Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 41(2), 2014, Pages 226–233
ISSN: 1223-6934

Generalized Hermite-Hadamard type integral inequalities for s-convex functions via fractional integrals

Mehmet Zeki Sarikaya and Fatma Ertugral

Abstract. In this paper, we have established Hermite-Hadamard type inequalities for the class of functions whose derivatives in absolute value at certain powers are s-convex functions by using fractional integrals depending on a parameter.

2010 Mathematics Subject Classification. Primary 26D07, 26D10 ; Secondary 26A33.
Key words and phrases. Hermite-Hadamard's inequalities, Riemann-Liouville fractional integral, s-convex functions, integral inequalities.

1. Introduction

Definition 1.1. The function $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$, is said to be *convex* if the following inequality holds

$$f(\lambda x+(1-\lambda)y)\leq\lambda f(x)+(1-\lambda)f(y)$$

for all $x,y\in[a,b]$ and $\lambda\in[0,1]$. We say that f is concave if $(-f)$ is convex.

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are very important in the literature (see, e.g.,[18, p.137], [12]). These inequalities state that if $f:I\to\mathbb{R}$ is a convex function on the interval I of real numbers and $a,b\in I$ with $a<b$, then

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2}. \tag{1}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hadamard's inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found (see, for example, [1, 2, 12, 13, 18]) and the references cited therein.

Definition 1.2. [4] Let s be a real number, $s\in(0,1]$. A function $f:[0,\infty)\to[0,\infty)$ is said to be $s-$*convex* (*in the second sense*), or that f belongs to the class K_s^2, if f

$$f(\lambda x+(1-\lambda)y)\leq\lambda^s f(x)+(1-\lambda)^s f(y)$$

for all $x,y\in[0,\infty)$ and $\lambda\in[0,1]$.

An $s-$convex function was introduced in Breckner's paper [4] and a number of properties and connections with $s-$convexity in the first sense are discussed in paper [11]. Of course, $s-$convexity means just convexity when $s=1$.

In [10], Dragomir and Fitzpatrick proved a variant of Hadamard's inequality which holds for $s-$convex functions in the second sense.

Received April 18, 2014. Revised October 30, 2014.

Theorem 1.1. [10] *Suppose that $f:[0,\infty)\to[0,\infty)$ is an s-convex function in the second sense, where $s\in(0,1)$, and let $a,b\in[0,\infty)$, $a<b$. If $f\in L^1([a,b])$, then the following inequalities hold:*

$$2^{s-1}f(\frac{a+b}{2})\leq\frac{1}{b-a}\int\limits_a^b f(x)dx\leq\frac{f(a)+f(b)}{s+1}. \tag{2}$$

The constant $k=\frac{1}{s+1}$ is the best possible in the second inequality in (2).

Meanwhile, Sarikaya et al.[21] presented the following important integral identity including the first-order derivative of f to establish many interesting Hermite-Hadamard type inequalities for convexity functions via Riemann-Liouville fractional integrals of the order $\alpha>0$.

Lemma 1.2. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $f'\in L\left[a,b\right]$, then the following equality for fractional integrals holds:*

$$\begin{aligned}&\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2\left(b-a\right)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right] \qquad (3)\\ &=\frac{b-a}{2}\int_0^1\left[(1-t)^{\alpha}-t^{\alpha}\right]f'\left(ta+(1-t)b\right)dt.\end{aligned}$$

It is remarkable that Sarikaya et al.[21] first give the following interesting integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 1.3. *Let $f:[a,b]\to\mathbb{R}$ be a positive function with $0\leq a<b$ and $f\in L_1\left[a,b\right]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right)\leq\frac{\Gamma(\alpha+1)}{2\left(b-a\right)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\leq\frac{f\left(a\right)+f\left(b\right)}{2} \tag{4}$$

with $\alpha>0$.

In the following we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper. More details, one can consult [14, 15, 17, 19].

Definition 1.3. Let $f\in L_1[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of order $\alpha>0$ with $a\geq0$ are defined by

$$J_{a+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x\left(x-t\right)^{\alpha-1}f(t)dt,\quad x>a$$

and

$$J_{b-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b\left(t-x\right)^{\alpha-1}f(t)dt,\quad x<b$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J_{a+}^{0}f(x)=J_{b-}^{0}f(x)=f(x)$.

For some recent results connected with fractional integral inequalities see ([3, 5, 6, 7, 8, 9, 16, 20, 22, 23, 24]).

The aim of this paper is to establish generalized Hermite-Hadamard type integral inequalities for the class of functions whose derivatives in absolute value at certain powers are s-convex functions by using Riemann-Liouville fractional integral and some other integral inequalities. The results presented in this paper provide extensions of those given in earlier works.

2. Main Results

For our results, we give the following important fractional integral identity [22]:

Lemma 2.1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $0\le a<b$. If $f'\in L[a,b]$, then the following equality for fractional integrals holds:*

$$-\frac{f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)}{(1-2\lambda)(b-a)}+\frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(b-a)^{\alpha+1}} \times\left[J^{\alpha}_{(\lambda b+(1-\lambda)a)^{+}}f(\lambda a+(1-\lambda)b)+J^{\alpha}_{(\lambda a+(1-\lambda)b)^{-}}f(\lambda b+(1-\lambda)a)\right] = \int_0^1\left[(1-t)^{\alpha}-t^{\alpha}\right]f'\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]dt \tag{5}$$

where $\lambda\in[0,1]\setminus\{\frac{1}{2}\}$ and $\alpha>0$.

Theorem 2.2. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $0\le a<b$. If $|f'|^{q}$, $q\ge1$ is s-convex on $[a,b]$, then the following inequality for fractional integrals holds:*

$$\Bigg|\frac{f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)}{(1-2\lambda)(b-a)}-\frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(b-a)^{\alpha+1}} \times\left[J^{\alpha}_{(\lambda b+(1-\lambda)a)^{+}}f(\lambda a+(1-\lambda)b)+J^{\alpha}_{(\lambda a+(1-\lambda)b)^{-}}f(\lambda b+(1-\lambda)a)\right]\Bigg| \le \left(\frac{2}{\alpha+1}\left[1-\frac{1}{2^{\alpha}}\right]\right)^{1-\frac{1}{q}}\left[|f'(\lambda a+(1-\lambda)b)|^{q}+|f'(\lambda b+(1-\lambda)a)|^{q}\right]^{\frac{1}{q}} \times\left[B_{1\backslash 2}(s+1,\alpha+1)-\frac{1}{2^{\alpha+s}(\alpha+s+1)}+\frac{1}{\alpha+s+1}-B_{1/2}(\alpha+1,s+1)\right]^{\frac{1}{q}} \tag{6}$$

where $\lambda\in[0,1]\setminus\{\frac{1}{2}\}$, $\alpha>0$, and B_x is the incomplete beta function defined as follows

$$B_x(m,n)=\int_0^x t^{m-1}(1-t)^{n-1},\quad m,n>0,\ 0<x\le1. \tag{7}$$

Proof. Firstly, we suppose that $q=1$. Using Lemma 2.1 and s-convexity of $|f'|^{q}$, we find that

$$\Bigg|\frac{f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)}{(1-2\lambda)(b-a)}-\frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(b-a)^{\alpha+1}} \times\left[J^{\alpha}_{(\lambda b+(1-\lambda)a)^{+}}f(\lambda a+(1-\lambda)b)+J^{\alpha}_{(\lambda a+(1-\lambda)b)^{-}}f(\lambda b+(1-\lambda)a)\right]\Bigg|$$

$$\le\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|\left|f'\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]\right|dt$$

$$\le\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|\left[t^{s}\left|f'(\lambda a+(1-\lambda)b)\right|+(1-t)^{s}\left|f'(\lambda b+(1-\lambda)a)\right|\right]dt$$

$$=\left|f'(\lambda a+(1-\lambda)b)\right|\left[\int_0^{\frac{1}{2}}\left[(1-t)^{\alpha}.t^{s}-t^{\alpha+s}\right]dt+\int_{\frac{1}{2}}^{1}\left[t^{\alpha+s}-(1-t)^{\alpha}.t^{s}\right]dt\right]$$

$$+\left|f'(\lambda b+(1-\lambda)a)\right|\left[\int_0^{\frac{1}{2}}\left[(1-t)^{\alpha+s}-t^{\alpha}(1-t)^{s}\right]dt+\int_{\frac{1}{2}}^{1}\left[t^{\alpha}(1-t)^{s}-(1-t)^{\alpha+s}\right]dt\right]$$

$$=\left[B_{1/2}(s+1,\alpha+1)-B_{1/2}(\alpha+1,s+1)-\frac{1}{2^{\alpha+s}(\alpha+s+1)}+\frac{1}{\alpha+s+1}\right]$$
$$\times\left[\left|f'(\lambda a+(1-\lambda)b)\right|+\left|f'(\lambda b+(1-\lambda)a)\right|\right]$$

Secondly, we suppose that $q>1$. Using Lemma 2.1 and power mean inequality, we obtain

$$\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|\left|f'\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]\right|dt$$
$$\leq\left(\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|dt\right)^{1-\frac{1}{q}} \tag{8}$$
$$\times\left(\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|\left|f'\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]\right|^{q}dt\right)^{\frac{1}{q}}.$$

Hence, using s-convexity of $|f'|^q$ and (8) we obtain

$$\left|\frac{f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)}{(1-2\lambda)(b-a)}-\frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(b-a)^{\alpha+1}}\right.$$
$$\left.\times\left[J^{\alpha}_{(\lambda b+(1-\lambda)a)^{+}}f(\lambda a+(1-\lambda)b)+J^{\alpha}_{(\lambda a+(1-\lambda)b)^{-}}f(\lambda b+(1-\lambda)a)\right]\right|$$
$$\leq\left(\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|dt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|\left|f'\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]\right|^{q}dt\right)^{\frac{1}{q}}$$
$$\leq\left(\int_0^{\frac{1}{2}}\left[(1-t)^{\alpha}-t^{\alpha}\right]dt+\int_{\frac{1}{2}}^{1}\left[t^{\alpha}-(1-t)^{\alpha}\right]dt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|\left[t^{s}\left|f'(\lambda a+(1-\lambda)b)\right|^{q}+(1-t)^{s}\left|f'(\lambda b+(1-\lambda)a)\right|^{q}\right]dt\right)^{\frac{1}{q}}$$
$$\leq\left(\frac{2}{\alpha+1}\left[1-\frac{1}{2^{\alpha}}\right]\right)^{1-\frac{1}{q}}\left[\left|f'(\lambda a+(1-\lambda)b)\right|^{q}+\left|f'(\lambda b+(1-\lambda)a)\right|^{q}\right]^{\frac{1}{q}}$$
$$\left[B_{1/2}(s+1,\alpha+1)-B_{1/2}(\alpha+1,s+1)-\frac{1}{2^{\alpha+s}(\alpha+s+1)}+\frac{1}{\alpha+s+1}\right]^{\frac{1}{q}}.$$

This completes the proof. □

Remark 2.1. If we take $s = 1$ in Theorem 2.2, then Theorem 2.2 reduces to Theorem 3 which is proved by Sarikaya and Budak in [22].

Remark 2.2 (Trapezoid Inequality)**.** If we take $s = 1$, $\alpha = 1$ and $\lambda = 0$ (or $\lambda = 1$) in Theorem 2.2, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{8}2^{\frac{q-1}{q}}\left[|f'(a)|^q+|f'(b)|^q\right]^{\frac{1}{q}}$$

where $q \geq 1$. Choosing $q = 1$ in last inequality, it follows that

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{8}\left[|f'(a)|+|f'(b)|\right]$$

which are proved by Dragomir and Agarwal in [13].

Theorem 2.3. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) with $0 \leq a < b$. If $|f'|^q$ s-convex on $[a, b]$ for same fixed $q > 1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}&\left|\frac{f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)}{(1-2\lambda)(b-a)}-\frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(b-a)^{\alpha+1}}\right.\\&\qquad\left.\times\left[J^{\alpha}_{(\lambda b+(1-\lambda)a)^+}f(\lambda a+(1-\lambda)b)+J^{\alpha}_{(\lambda a+(1-\lambda)b)^-}f(\lambda b+(1-\lambda)a)\right]\right|\\&\leq \left(\frac{2}{\alpha p+1}\right)^{\frac{1}{p}}\left(1-\frac{1}{2^{\alpha p}}\right)^{\frac{1}{p}}\left(\frac{|f'(\lambda a+(1-\lambda)b)|^q+|f'(\lambda b+(1-\lambda)a)|^q}{s+1}\right)^{\frac{1}{q}}\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $\alpha > 0$ and $\lambda \in [0,1]\setminus\{\frac{1}{2}\}$.

Proof. Using Lemma 2.1, s-convexity of $|f|^q$ and well-known Hölder's inequality, we obtain

$$\begin{aligned}&\left|\frac{f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)}{(1-2\lambda)(b-a)}-\frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(b-a)^{\alpha+1}}\right.\\&\qquad\left.\times\left[J^{\alpha}_{(\lambda b+(1-\lambda)a)^+}f(\lambda a+(1-\lambda)b)+J^{\alpha}_{(\lambda a+(1-\lambda)b)^-}f(\lambda b+(1-\lambda)a)\right]\right|\\&\leq \int_0^1 |(1-t)^{\alpha}-t^{\alpha}|\,|f'[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)]|\,dt\\&\leq \left(\int_0^1|(1-t)^{\alpha}-t^{\alpha}|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1|f'[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)]|^q\,dt\right)^{\frac{1}{q}}\\&\leq \left(\int_0^{\frac{1}{2}}[(1-t)^{\alpha}-t^{\alpha}]^p\,dt+\int_{\frac{1}{2}}^1[t^{\alpha}-(1-t)^{\alpha}]^p\,dt\right)^{\frac{1}{p}}\\&\qquad\times\left(\int_0^1 t^s\,|f'(\lambda a+(1-\lambda)b)|^q+(1-t)^s\,|f'(\lambda b+(1-\lambda)a)|^q\right)^{\frac{1}{q}}\end{aligned}$$

$$= \left(\frac{2}{\alpha p+1}\right)^{\frac{1}{p}}\left(1-\frac{1}{2^{\alpha p}}\right)^{\frac{1}{p}}\left(\frac{|f'(\lambda a+(1-\lambda)b)|^q+|f'(\lambda b+(1-\lambda)a)|^q}{s+1}\right)^{\frac{1}{q}}.$$

Here, we use

$$(c-d)^p \le c^p - d^p,$$

for any $c > d \ge 0$ and $p \ge 1$. □

Remark 2.3. If we take $s = 1$ in Theorem 2.3, then Theorem 2.3 reduces to Theorem 4 which is proved by Sarikaya and Budak in [22].

Remark 2.4 (Trapezoid Inequality)**.** If we take $s = 1$, $\alpha = 1$ and $\lambda = 0$ (or $\lambda = 1$) in Theorem 2.3, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{2}\left[\frac{2}{p+1}\left(1-\frac{1}{2^p}\right)\right]^{\frac{1}{p}}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}$$

which are proved by Dragomir and Agarwal in [13].

Theorem 2.4. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $0 \le a < b$. If $|f'|^q$ is a s-convex on $[a,b]$ for same fixed $q \ge 1$,then the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\left|\frac{f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)}{(1-2\lambda)(b-a)}-\frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(b-a)^{\alpha+1}}\right.\\
&\quad\times\left.\left[J^{\alpha}_{(\lambda b+(1-\lambda)a)^+}f(\lambda a+(1-\lambda)b)+J^{\alpha}_{(\lambda a+(1-\lambda)b)^-}f(\lambda b+(1-\lambda)a)\right]\right|\\
\le\;&\left[B_{1/2}(\alpha q+1,s+1)-B_{1\backslash 2}(s+1,\alpha q+1)+\frac{1}{\alpha q+s+1}\left(1-\frac{1}{2^{\alpha q+s}}\right)\right]^{\frac{1}{q}}\\
&\quad\times\left(|f'(\lambda a+(1-\lambda)b)|^q+|f'(\lambda b+(1-\lambda)a)|^q\right)^{\frac{1}{q}}
\end{aligned}$$

where $\lambda \in [0,1]\setminus\{\frac{1}{2}\}$ and $\alpha > 0$.

Proof. Using Lemma 2.1, s-convexity of $|f'|^q$, and well-known Hölder's inequality, we have

$$\begin{aligned}
&\left|\frac{f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)}{(1-2\lambda)(b-a)}-\frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(b-a)^{\alpha+1}}\right.\\
&\quad\times\left.\left[J^{\alpha}_{(\lambda b+(1-\lambda)a)^+}f(\lambda a+(1-\lambda)b)+J^{\alpha}_{(\lambda a+(1-\lambda)b)^-}f(\lambda b+(1-\lambda)a)\right]\right|\\
\le\;&\int_0^1 |(1-t)^\alpha-t^\alpha|\,|f'[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)]|\,dt\\
\le\;&\left(\int_0^1 1^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |(1-t)^\alpha-t^\alpha|^q\,|f'[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)]^q|\,dt\right)^{\frac{1}{q}}
\end{aligned}$$

$$= \left(\int_0^{\frac{1}{2}} \left[(1-t)^{\alpha} - t^{\alpha}\right]^q \left| f'\left[t(\lambda a + (1-\lambda)b) + (1-t)(\lambda b + (1-\lambda)a)\right]\right|^q dt \right.$$

$$\left. + \int_0^{\frac{1}{2}} \left[t^{\alpha} - (1-t)^{\alpha}\right]^q \left| f'\left[t(\lambda a + (1-\lambda)b) + (1-t)(\lambda b + (1-\lambda)a)\right]\right|^q dt \right)^{\frac{1}{q}}$$

$$\leq \left(\left| f'(\lambda a + (1-\lambda)b)\right|^q \int_0^{\frac{1}{2}} \left[(1-t)^{q\alpha} t^s - t^{q\alpha+s}\right] dt \right.$$

$$+ \left| f'(\lambda b + (1-\lambda)a)\right|^q \int_0^{\frac{1}{2}} \left[(1-t)^{q\alpha+s} - t^{q\alpha}(1-t)^s\right] dt$$

$$+ \left| f'(\lambda a + (1-\lambda)b)\right|^q \int_{\frac{1}{2}}^{1} \left[t^{q\alpha+s} - (1-t)^{q\alpha} t^s\right] dt$$

$$\left. + \left| f'(\lambda b + (1-\lambda)a)\right|^q \int_{\frac{1}{2}}^{1} \left[t^{q\alpha}(1-t)^s - (1-t)^{q\alpha+s}\right] dt \right)^{\frac{1}{q}}$$

$$= \left[B_{1/2}\left(\alpha q + 1, s+1\right) - B_{1\backslash 2}\left(s+1, \alpha q + 1\right) + \frac{1}{\alpha q + s + 1} - \frac{1}{2^{\alpha q + s}\left(\alpha q + s + 1\right)} \right]^{\frac{1}{q}}$$

$$\times \left[\left| f'(\lambda a + (1-\lambda)b)\right| + \left| f'(\lambda b + (1-\lambda)a)\right|\right]^{\frac{1}{q}}.$$

Here, we use $(A-B)^p \leq A^p - B^p$, for any $A > B \geq 0$ and $q \geq 1$. □

Remark 2.5. If we take $s = 1$ in Theorem 2.4, the Theorem 2.4 reduces the Theorem 5 which is proved by Sarikaya and Budak in [22].

Remark 2.6 (Trapezoid Inequality)**.** If we take $s = 1$, $\alpha = 1$ and $\lambda = 0$ (or $\lambda = 1$) in Theorem 2.4, we have

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{2} \left[\frac{1}{q+1} \left(1 - \frac{1}{2^{q+1}}\right)\right]^{\frac{1}{q}} \left(\frac{\left|f'(a)\right|^q + \left|f'(b)\right|^q}{2} \right)^{\frac{1}{q}}.$$

References

[1] A. G. Azpeitia, Convex functions and the Hadamard inequality, *Rev. Colombiana Math.* **28** (1994), 7–12.

[2] M. K. Bakula and J. Pečarić, Note on some Hadamard-type inequalities, *Journal of Inequalities in Pure and Applied Mathematics* **5** (2004), no. 3, art. 74.

[3] S. Belarbi and Z. Dahmani, On some new fractional integral inequalities, *J. Ineq. Pure and Appl. Math.* **10** (2009), no. 3, art. 86.

[4] W. W. Breckner, Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer funktionen in topologischen linearen Raumen, *Pupl. Inst. Math.* **23** (1978), 13–20.

[5] Z. Dahmani, New inequalities in fractional integrals, *International Journal of Nonlinear Scinece* **9** (2010), no. 4, 493–497.

[6] Z. Dahmani, On Minkowski and Hermite-Hadamard integral inequalities via fractional integration, *Ann. Funct. Anal.* **1** (2010), no. 1, 51–58.
[7] Z. Dahmani, L. Tabharit, S. Taf, Some fractional integral inequalities, *Nonl. Sci. Lett. A* **1** (2010), no. 2, 155–160.
[8] Z. Dahmani, L. Tabharit, S. Taf, New generalizations of Gruss inequality using Riemann-Liouville fractional integrals, *Bull. Math. Anal. Appl.* **2** (2010), no. 3, 93–99.
[9] J. Deng and J. Wang, Fractional Hermite-Hadamard inequalities for (α, m)-logarithmically convex functions. *J. Inequal. Appl.* 2013 (2013), art. 364.
[10] S. S. Dragomir and S. Fitzpatrik, The Hadamard's inequality for s-convex functions in the second sense, *Demonstration Math.* **32** (1999), no. 4, 687–696.
[11] H. Hudzik and L. Maligranda, Some remarks on $s-$convex functions, *Aequationes Math.* **48** (1994), 100–111.
[12] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000.
[13] S. S. Dragomir and R. P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, *Appl. Math. lett.* **11** (1998), no. 5, 91-95.
[14] R. Gorenflo and F. Mainardi, *Fractional calculus: integral and differential equations of fractional order*, Springer Verlag, Wien , 223-276, 1997.
[15] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204, Elsevier Sci. B.V., Amsterdam, 2006.
[16] M. A. Latif, S. S. Dragomir and A. E. Matouk,New inequalities of Ostrowski type for co-ordinated convex functions via fractional integrals, *J. Fract. Calc. Appl.* **2** (2012), 1–15.
[17] S. Miller and B. Ross, *An introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, USA, 1993, p.2.
[18] J. E. Pečarić, F. Proschan and Y. L. Tong, *Convex Functions, Partial Orderings and Statistical Applications*, Academic Press, Boston, 1992.
[19] I. Podlubni, *Fractional Differential Equations*, Academic Press, San Diego, 1999.
[20] M. Z. Sarikaya and H. Ogunmez, On new inequalities via Riemann-Liouville fractional integration, *Abstract and Applied Analysis* **2012** (2012), art. 428983, 10 pages.
[21] M. Z. Sarikaya, E. Set, H. Yaldiz and N. Basak, Hermite -Hadamard's inequalities for fractional integrals and related fractional inequalities, *Mathematical and Computer Modelling* **57** (2013) 2403–2407, DOI:10.1016/j.mcm.2011.12.048.
[22] M. Z. Sarikaya and H. Budak, Generalized Hermite -Hadamard type integral inequalities for fractional integrals, *RGMIA Research Report Collection* **17** (2014), art. 74, 12 pages.
[23] M. Tunc, On new inequalities for h-convex functions via Riemann-Liouville fractional integration, *Filomat* **27** (2013), no. 4, 559–565.
[24] Y. Zhang and J. Wang, On some new Hermite-Hadamard inequalities involving Riemann-Liouville fractional integrals, *J. Inequal. Appl.* 2013 (2013), art. 220.

(M. Z. Sarikaya, F. Ertugral) Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce-Turkey
E-mail address: sarikayamz@gmail.com, dolunay_sfm@windowslive.com, fatmaertugral4@gmail.com