymbol{x}$ we find some point with index in $\{0,\dots,\operatorname{ind}(\boldsymbol{x})\} \setminus \{2\}$. In any case, with degree exactly i.
If M is closed in $\mathbb{R}^3$, then $\partial(\partial_i M) \subset \overline{M} = M$ and $\overline{\partial_i M} \cap M = \overline{\partial_i M}$ for each $i \in \{0,1,2,3\}$. □

The exceptional behavior of $i = 2$ on the last statement of the previous proposition is only caused by $\mathbf{P}_0$: it has 2nd stratum empty. With less formalism, the origin in $\mathbf{P}_0$ is a vertex which is not attached to any edge.

We conclude this section with a corollary to the previous proposition which reflects the idea that ∂_i is a generalized boundary operator.

Corollary 4.4. *Let M be a pyramidal manifold. Then*

$$\partial_0 M = M^\circ \text{ and}$$
$$\partial^1 M = \partial M \cap M.$$

If M is closed in $\mathbb{R}^3$, $\partial^1 M = \partial M$.

Proof. After Proposition 4.3 it is enough to show one of the equalities. Let us check that $\partial_0 M = M^\circ$.

First of all, we will show that $\partial_0 M$ is open in M and hence a subset of M°. Given any point in $\partial_0 M$, we fix an adapted local chart (U, φ) and we want to prove that every point in U is actually in $\partial_0 M$. It is clear that $U \subset M$ ($U = \varphi^{-1}(V) = \varphi^{-1}(V \cap \mathbf{P}_{-3}) = U \cap M$). Given any point $\boldsymbol{y} \in U$, as V is open in $\mathbb{R}^3$, we can find an open ball $\widetilde{V}$ centered at $\varphi(\boldsymbol{y})$ fully contained in V. Fixing $\widetilde{U} = \varphi^{-1}(\widetilde{V})$ and

$$\widetilde{\varphi}\colon \boldsymbol{w} \in \widetilde{U} \longrightarrow \widetilde{\varphi}(\boldsymbol{w}) = \varphi(\boldsymbol{w}) - \varphi(\boldsymbol{y}) \in \mathbb{R}^3$$

we get an adapted local chart centered at $\boldsymbol{y}$ making $\deg(\boldsymbol{y}) = 0$.

We work similarly for the other inclusion. Taking $\boldsymbol{x} \in M^\circ$, we can find an open ball U centered at $\boldsymbol{x}$ fully contained in M°. We build an adapted local chart centered at $\boldsymbol{x}$ just by setting

$$\varphi\colon \boldsymbol{w} \in U \longrightarrow \varphi(\boldsymbol{w}) = \boldsymbol{w} - \boldsymbol{y} \in \mathbb{R}^3,$$

which completes the proof. □

5. Pyramidal manifolds and manifolds with corners

In this section we show the existing relations between pyramidal manifolds and some known objects of Differential Geometry such as manifolds with corners or the less demanding notions of manifold, manifold with boundary and manifold with edges. We will see that the concept of pyramidal manifold is a generalization in the 3D case of these and we will prove some results concerning the boundary of a pyramidal manifold.

We understand manifold (with or without boundary) on $\mathbb{R}^p$ in the sense of Spivak [18, Chapter 5]. We adapt to our language the definitions for the sake of clarity and extend them to the case of manifolds with edges and corners.

Definition 5.1. Let M be a subset of $\mathbb{R}^p$, $0 \leqslant n \leqslant p$ an integer, and $r \in \mathbb{N} \cup \{\infty\}$. We assume that whenever $\boldsymbol{x}$ is in M there exists a $\mathscr{C}^r$-diffeomorphism $\varphi_{\boldsymbol{x}}\colon U_{\boldsymbol{x}} \longrightarrow V_{\boldsymbol{x}}$ of an open neighborhood of $\boldsymbol{x}$ in $\mathbb{R}^p$ onto an open subset of $\mathbb{R}^p$ satisfying that $\varphi_{\boldsymbol{x}}(\boldsymbol{x}) = \mathbf{0}$ and $\varphi_{\boldsymbol{x}}(U_{\boldsymbol{x}} \cap M) = V_{\boldsymbol{x}} \cap \left(\Lambda_n^i \times \{0\}^{p-n}\right)$ for some $i \in \{0, 1, 2, n\}$.

We say that M is an *n-dimensional $\mathscr{C}^r$-manifold* (or *$\mathscr{C}^r$-manifold with boundary*, or *with edges*, or *with corners*) in $\mathbb{R}^p$ if $i = 0$ (or $i \leqslant 1$, or $i \leqslant 2$, or $i \in \{0, 1, 2, n\}$, respectively), for all $\boldsymbol{x} \in M$.

Remark 5.1. These charts centered at each point are the equivalent to the adapted charts of Theorem 3.1; that is why we will also call them adapted charts. In case of confusion we will specify *pyramidal adapted* local charts for the ones in Theorem 3.1. Following the same spirit of that theorem, there is an equivalent definition of manifold, manifold with boundary, with edges and with corners using analogous notions of pyramidal charts: instead of requiring the charts to send the point to the origin we ask all the charts to have i fixed. A local chart in this new sense is a chart with corners if $i = n$, with edges if $i = 2$, with boundary if $i = 1$ and just a local chart if $i = 0$.

Definition 5.2 (0th stratum and 1st skeleton)**.** Let M be a manifold with or without boundary, edges or corners. A point $\boldsymbol{x} \in M$ is called a boundary point if there exists no adapted chart with $i = 0$ for $\boldsymbol{x}$. It is called interior otherwise. Then the sets

$$\begin{aligned}
&\partial_0 M = \{\boldsymbol{x} \in M\colon\ \boldsymbol{x} \text{ is interior}\} \text{ and}\\
&\partial^1 M = \{\boldsymbol{x} \in M\colon\ \boldsymbol{x} \text{ is a boundary point}\}
\end{aligned}$$

are called, respectively, the $0th$ *stratum* and the $1st$ *skeleton* of M.

Those sets are well defined, disjoint and they cover M (see [11, Theorem 1.37]).

Remark 5.2. It follows from the definition of Λ-quadrant that $\Lambda^n_n \subset \Lambda^2_n \subset \Lambda^1_n \subset \Lambda^0_n$. We recall here that Λ^3_3 is diffeomorphic to $\mathbf{P}_3$ (Remark 3.2). Putting those two facts together we have, for a 3D $\mathscr{C}^r$-manifold $M \subset \mathbb{R}^3$, the following chain of implications:

M is a manifold $\Longrightarrow$ M is a manifold with boundary $\Longrightarrow$
M is a manifold with edges $\Longrightarrow$ M is a manifold with corners
$\Longrightarrow$ M is a pyramidal manifold.

We introduce the following twin propositions that state that strata are manifolds and the complementary of the skeletons are manifolds with corners (cf. [10]). As we have done just before, we fix $r \in \mathbb{N} \cup \{\infty\}$ and we assume that all the manifolds appearing in the following propositions are of class $\mathscr{C}^r$.

Proposition 5.1. *Let M be a pyramidal manifold. Then:*
0. *$\partial_0 M$ is a 3D manifold.*
1. *$\partial_1 M$ is a 2D manifold.*
2. *$\partial_2 M$ is a 1D manifold.*
3. *$\partial_3 M$ is a 0D manifold.*

Proof. This proposition follows from the definitions just by using the same coordinate charts. The proof can be summarized in the chain of equalities

$$\varphi(U \cap \partial_j M) = \varphi(U) \cap \partial_j(\mathbf{P}_{j-3}) = \varphi(U) \cap \left(\mathbb{R}^{3-j} \times \{0\}^j\right),$$

which hold for $j \in \{0, 1, 2\}$ and U small enough.

For the case $j = 3$ we have noted that $\mathbf{P}_3$ is diffeomorphic to Λ^3_3 (see Remark 3.2, where the diffeomorphism Φ is specified) so that

$$\varphi(U \cap \partial_3 M) = \Phi(\varphi(U)) \cap \partial_3(\Lambda^3_3) = \Phi(\varphi(U)) \cap \left(\mathbb{R}^0 \times \{0\}^3\right),$$

for U small enough. □

Proposition 5.2. *Let M be a pyramidal manifold. Then:*
0. *$M \smallsetminus \partial^0 M = \emptyset$. Hence, it is a 3D manifold with and without boundary, with and without edges and with and without corners.*
1. *$M \smallsetminus \partial^1 M$ is a 3D manifold.*
2. *$M \smallsetminus \partial^2 M$ is a 3D manifold with boundary.*
3. *$M \smallsetminus \partial^3 M$ is a 3D manifold with edges.*
4. *$(M \smallsetminus \partial^3 M) \cup \deg^{-1}(3)$ is a 3D manifold with corners.*

Proof.
0. $M \smallsetminus \partial^0 M = \emptyset$ by Proposition 4.3. The empty set is indeed a manifold with and without boundary, edges and corners.
1. Again by Proposition 4.3, $M \smallsetminus \partial^1 M = \partial_0 M$. This is just a repetition of the statement 0 in Proposition 5.1.
2. In this case we have to modify the coordinate charts only for the points in $\partial_0 M$, being the statement clear in the other case (note that $M \smallsetminus \partial^2 M = \partial_0 M \sqcup \partial_1 M$). This problem arises from the fact that given $\boldsymbol{x} \in \partial_0 M$ and (U, φ) an adapted local chart centered at $\boldsymbol{x}$, $\varphi(U)$ is not a subset of Λ^1_3. We skip this problem by shrinking $\varphi(U)$ to an open ball $B(\varepsilon) \subset \varphi(U)$ of radius ε. We fix $\widetilde{U} = \varphi^{-1}(B(\varepsilon))$ and

$$\widetilde{\varphi}\colon \boldsymbol{w} \in \widetilde{U} \longrightarrow \widetilde{\varphi}(\boldsymbol{w}) = \varphi(\boldsymbol{w}) + (\varepsilon, \varepsilon, \varepsilon) \in \mathbb{R}^3,$$

thus getting a local chart with boundary at $\boldsymbol{x}$ and making $M \setminus \partial^2 M$ be a manifold with boundary.

3. The same situation arises here, with similar solution.
4. We only add to the previous case the points locally like $\mathbf{P}_3$, which is diffeomorphic to Λ_3^3 as we have previously said in the Remark 3.2.

□

After those two propositions, we are able to properly state the implications sketched in Remark 5.2. As it has been the practice throughout this section, we fix $r \in \mathbb{N} \cup \{\infty\}$ and we assume that all the manifolds appearing in the following statement are of class $\mathscr{C}^r$, and, in this particular case, also 3D.

Corollary 5.3. *Let M be a subspace of $\mathbb{R}^3$. Then we have the following equivalences:*
(1) *M is a manifold if and only if M is a pyramidal manifold with empty 1st skeleton ($\partial^1 M = \emptyset$).*
(2) *M is a manifold with boundary if and only if M is a pyramidal manifold with empty 2nd skeleton ($\partial^2 M = \emptyset$).*
(3) *M is a manifold with edges if and only if M is a pyramidal manifold with empty 3rd skeleton ($\partial^3 M = \emptyset$).*
(4) *M is a manifold with corners if and only if M is a pyramidal manifold with 3rd skeleton reduced to $\deg^{-1}(3)$ ($\partial^3 M = \deg^{-1}(3)$).*

Proof. The implications from right to left are just consequences of Proposition 5.2. Before going further, let us point out that $\Lambda_3^i = \mathbf{P}_{i-3}$, and thus it is a $\mathscr{C}^\infty$-pyramidal manifold with $\partial^{i+1}\mathbf{P}_{i-3} = \emptyset$ for $i \in \{0,1,2\}$. Furthermore, there exists a $\mathscr{C}^\infty$-diffeomorphism Φ from $\mathbf{P}_3$ onto Λ_3^3 (see Remark 3.2), and hence Λ_3^3 is a $\mathscr{C}^\infty$-pyramidal manifold with $\partial^3(\Lambda_3^3) = \partial^3(\mathbf{P}_3) = \deg^{-1}(3)$ where the degree might be taken both in Λ_3^3 or in $\mathbf{P}_3$.

In either case, if $\boldsymbol{x} \in M$ we fix (U,φ) a local chart (with or without boundary, edges or corners) at $\boldsymbol{x}$. Post-composing with Φ if necessary, we have that $\varphi(\boldsymbol{x}) \in \Lambda_3^i$ for some $i \in \{0,1,2,3\}$. There exists a, now pyramidal adapted, local chart (V,ψ) centered at $\varphi(\boldsymbol{x})$. Just by taking $\widetilde{U} = U \cap \varphi^{-1}(V)$ and $\widetilde{\varphi}(\boldsymbol{y}) = \psi \circ \varphi(\boldsymbol{y})$ defined for each $\boldsymbol{y} \in \widetilde{U}$, we have that $(\widetilde{U},\widetilde{\varphi})$ is a pyramidal adapted local chart centered at $\boldsymbol{x}$. What is more, $\operatorname{ind}(\boldsymbol{x}) = \operatorname{ind}(\varphi(\boldsymbol{x}))$, $\deg(\boldsymbol{x}) = \deg(\varphi(\boldsymbol{x}))$ if it is defined and, since $\varphi(\boldsymbol{x}) \in \Lambda_3^i$, we have that $\operatorname{ind}(\boldsymbol{x}) \leqslant i$. Repeating this process for each $\boldsymbol{x} \in M$ we get that M is a pyramidal manifold with $\partial^{i+1}M = \emptyset$ if $i \in \{0,1,2\}$ or that $\partial^3 M = \deg^{-1}(3)$ for $i = 3$. □

It is not true in general that the closure of each connected component of $\partial_1 M$ is a 2D manifold with corners. As counterexamples we find $\mathbf{P}_0$ and $\mathbf{P}_1$. We can only ensure the following local behavior.

Proposition 5.4. *Let M be a pyramidal manifold. Let (U,φ) be an adapted local chart centered at $\boldsymbol{x} \in \partial^1 M$ such that $U \cap \partial^1 M$ is connected. Then $U \cap \partial_1 M$ has $\operatorname{ind}(\boldsymbol{x})$ connected components if $\operatorname{ind}(\boldsymbol{x}) < 3$, otherwise it has $\deg(\boldsymbol{x})$ connected components if $\deg(\boldsymbol{x}) > 0$ and it is connected if $\deg(\boldsymbol{x}) = 0$.*

Moreover, if $\boldsymbol{x} \notin \deg^{-1}(\{0,1\})$, given any one of those connected components, its closure relative to U is a 2D manifold with corners.

Proof. As this is a local statement, it is enough to prove it for the pyramids, because the number of connected components and the fact that it is a 2D manifold with boundary are stable under diffeomorphisms.

We fix a pyramid $M = \mathbf{P}_i$ for some $i \geqslant -2$. Given any neighborhood U of $(0,0,0)$ in $\mathbf{P}_i$ such that $U \cap \partial \mathbf{P}_i$ is connected, it follows that $U \cap (\partial_1 \mathbf{P}_i)$ has the required number of connected components (one for each surface π_k^i).

We suppose now that $\boldsymbol{x} \notin \deg^{-1}(\{0,1\})$. The surfaces π_k^i are planes except for $i = 2$ when they are both diffeomorphic to a triangular sector of the plane $x_2 = 0$ just by projection (we call this map ρ). We fix S one of such connected components. Its closure relative to U is nothing but the piece of plane π_k^i which meets V (post-composing with ρ if necessary). It is now enough to see that $\pi_k^i \cap V$ is diffeomorphic to an open subspace of $\Lambda_2^2 \times \{0\}$. This is conquered just by considering the restriction to $\pi_k^i \cap U$ of the linear map which sends π_k^i onto the plane $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \colon\ x_3 = 0\}$. We then have that $\pi_k^i \cap U$ is diffeomorphic to an open submanifold of a 2D manifold with corners, and then it is also a manifold with corners (see [18]). $\square$

It is important to be aware that each connected component of $U \cap \partial_1 M$ does not have to be part of a different connected component of $\partial_1 M$. A tear-based bounded cylinder serves as a counterexample (by tear we mean the topological disc $\mathbf{P}_1 \cap \{x_3 = 1\}$).

Remark 5.3. Even though we have no intrinsic 2D local charts with corners for $\boldsymbol{x} \in \deg^{-1}(\{0,1\})$ we might construct them explicitly. For $M = \mathbf{P}_i$, $i \in \{0,1\}$, we consider $\partial \mathbf{P}_i \cap \{x_2 \leqslant 0\}$ and $\partial \mathbf{P}_i \cap \{x_2 \geqslant 0\}$. Those two connected subspaces of the boundary are 2D manifolds with boundary just by projecting to the plane $x_2 = 0$.

6. Bodies

The problems arising from the Gurtin-Kellogg definition of body have already been explained in the introduction. The previous opening discussion about pyramidal manifolds leads us to seek a definition of body in terms of pyramidal manifolds.

Remark 6.1. We will think of bodies as deformations of locally convex polytopes. In particular, we exclude from the definition those subspaces of $\mathbb{R}^3$ which are not locally deformable to a convex body. Extending the definition to those regions is a fundamental future line of work.

6.1. Regions.

We are going to introduce a definition appearing in Gurtin's book [6], with some modifications.

Definition 6.1 (Closed region)**.** A subset R of $\mathbb{R}^n$ is called a *closed region* in $\mathbb{R}^n$ whenever there exists an open and connected set $D \subset \mathbb{R}^n$ such that $\overline{D} = R$.

We remark that for Gurtin the term connected refers to $\mathscr{C}^1$-path connectedness. We use instead the general notion of connectedness. Those two concepts are equivalent for open subsets of $\mathbb{R}^n$ as a corollary of Whitney's approximation theorem (see [11, Theorem 6.26], for instance). An open and connected set, like D in the definition above, is often called a *domain* as we have already said in the introduction for $n = 3$. It is important to observe that, given any two points in a closed region $R = \overline{D}$, it is possible to join them by a $\mathscr{C}^1$ (in fact, $\mathscr{C}^\infty$) path whose interior is fully contained in D.

We start relating Gurtin's terminology with that of pyramidal manifolds by the following definition.

Definition 6.2 (Quasi-body)**.** We call *quasi-body* a $\mathscr{C}^1$-differentiable pyramidal manifold which is non-empty and connected.

This term is justified by the next proposition, which shows that the only property that a quasi-body lacks to be a closed region is to be closed in $\mathbb{R}^3$.

Proposition 6.1. *Let M be a quasi-body. The following statements are equivalent:*
(1) *M is closed in $\mathbb{R}^3$.*
(2) *M is a closed region in $\mathbb{R}^3$. What is more, M° is a domain and $M = \overline{M^\circ}$.*

Proof. It clearly suffices to prove the implication $1 \Rightarrow 2$, since the other one holds trivially. Assume that the quasi-body M is closed in $\mathbb{R}^3$.

a) We first observe that $M = \overline{M^\circ}$. This fact follows from Corollary 4.4 and Proposition 4.3:
$$\overline{M^\circ} = \overline{(\partial_0 M)} = \partial^0 M = M.$$
b) We prove now that M° is connected. For doing so, we write M° as the union of two sets A and B which are both open and closed (in M°) and disjoint. We want to argue that either A or B is the empty set. In order to do so we are going to prove that $M = \overline{A} \cup \overline{B}$ and that the closed subsets $\overline{A}$ and $\overline{B}$ of M are disjoint.

b.i: First of all $\overline{A} \cup \overline{B} = \overline{A \cup B} = \overline{M^\circ} = M$ as seen in the previous step.

b.ii: Now we are going to prove that $\overline{A}$ and $\overline{B}$ are disjoint. It is enough to prove that ∂A and ∂B are disjoint, because A is open in M and disjoint from B. Then $A \cap \overline{B} = \emptyset$, and by symmetry $\overline{A} \cap B = \emptyset$, which yields to
$$\overline{A} \cap \overline{B} = \partial A \cap \partial B.$$
Now, given $\boldsymbol{x} \in \partial A$ we have that $\boldsymbol{x} \notin A$ because A is open in M and $\boldsymbol{x} \notin B$ because $\partial A \cap B \subset \overline{A} \cap B = \emptyset$. Thus $\boldsymbol{x} \in M \setminus M^\circ = \partial M$. We are about to prove that $\boldsymbol{x} \notin \partial B$. Let (U, φ, n) be an adapted local chart for $\boldsymbol{x}$. Fix ε such that $0 < \varepsilon < \mathrm{d}\left((0,0,0), \partial\left(\varphi(U)\right)\right)$ and define $B = B(\varepsilon)$, $V = \varphi^{-1}(B)$ and $W = \varphi^{-1}(B \cap \mathbf{P}_n^\circ)$. We have on one hand that V is an open neighborhood of $\boldsymbol{x}$ and then $V \cap A \neq \emptyset$. On the other hand, we have that $B \cap \mathbf{P}_n^\circ$ is convex (it is the intersection of two convex sets) and thus connected, from which we infer that W is also connected.
We observe now that $W = V \cap M^\circ$ just by checking the degrees. In particular, $W \subset M^\circ$ and hence fully contained either on A or on B. But $W \cap A = V \cap M^\circ \cap A = V \cap A \neq \emptyset$. This proves $W \subset A$ and in particular $W \cap B = \emptyset$. In order to conclude we just see that the open neighborhood V of $\boldsymbol{x}$ does not intersect B and then $\boldsymbol{x} \notin \partial B$:
$$V \cap B = V \cap (M^\circ \cap B) = (V \cap M^\circ) \cap B = W \cap B = \emptyset.$$
b.iii: We have that $\overline{A}$ and $\overline{B}$ are disjoint, both closed in $M = \overline{A} \cup \overline{B}$, which implies that one of them is empty, then either $A = \emptyset$ or $B = \emptyset$. □

6.2. Kellogg's terminology. In this subsection we reproduce the basic definitions of Kellogg in [8, IV.7-9] which are relevant for our purposes. The inclusion of these terms is justified not only for the article to be self-contained but also because we have translated the definitions into a more contemporary language with the help of [4].

Definition 6.3 (Piecewise $\mathscr{C}^1$-curve in $\mathbb{R}^n$)**.** Let $[a,b] \subset \mathbb{R}$ be a non-degenerated interval, and let n be an integer greater than or equal to 2. We say that a continuous function $\gamma\colon [a,b] \longrightarrow \mathbb{R}^n$ is a *piecewise $\mathscr{C}^1$-curve in $\mathbb{R}^n$* when:
(1) $\gamma|_{[a,b)}$ is injective.
(2) There exists $a = t_0 < \cdots < t_k = b$ a partition of $[a,b]$ such that $\gamma|_{[t_i,t_{i+1}]}$ is of class $\mathscr{C}^1$and $\gamma'|_{(t_i,t_{i+1})} \neq 0$ for all $0 \leqslant i \leqslant k-1$.

The curve γ is said to be *closed* whenever $\gamma(a) = \gamma(b)$.

The points $\{\gamma(t_j)\}_{j=0}^k$ are called *vertices* of the curve, while for each $j \in \{0, \ldots, k-1\}$ the set $\{\gamma(t) : t \in [t_j, t_{j+1}]\}$ will be referred to as an *arc* of γ.

Definition 6.4 (Regular region of the plane)**.** We say that $R \subset \mathbb{R}^2$ is a *regular region of the plane* if it is a bounded closed region whose boundary is the image of a piecewise $\mathscr{C}^1$-curve in $\mathbb{R}^2$.

Definition 6.5 (Regular surface element)**.** A subset S of $\mathbb{R}^3$ is called a *regular surface element* when it is the graph of a regular region of one of the coordinate planes by a $\mathscr{C}^1$-map.

A point in S is called a *vertex* when it is the image of a vertex of the piecewise $\mathscr{C}^1$-curve bounding the regular region whose graph is S. Analogously, the image of an arc is called an *arc* on S.

Definition 6.6 (Piecewise regular surface)**.** A subset $S \subset \mathbb{R}^3$ is called a *piecewise regular surface* when there exists a finite number of regular surface elements $\{S_i\}_{i=1}^n$, called faces, such that $S = \bigcup_{i=1}^n S_i$ and fulfill the following conditions:

s1: $S_i \cap S_j$ is a common vertex, or a common arc of the two faces, or the empty set for each pair of different indices in $\{1, \ldots, n\}$.

s2: $S_i \cap S_j \cap S_k$ is empty or a common vertex of the three faces for each triple of different indices in $\{1, \ldots, n\}$.

s3: For each pair of different indices i and j in $\{1, \ldots, n\}$, there exist m, a positive integer smaller than n, and σ, a permutation of $\{1, \ldots, n\}$, such that $\sigma(1) = i$, $\sigma(m) = j$ and $S_{\sigma(k)} \cap S_{\sigma(k+1)}$ is a common arc of the faces $S_{\sigma(k)}$ and $S_{\sigma(k+1)}$ for each k in $\{1, \ldots, m\}$.

s4: For each $\boldsymbol{x} \in S$, let m be the cardinality of $\{S_i : \boldsymbol{x} \text{ is a vertex of } S_i\}$. Then there exists σ, a permutation of $\{1, \ldots, n\}$, such that for each $k \in \{1, \ldots, m-1\}$ there exists e_k, a common arc of $S_{\sigma(k)}$ and $S_{\sigma(k+1)}$, such that $\boldsymbol{x}$ is an extremum of e_k.

The piecewise regular surface is said to be closed whenever every arc of every face is an arc of exactly two faces.

It is already possible to see the will of Kellogg of defining *regular regions* in a recursive way. The following definitions generalize this idea to the 3D case.

Definition 6.7 (Element of regular region of the space)**.** An *element of regular region of the space* is a subset R of $\mathbb{R}^3$ such that

(1) R is a bounded and closed region of $\mathbb{R}^3$, and
(2) ∂R is a closed piecewise regular surface.

Again, we call the vertices and arcs of ∂R vertices and arcs of R.

Definition 6.8 (Regular region of the space)**.** A subset R of $\mathbb{R}^3$ is called a *regular region of the space* when there exists a finite number of elements of regular region of the space, $\{R_i\}_{i=1}^n$, such that $R = \bigcup_{i=1}^n R_i$ and $R_i \cap R_j$ is either empty, a common vertex, a common arc, or a common face of both elements for each pair of different indices.

7. Body as a pyramidal manifold

We introduce now our proposal of a local definition of body in CM.

Definition 7.1 (Body)**.** We say that $\mathscr{B} \subset \mathbb{R}^3$ is a *body* when it is a quasi-body which is closed in $\mathbb{R}^3$.

The following theorem is the main result of this paper.

Theorem 7.1. *Let $\mathscr{B}$ be a body. Then each connected component of $\partial\mathscr{B}$ is a piecewise regular surface. The number of those piecewise regular surfaces is countable, and finite when $\mathscr{B}$ is bounded.*

It is crucial to distinguish the connected components, since a disconnected boundary does not constitute a piecewise regular surface (see condition **s3** in Definition 6.6).

Proof. The proof is structured into five steps, to facilitate its comprehension.

Step 1: Suppose that $\mathscr{B}$ is a bounded body whose boundary is connected.

Let $V(\mathscr{B})$ be the set of vertices of $\mathscr{B}$. We are going to add points to the set $V(\mathscr{B})$ at each step. First of all, we initialize in the following way: $V(\mathscr{B}) = \partial^3\mathscr{B}$. This set is finite as a consequence of Proposition 4.3 and the compactness of $\mathscr{B}$. For each $\boldsymbol{x}$ in the boundary of $\mathscr{B}$, we can find a pyramidal adapted local chart centered at $\boldsymbol{x}$ which defines a finite number of 2D local charts with corners of $\partial^1 B$ (see Proposition 5.4 and Remark 5.3). Fix one of those charts with corners, (U,φ). Note that U can be reduced if necessary for it to be the graph of a coordinate function (see [4, Proposition 3]). We assume without loss of generality that $U = \{(x_1, g(x_1,x_3), x_3)\}$. In this case we take φ such that $\varphi(x_1,x_2,x_3) = (x_1,x_3)$. We make U even smaller so that $\varphi(U)$ is a square ball in the corresponding coordinate plane.

Let $\mathcal{U}$ be the covering of $\partial^1\mathscr{B}$ given by the charts constructed as before for each point in the boundary and each of those local charts. The set $\partial^1\mathscr{B}$ is closed in $\mathscr{B}$ and hence compact. We take a finite subcovering of $\mathcal{U}$ which is minimal in the sense that we cannot extract any proper subcovering. We denote it by $\{(U_i,\varphi_i)\}_{i=1}^n$.

Step 2: We want a covering $\{V_i\}_{i=1}^n$ with an additional property: the intersection of the interior of any two different elements of the covering is empty. This is conquered by setting

$$\begin{aligned} V_1 &= \overline{U_1}, \\ V_i &= \overline{\left(U_i \setminus \bigcup_{j=1}^{i-1} U_j\right)} \quad \text{for } i > 1. \end{aligned}$$

We extend by continuity the maps φ_i to V_i. We add to the set of vertices those which are preimage of a vertex in $\varphi_i(V_i)$ for each i. Each $V_i \cap \partial^1 M$ is a compact 2D manifold with corners. This ensures that $V(\mathscr{B})$ remains finite.

We want to work with a refinement of $\{V_i\}_{i=1}^n$ formed by closed regions. We take the closure of each connected component of the interior of each V_i. The number of such sets is finite, because of the finiteness of $\{(U_i,\varphi_i)\}_{i=1}^n$ and $V(\mathscr{B})$. We denote this collection by $\{W_j\}_{j=1}^m$, which is still a covering of $\partial_1\mathscr{B}$. We call ψ_j the restriction to W_j of φ_i, where $W_j \subset V_i$.

Step 3: We are going to distinguish two different kinds of vertices on $\mathscr{B}$. On one hand, those which are in the interior of some W_j. We denote the set of such vertices as $V_{\text{int}}(\mathscr{B})$ that so far is empty. On the other hand, we work with the vertices which are in the boundary of some W_j. The set of such vertices will be denoted by $V_{\text{ext}}(\mathscr{B})$ and it is currently constituted by the points in $V(\mathscr{B})$. We will also build at this step the set of arcs of $\partial\mathscr{B}$, which will be denoted by $A(\mathscr{B})$. We similarly distinguish between interior and exterior arcs ($A_{\text{int}}(\mathscr{B})$ and $A_{\text{ext}}(\mathscr{B})$).

We are going to give the structure of an element of regular surface to each (W_j, ψ_j), where j varies from 1 to m. We first observe that $\psi_j(W_j)$ is a closed region of the plane, and thus $\mathscr{C}^1$-path connected (in fact, $\mathscr{C}^\infty$-path connected). We denote by $\{\boldsymbol{y}_{j,i}\}_{i=1}^{k(j)}$ the set of images by ψ_j of $V(\mathscr{B}) \cap W_j$. We pick up a point $\boldsymbol{y}_{j,0}$ in the interior of $\psi_j(W_j)$. We are going to build arcs connecting $\boldsymbol{y}_{j,0}$ with each of the points in $\{\boldsymbol{y}_{j,i}\}_{i=0}^{k(j)}$ in a recursive way. For convenience, we denote $W_{j,0} = W_j^\circ$ and we construct at each step a domain $W_{j,l}$ which has the points $\{\boldsymbol{y}_{j,i}\}_{i=0}^{k(j)}$ on its boundary. Suppose we have built a $\mathscr{C}^1$-path for some l (the case $l = 0$ is trivial). We can build a $\mathscr{C}^1$-path

$$\gamma_{j,l+1} \colon [0,1] \longrightarrow \psi(W_j)$$

joining $\boldsymbol{y}_{j,0}$ and $\boldsymbol{y}_{j,l+1}$ such that $\gamma_{j,l+1}((0,1))$ is fully contained in $W_{j,l}$. We set $W_{j,l+1} := (\psi(W_{j,l}))^\circ \smallsetminus \gamma_{j,l+1}([0,1])$, which is still a domain whose closure contains the points $\{\boldsymbol{y}_{j,i}\}_{i=1}^{k(j)}$. We repeat this process until we have constructed $\gamma_{j,k(j)}$.

We add to $V_{\mathrm{int}}(\mathscr{B})$ the points $\{\psi_j^{-1}(\boldsymbol{y}_{j,0})\}_{j=1}^m$. We add to $A_{\mathrm{int}}(\mathscr{B})$ the set of arcs built for W_j:

$$\left\{\psi_j^{-1}\left(\gamma_{j,l}([0,1])\right) : \quad 1 \leqslant l \leqslant k(l)\right\}_{j=1}^m .$$

The exterior arcs are those paths joining the points $\{\boldsymbol{y}_{j,i}\}_{i=1}^{k(j)}$ on ∂W_j. We repeat this process until we treat all the elements W_j.

Step 4: For each j, the set $(\psi_j(W_j))^\circ \smallsetminus \bigcup_{l=1}^{k(j)} \gamma_{j,l}([0,1])$ has $k(j)$ connected components that we can index by $l \in \{1, \ldots, k(j)\}$. We denote by $S_{j,l}$ the preimage by ψ_j of the closure of the lth connected component of that set. The set of faces is given by all those $S_{j,l}$ varying j and l until we have covered all the possible cases. We remark that this set, denoted by $C(\mathscr{B})$, is finite as well. By construction $\bigcup_{S_{j,l} \in C(\mathscr{B})} S_{j,l} = \partial_1\mathscr{B}$. What is more, each $S_{j,l}$ is the image of a triangle, and then an element of regular surface which exactly three arcs (two interior and one exterior) and three vertices (two exterior and one interior). We have endowed $\partial\mathscr{B}$ with the structure of a piecewise regular surface. Let us check the conditions:

- **s1:** $S_{i,j} \cap S_{k,l}$ has to occur in the boundary of those sets by construction. If $i = k$ they share the point $\psi_i^{-1}(\boldsymbol{y}_{i,0})$; in case they share another vertex, the whole intersection will just be the arc joining those two vertices. If $i \neq k$, $S_{i,j}$ and $S_{k,l}$ have an unique exterior arc that could be the same or not. In the first case the intersection is that arc, in the second case the intersection is a common vertex of those two arcs or the empty set.
- **s2:** $S_{i,j} \cap S_{k,l} \cap S_{m,n}$ is the point $\psi_i^{-1}(\boldsymbol{y}_{i,0})$ if $i = k = m$. If one the indices i, j and m is not equal to the others, the intersection has to occur in the exterior arcs of the faces. By repeating the last argument in **s1** twice we get that the intersection can only be a common vertex or the empty set.
- **s3:** Since $\partial\mathscr{B} \smallsetminus V(\mathscr{B})$ is a connected topological manifold with boundary, we can always join two points by an arc in $\partial\mathscr{B}$ (recall that $V(\mathscr{B})$ is finite). By taking two points in the interior of $S_{i,j}$ and $S_{k,l}$ we build such a path and we take care of the faces and arcs that the path crosses—avoiding repetition—to construct the desired faces and arcs.
- **s4:** Given an element of $V(\mathscr{B})$ we distinguish two cases: interior and exterior. If the vertex, say $\psi_j^{-1}(\boldsymbol{y}_{j,0})$, is interior we just select an arbitrary element

$S_{j,l}$ and continue in a fixed direction (clockwise for instance) on the coordinate associated plane until we return to $S_{j,l}$. If the vertex is exterior, we take some $S_{j,l}$ such that the point is a vertex of this face and we continue through the unique $S_{j,k}$ different from $S_{j,l}$ sharing that vertex. After that, we just keep moving along all the $\psi_j^{-1}(W_j)$ with no empty intersection with the selected vertex. This process finishes coming back to $S_{j,l}$ due to the finiteness of $C(\mathscr{B})$.

Step 5: We extend the proof to the general case. If $\partial^1\mathscr{B}$ is still bounded but not connected, we repeat the process for each connected component. The number of those connected components is finite.

If $\mathscr{B}$ is not bounded, we take the net

$$\{K_{ijk} = [i, i+1] \times [j, j+1] \times [k, k+1]\}_{i,j,k \in \mathbb{Z}}.$$

Every $\mathscr{B} \cap K_{ijk}$ is a bounded body and we can apply the previous process for $\partial^1\mathscr{B} \cap K_{ijk} \subset \partial(\mathscr{B} \cap K_{ijk})$. We lose the finiteness of the number of regular surfaces, but not its countability.

Now the proof of Theorem 7.1 is complete. □

We remark here that a bounded body fulfills the conditions of Definition 6.8 (this cannot be expected for an unbounded space). What is more, if its boundary is connected, a bounded body fulfills the conditions of Definition 6.7.

8. Deformations

Once we have established a definition of body in terms of Differential Geometry we keep translating the definitions of Gurtin [6] into this language. We will prove some results regarding deformations of the whole space and boundary invariance.

Proposition 8.1. *Let $\mathscr{B}$ be a body and let $\mathsf{f}\colon \mathscr{B} \longrightarrow \mathbb{R}^3$ be an injective and differentiable map such that* $\det \mathrm{D}\mathsf{f}(\boldsymbol{x}) \neq 0$ *for each $\boldsymbol{x} \in \mathscr{B}$. Then:*
(1) $\mathsf{f}(\mathscr{B})$ *is a quasi-body.*
(2) $\mathrm{ind}(\boldsymbol{x}) = \mathrm{ind}(\mathsf{f}(\boldsymbol{x}))\ \forall \boldsymbol{x} \in \mathscr{B}$.
(3) $\deg(\boldsymbol{x}) = \deg(\mathsf{f}(\boldsymbol{x}))\ \forall \boldsymbol{x} \in \mathrm{ind}^{-1}(3)$.

Proof. Since $\mathscr{B}$ is closed in $\mathbb{R}^3$, there exists a differentiable extension of f, namely $\widetilde{\mathsf{f}}\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, by [9, Appendix 3, Theorem 2].

Set $N := \mathsf{f}(\mathscr{B})$ and let $\boldsymbol{y}$ be a point in N. Let $\boldsymbol{x} \in \mathscr{B}$ be its unique preimage in $\mathscr{B}$ and let (U_1, φ_1, n) be an adapted local chart centered at $\boldsymbol{x}$. By the inverse function theorem, there exists U an open neighborhood of $\boldsymbol{x}$, $U \subset U_1$, such that $g := \widetilde{\mathsf{f}}|_U : U \longrightarrow \widetilde{W} = \widetilde{\mathsf{f}}(U)$ is a diffeomorphism.

By restricting the local chart to $(U, \varphi = \varphi_1|_U, n)$ we construct an adapted local chart on N centered at $\boldsymbol{y}$. Set $W := \widetilde{\mathsf{f}}(U) \subset \widetilde{W}$ and $\psi\colon = \varphi \circ g^{-1}\colon W \longrightarrow \varphi(U)$. The mapping ψ is a diffeomorphism because it is the composition of two diffeomorphisms. What is more, $\psi(W \cap N) = \varphi(U \cap \mathscr{B}) = \varphi(U) \cap \mathbf{P}_n$ and $\psi(\boldsymbol{y}) = (0, 0, 0)$. This proves that (W, ψ, n) is an adapted local chart for $\boldsymbol{y}$, and so the index and degree (in case it is defined) of $\boldsymbol{x}$ and $\boldsymbol{y}$ are the same. □

This result and the following corollary show that the approach of taking charts to pyramids has the advantage that we can distinguish between points on the boundary of a body by inspecting their degrees. The classification remains the same after a deformation, so that a point having a certain degree cannot evolve into a point with a different degree.

Corollary 8.2 (Stratum invariance)**.** *Let $\mathscr{B}$ be a body and let* $\mathsf{f}\colon \mathscr{B} \longrightarrow \mathbb{R}^3$ *be an injective and differentiable map such that* $\det \mathrm{Df}(\boldsymbol{x}) \neq 0\ \forall \boldsymbol{x} \in \mathscr{B}$*. Then, for each* $i \in \{0, 1, 2, 3\}$,

$$\mathsf{f}(\partial_i \mathscr{B}) = \partial_i(\mathsf{f}(\mathscr{B})) \quad \textit{and} \quad \mathsf{f}(\partial^i \mathscr{B}) = \partial^i(\mathsf{f}(\mathscr{B})).$$

This result is a generalization (in $\mathbb{R}^3$) of [14, Theorem 1.2.12]. As a particular case we have that

$$\mathsf{f}(\partial \mathscr{B}) = \partial(\mathsf{f}(\mathscr{B})) \quad \text{and} \quad \mathsf{f}(\mathscr{B}^\circ) = (\mathsf{f}(\mathscr{B}))^\circ .$$

In the hypotheses of the two previous results, $\mathsf{f}(\mathscr{B})$ only needs to be closed in $\mathbb{R}^3$ to be a body. We are going to require this extra condition and the fact that $\det \mathrm{Df}(\boldsymbol{x}) > 0\ \forall \boldsymbol{x} \in \mathscr{B}$ in the definition of deformation. In order to use a more common language, let us remark that this last condition is often summarized by saying that f is orientation-preserving.

Definition 8.1 (Deformation)**.** Let $\mathscr{B}$ be a body. A map $\mathsf{f}\colon \mathscr{B} \longrightarrow \mathbb{R}^3$ is said to be a *deformation of* $\mathscr{B}$ if it satisfies the following conditions:

d1: f is a $\mathscr{C}^1$-diffeomorphism between $\mathscr{B}$ and $\mathsf{f}(\mathscr{B})$,
d2: f is orientation-preserving, and
d3: $\mathsf{f}(\mathscr{B})$ is closed in $\mathbb{R}^3$.

Previous results help to easily recognize whether a given body is deformable into another configuration or not. Figure 2 shows an example.

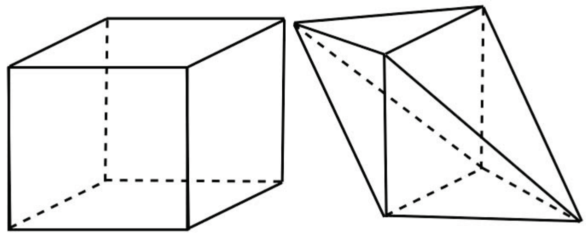

FIGURE 2. These two bodies cannot be deformed one into another. This fact can be derived from the invariance of the degree (Corollary 4.2).

Definition 8.1 coincides with the one given by Gurtin in [6], except in condition **d3**. Gurtin asks $\mathsf{f}(\mathscr{B})$ to be a closed region of the space, but from Proposition 6.1 we know that this fact and the closed nature of $\mathsf{f}(\mathscr{B})$ are equivalent provided $\mathsf{f}(\mathscr{B})$ is a quasi-body (which in turn is ensured by Proposition 8.1). In this new language it is very easy to prove some results concerning deformations that were not clear in the Gurtin-Kellogg language.

Proposition 8.3. *Let* $\mathsf{f}\colon \mathscr{B} \longrightarrow \mathbb{R}^3$ *be a deformation of a body* $\mathscr{B}$*. Then* $\mathsf{f}(\mathscr{B})$ *is a body.*

Proof. $\mathsf{f}(\mathscr{B})$ is a quasi-body by Proposition 8.1, non-empty because $\mathscr{B} \neq \emptyset$, and closed in $\mathbb{R}^3$ by definition of deformation. □

We are going to remind the concept of part of a body.

Definition 8.2 (Part of a body)**.** A subset $\mathscr{P}$ of a body $\mathscr{B}$ is called a *part* of $\mathscr{B}$ if it is itself a body and it is bounded.

Proposition 8.4. *Let* $\mathsf{f}\colon \mathscr{B} \longrightarrow \mathbb{R}^3$ *be a deformation of a body* $\mathscr{B}$ *and let* $\mathscr{P}$ *be a part of* $\mathscr{B}$*. Then*
(1) $\mathsf{f}^{-1}\colon \mathsf{f}(\mathscr{B}) \longrightarrow \mathbb{R}^3$ *is a deformation of* $\mathsf{f}(\mathscr{B})$*,*
(2) $\mathsf{f}|_{\mathscr{P}}\colon \mathscr{P} \longrightarrow \mathbb{R}^3$ *is a deformation of* $\mathscr{P}$*, and*
(3) $\mathscr{B} = \mathbb{R}^3$ *if, and only if,* $\mathsf{f}(\mathscr{B}) = \mathbb{R}^3$*.*

Proof. First of all we see that $\mathsf{f}(\mathscr{B})$ is a body from Proposition 8.3.
(1) **d1-2:** By the inverse function theorem, f^{-1} is a diffeomorphism between $\mathsf{f}(\mathscr{B})$ and $\mathscr{B}$. Moreover,

$$\det \mathsf{Df}^{-1}(\boldsymbol{y}) = [\det \mathsf{Df}(\mathsf{f}^{-1}(\boldsymbol{y}))]^{-1} > 0 \ \forall \boldsymbol{y} \in \mathsf{f}(\mathscr{B}).$$

d3: $\mathsf{f}^{-1}(\mathsf{f}(\mathscr{B})) = \mathscr{B}$, which is closed in $\mathbb{R}^3$.
(2) The only non-trivial condition is **d3**. $\mathscr{P}$ is compact and f continuous, so $\mathsf{f}(\mathscr{P})$ is compact and hence closed in $\mathbb{R}^3$ as it is a closed subset of the closed space $\mathsf{f}(\mathscr{B})$.
(3) If $\mathscr{B} = \mathbb{R}^3$, $\mathsf{f}(\mathbb{R}^3)$ is a body with $\partial^1(\mathsf{f}(\mathbb{R}^3)) = \mathsf{f}(\partial^1\mathbb{R}^3) = \mathsf{f}(\emptyset) = \emptyset$. The only subsets of $\mathbb{R}^3$ with empty boundary are $\mathbb{R}^3$ and $\emptyset$. But $\partial^1(\mathsf{f}(\mathbb{R}^3)) = \partial(\mathsf{f}(\mathbb{R}^3))$ by Proposition 4.4 and, being $\mathsf{f}(\mathbb{R}^3) \neq \emptyset$, the only choice is $\mathsf{f}(\mathbb{R}^3) = \mathbb{R}^3$. The other implication follows by applying the same procedure to its inverse. □

The higher idea of motion of a body is also a well known concept in Differential Geometry.

Definition 8.3 (Motion of a body)**.** Let $\mathscr{B}$ be a body. We say that the map $\mathsf{x}\colon \mathscr{B} \times \mathbb{R} \longrightarrow \mathbb{R}^3$ is a *motion of* $\mathscr{B}$ if it is an isotopy of class $\mathscr{C}^3$ such that, for each $t \in \mathbb{R}$, the map $\mathsf{x}_t(\boldsymbol{x}) = \mathsf{x}(\boldsymbol{x}, t)$ is a deformation of $\mathscr{B}$.

9. Differential forms

In this section we introduce the language of differential forms on pyramidal manifolds in order to generalize the proof of Stokes theorem for bodies. For doing that we will not only need to establish the notions of differential forms on pyramidal manifolds, but also on 2D manifolds with corners (think of the boundary of a pyramidal manifold). We will use the term Λ*-pyramid* to mean both the pyramids in $\mathbb{R}^3$ and the Λ-quadrants of orders 1 and n in $\mathbb{R}^n$. We will use as well the term Λ*-manifold of dimension* n *in* $\mathbb{R}^p$ (or simply Λ*-manifold*) to embrace the notions of pyramidal manifold in $\mathbb{R}^3$ and n-dimensional manifold with corners in $\mathbb{R}^p$.

A differentiable function $\mathsf{f}\colon U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ defines the linear map $\mathsf{Df}(\boldsymbol{x})\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$, for each $\boldsymbol{x} \in U$, and then a linear map between tangent spaces $\mathsf{f}_*\colon T_{\boldsymbol{x}}\mathbb{R}^n \longrightarrow T_{\mathsf{f}(\boldsymbol{x})}\mathbb{R}^m$ by $\mathsf{f}_*(v_{\boldsymbol{x}}) = (\mathsf{Df}(\boldsymbol{x})(v))_{\mathsf{f}(\boldsymbol{x})}$. This linear map induces a linear map between the vector spaces of k-linear forms $\mathsf{f}^*\colon \Lambda^k(T_{\mathsf{f}(\boldsymbol{x})}\mathbb{R}^m) \longrightarrow \Lambda^k(T_{\boldsymbol{x}}\mathbb{R}^n)$ for each $k \geqslant 1$.

Definition 9.1 (Tangent space to a Λ-manifold)**.** Let $\boldsymbol{x}$ be point of a Λ-manifold M of dimension n in $\mathbb{R}^p$. Given an adapted local chart (U, φ) at $\boldsymbol{x}$, we define the *tangent space* to M at $\boldsymbol{x}$ by $T_{\boldsymbol{x}}M = \varphi_*^{-1}(\mathbb{R}^n \times \{0\}^{p-n})_0$ if M is an n-dimensional Λ-manifold in $\mathbb{R}^p$.

This definition is independent of the adapted local chart. The proof is a repetition, *mutatis mutandis*, of the same statement for manifolds. The vector space of linear forms of degree k on $T_{\boldsymbol{x}}M$ is denoted by $\Lambda^k(T_{\boldsymbol{x}}M)$.

It is possible to talk about vector fields and differential forms on Λ-pyramids. This is done just by considering them to be subspaces of the corresponding $\mathbb{R}^p$ with the differentiable structure given by local extensions. This allows us to extend those definitions to Λ-manifolds.

Definition 9.2. Let M be a Λ-manifold of dimension n in $\mathbb{R}^p$. Let $TM = \bigcup_{\boldsymbol{x}\in M} T_{\boldsymbol{x}}M$ and $\Lambda^k TM = \bigcup_{\boldsymbol{x}\in M} \Lambda^k T_{\boldsymbol{x}}M$.

(1) We say that a map $\mathrm{F}\colon M \longrightarrow TM$ such that $\mathrm{F}(\boldsymbol{x}) \in T_{\boldsymbol{x}}M$ for each $\boldsymbol{x} \in M$ is a *vector field of class $\mathscr{C}^r$ on M* if for each adapted chart (U,φ) on M the map $\boldsymbol{y} \in \varphi(U \cap M) \longrightarrow \varphi_*\mathrm{F}(\varphi^{-1}(\boldsymbol{y})) = \sum_{i=1}^n \lambda_i(\boldsymbol{y})(e_i)_{\boldsymbol{y}}$ is of class $\mathscr{C}^r$, that is the components $\lambda_1,\dots,\lambda_n$ are functions of class $\mathscr{C}^r$.

(2) We say that a map $\omega\colon M \longrightarrow \Lambda^k TM$ such that $\omega(\boldsymbol{x}) \in \Lambda^k T_{\boldsymbol{x}}M$ for each $\boldsymbol{x} \in M$ is a *k-form of class $\mathscr{C}^r$ on M* when for every adapted chart (U,φ) on M, the components of $(\varphi^{-1})^*\omega$ are differentiable functions of class $\mathscr{C}^r$ on $\varphi(U \cap M)$.

(3) Given ω a k-form on M, we define $d\omega$ to be the unique $(k+1)$-form on M such that, for each adapted chart (U,φ) on M, $(\varphi^{-1})^*(d\omega) = d((\varphi^{-1})^*\omega)$.

Before stating the classical theorems of integration we need to show that every pyramidal manifold is orientable. We say that a manifold with corners or pyramidal manifold is *orientable* if it admits an atlas of adapted local charts such that the transition functions are orientation-preserving—this term used as in the previous section—. Once we have fixed such an atlas we say that the manifold has been oriented and that those adapted local charts are oriented.

Proposition 9.1. *Every pyramidal manifold M is orientable. The atlas given by the adapted local charts which are orientation-preserving is an orientation of M which will be called its* natural orientation.

Proof. Let (U,φ) be a connected adapted local chart in M. $\mathrm{D}\varphi$ has constant sign in U because U is connected and φ is a $\mathscr{C}^r$-diffeomorphism. If its sign is positive, the adapted local chart belongs to the natural orientation of $\mathbb{R}^3$—the one given by adapted local charts which preserve the orientation—. If the sign is negative, we take the symmetric adapted chart $(U,\varphi\circ \mathrm{r})$, where

$$\mathrm{r}(x_1,x_2,x_3) = (x_1,-x_2,x_3).$$

Note that $\mathrm{r}(\mathbf{P}_i) = \mathbf{P}_i$ for all $i \neq -1$ because roots of unity on the plane are symmetric with respect to the x_1-axis (the case $i=-1$ is an exception and we need to take $\mathrm{r}(x_1,x_2,x_3) = (-x_1,x_2,x_3)$). Thus, the new adapted local chart is well defined and its differential has the opposite sign to $\mathrm{D}\varphi$. We are then able to cover M by adapted charts in this special atlas or orientation-preserving adapted charts. □

From now on we will assume that every pyramidal manifold is endowed with the natural orientation. The following theorem remains true for pyramidal manifolds, the proof being the same that the classical one (which can be found in [11, Proposition 15.24]). In this theorem we mean by smooth components of the boundary those given in Proposition 5.1.

Theorem 9.2. *Let M be an Λ-manifold of dimension at least* 1*. If M is orientable and $\partial^1 M \neq \emptyset$ then the smooth components of $\partial^1 M$ are orientable.*

We generalize here the concept of Stokes' orientation to pyramidal manifolds. Let M be as in the hypotheses of the previous theorem. The Stokes' orientation on $\partial^1 M$ is the one given by the restriction to the boundary of the adapted charts in the oriented atlas of M if its dimension is even, and the opposite if its dimension is odd.

Definition 9.3 (Outward unit normal vector)**.** Let M be an oriented pyramidal manifold. For each $\boldsymbol{x} \in \partial_1 M$, $T_{\boldsymbol{x}}(\partial M)$ is a 1-codimensional vector subspace of $T_{\boldsymbol{x}}M$ which divides $T_{\boldsymbol{x}}M$ in two connected components. We take on $T_{\boldsymbol{x}}M$ the unique orthogonal vector to $T_{\boldsymbol{x}}(\partial M)$ of length one which is on the same connected component of $T_{\boldsymbol{x}}M \smallsetminus T_{\boldsymbol{x}}(\partial M)$ than $\varphi_*^{-1}\big((0,0,-1)_{(0,0,0)}\big)$. This vector is called the *outward unit normal vector* to M at $\boldsymbol{x}$. If $\boldsymbol{x}$ is an element of M of index 2 we can produce two different vectors corresponding to π_0^{-1} and to π_1^{-1} respectively. We define the *outward unit normal vector* to M at $\boldsymbol{x} \in \partial_2 M$ to be the normalized sum of those two vectors.

As in the previous lines, we are understanding $T_{\boldsymbol{x}}(\partial M)$ to be the tangent space at $\boldsymbol{x}$ of the smooth component of ∂M given in Proposition 5.1. The outward unit normal vector is independent of the selected adapted local chart. It is possible to talk about the vector field which assigns to each point its outward unit normal vector. We will say that it is the *outward unit normal vector field* to $\partial_1 M \cup \partial_2 M$. This vector field is $\mathscr{C}^1$-differentiable in the complementary of a subset of the boundary of zero 2D-measure $(\partial^2 M)$ (see [11, Problem 8-4; Lemma 13.16 in first ed.]).

It is also possible to talk about the outward unit normal vector to M at a point $\boldsymbol{x} \in \deg^{-3}(M)$ by the same procedure. But we prefer not to do so and to have it just defined for $\boldsymbol{x} \in \partial_1 M \cup \partial_2 M$.

10. Integration

Differential forms are the basic tool to integrate over pyramidal manifolds independently of the local charts. This section shows the extension to pyramidal manifolds of the theory of integration over manifolds with corners (see [11]). The main result is a proof of the Stokes theorem for Λ-manifolds.

From now on we will not need adapted charts, but pyramidal, with or without boundary, edges and corners. We will use the denomination Λ-chart to cover all those terms. We define at any oriented Λ-manifold what an oriented Λ-chart is: a chart which after an orientation-preserving map is an adapted oriented local chart.

We first introduce the meaning of integral of a form over an open subset of a Λ-pyramid.

Definition 10.1. Let U be an open subset of a Λ-pyramid P. Let ω be an n-form on U and let $f\colon U \longrightarrow \mathbb{R}$ be a continuous function such that $\omega = f\ dx^1 \wedge \cdots \wedge dx^n$. The *integral of* ω *over* U is defined as

$$\int_U \omega := \int_U f\ dx^1 \cdots dx^n,$$

whenever the last integral exists.

That existence is guaranteed for instance when ω has compact support. We are going to denote the support of ω by $\operatorname{supp}\omega$. The integration on a Λ-pyramid gives a way of integrating over a manifold. The basic tool is the same used in the theory of differential forms on manifolds: $\mathscr{C}^\infty$ partitions of unity. The existence of a partition of unity is guaranteed for Λ-manifolds since they are paracompact (see [9, Appendix 3, Theorem 1]).

Definition 10.2. Let M be an oriented n-dimensional Λ-manifold and let ω be a differential n-form on M. Let $\{(U_\alpha, \varphi_\alpha\}_{\alpha\in\Theta}$ be a covering of $\operatorname{supp}\omega$ by oriented Λ-charts in M with a given smooth subordinated partition of unity $\{\Psi_\alpha\}_{\alpha\in\Theta}$. We define the *integral of* ω *over* M as

$$\int_M \omega := \sum_{\alpha\in\Theta} \int_{\varphi_\alpha(U_\alpha)\cap P_{k(\alpha)}} \left(\varphi_\alpha^{-1}\right)^* \Psi_\alpha \omega,$$

where $\varphi_\alpha(U_\alpha) \cap P_{k(\alpha)} = \varphi_\alpha(U_\alpha \cap M)$, whenever the integrals and the sum of the right hand side exist.

If ω has compact support, then the integrals and the sum are well defined. Exactly as in the case of manifolds with corners, we have that this definition is independent of the Λ-charts and of the partition of unity (see [11, page 418]).

Recall that the boundary of an n-dimensional Λ-pyramid is not a Λ-manifold but the union of $(n-1)$-dimensional manifolds with corners. We are going to label those manifolds with corners. In the case of Λ_n^n, $\partial(\Lambda_n^n) = H_1 \cup \cdots \cup H_n$, where

$$H_i = \{(x_1, \ldots, x_n) \in \Lambda_n^n \colon \ x_i = 0\}$$

is for each i an $(n-1)$-dimensional manifold with corners. For a pyramid $\mathbf{P}_k$ in $\mathbb{R}^3$, $\partial \mathbf{P}_k = H_1 \cup \cdots \cup H_m$, where H_i denotes in that case $\pi_{i-1}^k \cap \mathbf{P}_k$ (see Definition 3.1). We will call these H_i the manifolds with corners of the boundary of the Λ-pyramid. Now we can define the integral of a lower dimensional form along the boundary of a Λ-pyramid.

Definition 10.3. Let M be an oriented n-dimensional Λ-manifold. Let ω be an $(n-1)$-differential form on M. Let $\{(U_\alpha, \varphi_\alpha\}_{\alpha\in\Theta}$ be a covering of $\operatorname{supp}\omega$ by oriented Λ-charts in M with a given smooth subordinated partition of unity $\{\Psi_\alpha\}_{\alpha\in\Theta}$. We define the *integral of ω over ∂M* as

$$\int_{\partial M} \omega := \sum_{\alpha\in\Theta} \sum_{i=1}^{m_\alpha} \int_{H_i} \left(\varphi_\alpha^{-1}\right)^* \Psi_\alpha \omega,$$

whenever the integrals and the sums are defined and where m_α denotes the number of manifolds with corners of $\partial P_{k(\alpha)}$, with $\varphi_\alpha(U_\alpha \cap M) = \varphi_\alpha(U_\alpha) \cap P_{k(\alpha)}$.

The main result in this section is a generalization of the Stokes theorem to Λ-manifolds. The most general case that is found in the literature is the one for manifolds with corners. The proof can be found in [11, Theorem 16.25] or in [12, Theorem 14]. We are going to use this result to generalize it, in the 3D case, to pyramidal manifolds.

Theorem 10.1 (Stokes theorem for manifolds with corners)**.** *Let M be an oriented n-dimensional manifold with corners in $\mathbb{R}^p$ and let ω be a differential $(n-1)$-form with compact support in $\mathbb{R}^p$ which is a subset of the closure of M. Then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

We understand the orientation on the boundary to be the induced Stokes' orientation. If $\partial M = \emptyset$, the integral on the right (and hence on the left) hand side vanishes.

Theorem 10.2 (Stokes theorem for bodies)**.** *Let $\mathscr{B}$ be a body and let ω be an $(n-1)$-form with compact support in $\mathscr{B}$. Then*

$$\int_{\mathscr{B}} d\omega = \int_{\partial\mathscr{B}} \omega.$$

Proof. The idea of the proof is to take the integrals over a manifold with corners similar to $\mathscr{B}$ and to apply previous Theorem 10.1. We could take $(\mathscr{B} \smallsetminus \partial^3\mathscr{B}) \cup \deg^{-1}(3)$, which is a manifold with corners in $\mathbb{R}^3$ by Proposition 5.2, but we prefer for simplicity take $\mathcal{D} = \mathscr{B} \smallsetminus \partial^3\mathscr{B}$ which is also a manifold with corners (it is a manifold with edges). We assume without loss of generality that both $\mathscr{B}$ and $\mathcal{D}$ are endowed with the natural orientation.

Since $\operatorname{supp}\omega$ is compact we can find a finite family of pyramidal charts in $\mathscr{B}$, $\{(U_i, \varphi_i)\}_{i=1}^m$, covering $\operatorname{supp}\omega$. We require that each U_i is an open ball in $\mathbb{R}^3$. We take $\{\Psi_i\}_{i=1}^m$ a subordinated partition of unity. From the compactness of $\operatorname{supp}\omega$ we have that the following integrals and sums are convergent:

$$\int_{\mathscr{B}} d\omega = \sum_{i=1}^{m} \int_{\mathscr{B}} \Psi_i \; d\omega.$$

We consider, for each i, the open set $\widetilde{U}_i = U_i \cap \mathcal{D}$ and the restricted maps $\widetilde{\varphi}_i = \varphi_i|_{\widetilde{U}_i}$ and $\widetilde{\Psi}_i = \Psi_i|_{\mathcal{D}}$. As $\partial^3\mathscr{B}$ is a countable set of isolated points by Proposition 4.3, $(\widetilde{U}_i, \widetilde{\varphi}_i)$ is an oriented pyramidal chart in $\mathcal{D}$. We are going to check that $\{\widetilde{\Psi}_i \colon \mathcal{D} \longrightarrow \mathbb{R}\}_{i=1}^m$ is a smooth partition of unity subordinated to $\{\widetilde{U}_i\}_{i=1}^m$. The smoothness is preserved because $\widetilde{U}_i$ is an open subset of U_i. For $\{\widetilde{\Psi}_i\}_{i=1}^m$ to be a subordinated partition of unity, it has to fulfill the following conditions (see [11, page 43]): each map takes values in $[0,1]$, the support of each $\widetilde{\Psi}_i$ is a subset of $\widetilde{U}_i$ which is locally finite, and $\sum_{i=1}^m \widetilde{\Psi}_i(\boldsymbol{x}) = 1$ for each point $\boldsymbol{x} \in \mathcal{D}$. The first and last conditions hold because each $\widetilde{\Psi}_i$ is nothing but the restriction of Ψ_i to its domain. The statements about the support follow from the fact that $\operatorname{supp}\widetilde{\Psi}_i = (\operatorname{supp}\Psi_i) \cap \mathcal{D}$.

$\varphi(U_i \cap \partial^3\mathcal{D})$ is a set of isolated points and so is, consequently, its intersection with $\mathbf{P}_{k(i)}$ for every i. Thus

$$\begin{aligned}
\int_{\mathscr{B}} \Psi_i \; d\omega &= \int_{U_i \cap \mathscr{B}} \Psi_i \; d\omega = \int_{\varphi_i(U_i) \cap \mathbf{P}_{k(i)}} \left(\varphi_i^{-1}\right)^* \; \Psi_i \; d\omega \\
&= \int_{\varphi_i(U_i \cap \partial^3\mathscr{B}) \cap \mathbf{P}_{k(i)}} \left(\varphi_i^{-1}\right)^* \; \Psi_i \; d\omega + \int_{\varphi_i(U_i \cap \mathcal{D}) \cap \mathbf{P}_{k(i)}} \left(\varphi_i^{-1}\right)^* \; \Psi_i \; d\omega \\
&= 0 + \int_{\widetilde{\varphi}_i(\widetilde{U}_i) \cap \mathbf{P}_{k(i)}} \left(\widetilde{\varphi}_i^{-1}\right)^* \; \widetilde{\Psi}_i \; d\omega = \int_{\widetilde{U}_i} \widetilde{\Psi}_i \; d\omega = \int_{\mathcal{D}} \widetilde{\Psi}_i \; d\omega.
\end{aligned}$$

This procedure can be repeated for $\partial\mathscr{B}$ and $\partial\mathcal{D}$ because, for every i, $\varphi(U_i \cap \partial^3\mathcal{D})$ has zero 2D-measure. So we have that

$$\int_{\mathscr{B}} d\omega = \int_{\mathcal{D}} d\omega \qquad \text{and} \qquad \int_{\partial\mathscr{B}} \omega = \int_{\partial\mathcal{D}} \omega.$$

Now we observe that both $\mathcal{D}$ and ω are in the hypotheses of Stokes theorem for manifolds with corners and hence

$$\int_{\mathcal{D}} d\omega = \int_{\partial\mathcal{D}} \omega,$$

which proves that

$$\int_{\mathscr{B}} d\omega = \int_{\partial\mathscr{B}} \omega,$$

as desired. □

Remark 10.1. The classical theorems of integration hold for bodies as they are special cases of the Stokes theorem. These results include the divergence theorem, Green's theorem and the classical version of Stokes theorem. See [12, Section 6.3] for the explicit calculations.

11. Other dimensions

A pyramidal manifold is no more than a subset of $\mathbb{R}^3$ which locally looks like some cone over a topological disc of the plane $x_3 = 1$ with certain corners. It is possible to enlarge the definition of pyramidal manifold to other dimensions following the same spirit. Basically, we will define the $(n+1)$-dimensional pyramids as the cones over all possible n-dimensional pyramidal manifolds homeomorphic to the n-dimensional closed disc D^n. An $(n+1)$-dimensional pyramidal manifold will be a topological space which looks locally like some $(n+1)$-dimensional pyramid. For doing that we need a definition of what a 1D pyramid is, but homeomorphic objects to the 0D discs are just points.

The next definition provides the first inductive step to define the $(n+1)$-dimensional pyramidal manifolds.

Definition 11.1 (0D pyramid and pyramidal manifold)**.** We define $D^0 = \{0\}$ to be the *0D pyramid*. We say that $M \subset \mathbb{R}^p$ is a *$\mathscr{C}^r$-differentiable 0D pyramidal manifold in $\mathbb{R}^p$*, where $r \in \mathbb{N} \cup \{\infty\}$, if for each $\boldsymbol{x}$ in M there exists a $\mathscr{C}^r$-diffeomorphism $\varphi_{\boldsymbol{x}} \colon U_{\boldsymbol{x}} \longrightarrow V_{\boldsymbol{x}}$ of an open neighborhood of $\boldsymbol{x}$ in $\mathbb{R}^p$ onto an open subset of $\mathbb{R}^p$ satisfying that $\varphi_{\boldsymbol{x}}(U_{\boldsymbol{x}} \cap M) = V_{\boldsymbol{x}} \cap \big(D^0 \times \{0\}^{p-0}\big)$. That is, M is a set of isolated points.

Now we need to define what a (not bounded) cone over a figure is. Topologically the cone over X is given by $X \times [0,\infty)$ quotiented by relating all the points with second component equal to 0. As we are working with subspaces of some Euclidean space, we would like to give a more explicit definition of the cone. For $X \subset \mathbb{R}^p$ and $t \in \mathbb{R}$ we define $t \cdot X$ to be the empty set if $t < 0$, the origin of $\mathbb{R}^p$ if $t = 0$ and otherwise the set

$$\Big\{(x_1,\dots,x_p) \in \mathbb{R}^p \colon \quad \Big(\frac{x_1}{t},\dots,\frac{x_p}{t}\Big) \in X\Big\}.$$

Definition 11.2 (Cone over a figure)**.** Set $X \subset \mathbb{R}^p$. We define the *cone over X* to be the subset of $\mathbb{R}^{p+1}$ given by

$$\{(x_1,\dots,x_{p+1}) \in \mathbb{R}^{p+1} \colon \quad (x_1,\dots,x_p) \in x_{p+1} \cdot X\}.$$

Definition 11.3 (Pyramid and pyramidal manifold)**.** We say that a subset M of $\mathbb{R}^p$ is a *$\mathscr{C}^r$-differentiable $(n+1)$-dimensional pyramidal manifold in $\mathbb{R}^p$*, where $r \in \mathbb{N} \cup \{\infty\}$ and $0 \leqslant n \leqslant p-1$ is an integer, when for each $\boldsymbol{x}$ in M there exist

- a $\mathscr{C}^r$-diffeomorphism $\varphi_{\boldsymbol{x}} \colon U_{\boldsymbol{x}} \longrightarrow V_{\boldsymbol{x}}$ of an open neighborhood of $\boldsymbol{x}$ in $\mathbb{R}^p$ onto an open subset of $\mathbb{R}^p$, and
- an $(n+1)$-dimensional pyramid $\mathbf{P}_{\boldsymbol{x}}$

satisfying that $\varphi_{\boldsymbol{x}}(U_{\boldsymbol{x}} \cap M) = V_{\boldsymbol{x}} \cap (\mathbf{P}_{\boldsymbol{x}} \times \{0\}^{p-(n+1)})$.

Here an $(n+1)$-dimensional pyramid is a cone over a $\mathscr{C}^r$-differentiable n-dimensional pyramidal manifold in $\mathbb{R}^n$ which is homeomorphic to D^n.

This definition is recursive and the first inductive step has been stated in Definition 11.1. We are interested in the classification of pyramidal manifolds of the lowest class $r = 1$. We omit the degree of differentiability when we have $r = 1$. The 1D pyramidal manifolds are $\mathscr{C}^1$-curves (open or closed). Hence the only 2D pyramid up to diffeomorphism is Λ_2^2, in this way the 2D pyramidal manifolds are exactly the 2D pyramidal manifolds with corners. Following the induction we conclude that the only 3D pyramids are precisely those defined in Definitions 3.1 and 3.2. Consequently, Definitions 11.3 and 3.3 agree for 3D pyramidal manifolds in $\mathbb{R}^3$.

This article has thus studied pyramids and pyramidal manifolds for dimensions $n \in \{0,1,2,3\}$. The first original classification is the one developed in the previous sections: 3D pyramidal manifolds which are catalogued by degree and index. Classification of higher dimensional pyramidal manifolds is an open question.

We might equally give a definition of bodies for other dimensions. We remark again that we are only studying bodies which are locally deformable to convex spaces.

Definition 11.4 (Body)**.** We call *n-dimensional body in $\mathbb{R}^p$* a $\mathscr{C}^1$-differentiable n-dimensional pyramidal manifold in $\mathbb{R}^p$ which is non-empty, connected and closed in $\mathbb{R}^p$.

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