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Sinc-Legendre collocation method for the non-linear Burgers' fractional equation

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Abstract. This paper deals with the numerical solution of the nonlinear fractional Burgers' equation. The fractional derivatives are described based on the Caputo sense. We construct the solution using different approach, that is based on using collocation techniques. The solution is based on using the Sinc method, which builds an approximate solution valid on the entire spatial domain, and in the time domain, we use the shifted Legendre polynomials to replace the time fractional derivatives. The error in the approximation is shown to converge to the exact solution at an exponential rate. Illustrative examples are given with an applications from traffic flow, and the numerical results are shown to demonstrate the efficiency of the newly proposed method.

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1. Introduction

Nonlinear partial differential equations appear in many branches of chemistry, physics, engineering and applied mathematics. The Burgers equation [6, 7], which is a nonlinear partial differential equation of second order, is used in disciplines as a simplified model for turbulence, boundary layer behavior, shock wave formation and mass transport. Due to the recent development of new modeling approaches, reaction-diffusion equations are the subject of new mathematical interest concerning chemical reactions and electro-chemistry of corrosion, it has turned that many phenomena in engineering, chemistry and other sciences can be described very successfully by models using mathematical tools from fractional calculus. For better understanding the phenomena that a given nonlinear fractional partial differential equation describes, the solutions of differential equations of fractional order is much involved. The fractional Burgers equation [11] describes the physical processes of unidirectional propagation of weakly nonlinear acoustic wave through a gas-filled pipe. Fractional derivatives provide more accurate models of real world problems than integer order derivatives do.

In recent years, there has been a growing interest in the field of fractional calculus. Oldham and Spanier [14], Miller and Ross [10], Momani [12, 13] and Podlubny [15] provide the history and a comprehensive treatment of this subject. Fractional calculus is the field of mathematical analysis, which deals with the investigation and applications of integrals and derivatives of arbitrary order, which can be real or complex. The idea appeared in a letter by Leibniz to L'Hospital in 1695. The subject of

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fractional calculus has gained importance during the past three decades due mainly to its demonstrated applications in different areas of physics and engineering. Several fields of applications of fractional differentiation and fractional integration are already well established, some others just started. Many applications of fractional calculus can be found in turbulence and fluid dynamics, stochastic dynamical systems, plasma physics and controlled thermonuclear fusion, nonlinear control theory, image processing, nonlinear biological systems, for more see [18] and the references therein. Indeed, it provides several potentially useful tools for solving differential equations. It is important to solve time fractional partial differential equations. It was found that fractional time derivatives arise generally as infinitesimal generators of the time evolution when taking along time scaling limit. Hence, the importance of investigating fractional equations arises from the necessity to sharpen the concepts of equilibrium, stability states, and time evolution in the long time limit. In general, there exists no method that yields an exact solution for nonlinear fractional partial differential equations. There has been some attempt to solve linear problems with multiple fractional derivatives. In [1], an approximate solution based on the decomposition method is given for the generalized fractional diffusion-wave equation. In [30], the authors used the Sinc-Legendre collocation method to a numerical solution for a class of fractional convection-diffusion equation. The survey paper [22] discusses the application of Sinc methods to fractional differential equations. In the present paper, we consider the fractional Burgers' equation:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + u(x,t)\frac{\partial u(x,t)}{\partial x} = \epsilon\frac{\partial^2 u(x,t)}{\partial x^2}, \quad (x,t) \in (a,b) \times (0,T) \tag{1}$$

with the following initial and boundary conditions

$$u(x,0) = u_0(x), \quad x \in \mathbb{R} \tag{2}$$

$$u(a,t) = \gamma(t), \quad u(b,t) = \delta(t), \quad t \geq 0 \tag{3}$$

where ϵ is the coefficient of the kinematic viscosity, T is the total time, and $u_0(x), \gamma(t)$, and $\delta(t)$ are given functions of the variables. We construct the solution of Equations (1)-(3) using a new approach. This approach is based on the use of collocation techniques. The main idea consists of reducing the solution of the problem to a set of algebraic equations by expanding the required solution as the elements of Legendre polynomials in time direction, and the Sinc function basis in the space direction. To the best of the author's knowledge, such approach has not been used for solving non-linear fractional partial differential equations. But, it has been used for linear fractional PDEs [30]. One important application of equation (1) when $\alpha = 1$, which was used as a simple model of turbulence in an extensive study by Burgers'. Regarding the velocity field of a fluid, the essential ingredient is the competition between the dissipative term, ϵu_{xx} the coefficient of which is the kinematic viscosity, and the nonlinear term uu_x. Equation (1), which appears as a mathematical model for many physical events, such as gas dynamics, turbulence and shock wave theory [28]. Many researchers have used various numerical methods to solve Burgers' equation [26, 25, 32, 31]. In [19], numerical solutions of Burgers equation defined by a new generalized time-fractional derivative are discussed. The approximate solution of time and/or space fractional Burgers equations are obtained by several methods, such as Adomian decomposition [11], variational iteration [9], homotopoy perturbation analysis [20]. Lund [27], uses Sinc-Galerkin method to find a numerical solution of the nonlinear advection-diffusion equation (Burgers' equation). The method results in an iterative scheme of an error of order $O(\exp(-c/h))$ for some positive constants c, h.

In [29], the Burgers' equation is transformed into an equivalent integral equation, and a Sinc-collocation procedure is developed for the integral equation. In [33] a comparison between Cole-Hopf transformation and the decomposition method is made for solving Burgers' equation. Fractional calculus has been used as a model for many physical processes, for this reason a reliable and efficient technique for the solution of nonlinear fractional differential equations is sought. We construct a solution of Equations (1)-(3) using a new approach. This approach is based on the use of collocation techniques. The main idea consists of reducing the solution of the problem to a set of algebraic equations by expanding the required solution as the elements of Legendre polynomials in time direction, and the Sinc function basis in the space direction. Many definitions and studies for the fractional calculus may be found in the literature. These definitions include, Riemman-Liouville, Weyl, Reize, Campos, Caputa, and Nishimoto fractional operator. The Riemann-Liouville definition of fractional derivative operator J_a^α which is defined in [23, 14, 16]. The Riemann-Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differentiation operator D^α proposed by Caputo's (see, [23]). Sinc function that will be used in this paper, are discussed in Stenger [29] and by Lund [27]. The paper is organized as follows: In section 2, we recall notations and definitions of the Sinc function, and derive some formulas that will be needed for developing our method. In section 3, we introduce some necessary definitions of the fractional calculus theory. In section 4, the fractional-order shifted Legendre functions and their properties are obtained. Section 5 is devoted to the solution of the fractional Burgers equation using the proposed method. In section 6, we apply the newly method to specific problems, compare the results, and the accuracy of the proposed schemes is demonstrated. Also, a conclusion is given in the last section.

2. Sinc function properties

The goal of this section is to recall notations and definitions of the Sinc function that will be used in this paper. These are discussed in [29, 27, 4]. Let f be a function defined on $\mathbb{R}$ and $h > 0$ a step size. Then Whittaker cardinal function is defined by the series $C(f, h, x) = \sum_{k=-\infty}^{\infty} f(kh)S(k, h)(x)$, whenever this series converges, and where the k-th Sinc function is defined as

$$S(k, h)(x) = \text{sinc}\,[(x - kh)/h] = \frac{\sin[\pi(x - kh)/h]}{\pi(x - kh)/h}$$

The properties of Sinc functions have been extensively studied in [29, 27]. A comprehensive survey of these approximation properties is found in [2, 3]. Now, for positive integer N, define

$$C_N(f, h, x) = \sum_{k=-N}^{N} f(kh)S(k, h)(x) \tag{4}$$

Definition 2.1. Let $d > 0$, and let D_d denote the region $\{z = x + iy|y| < d\}$ in the complex plane $\mathbb{C}$, and ϕ the conformal map of a simply connected domain D in the complex plane domain onto D_d such that $\phi(a) = -\infty$ and $\phi(b) = \infty$, where a and b are the boundary points of D. Let ψ denote the inverse map of ϕ, and let the arc Γ, with end points a and b $(a, b \notin \Gamma)$, be given by $\Gamma = \psi(-\infty, \infty)$. For $h > 0$, let the points x_k on Γ be given by $x_k = \psi(kh), z \in Z$ and $\rho(z) = \exp(\phi(z))$.

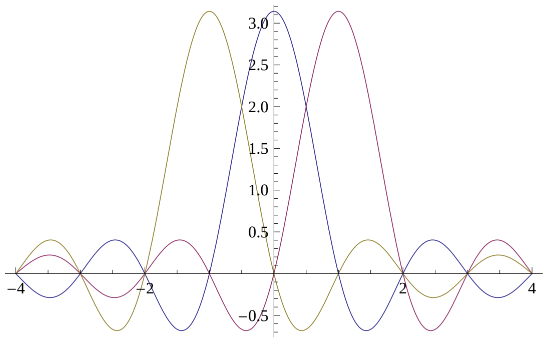

FIGURE 1. The $k-th$ Sinc function $S(k,h)(x), k=-1,0,1$ and $h=1$.

Hence, the numerical process developed in the domain containing the whole real line can be carried over to infinite interval by the inverse map.

Definition 2.2. Let $\mathbf{B}(D)$ be the class of functions f that are analytic in D and satisfy $\int_{\psi(L+u)} |F(z)dz| \to 0,\ u \to \mp\infty$ where $L = \{iy : |y| < d \leq \pi/2\}$, and on the boundary ∂D satisfy $T(F) = \int_{\partial D} |f(z)dz| < \infty$. Corresponding to the number α, let $\mathbf{L}_\alpha(D)$ denote the family of all functions f that are analytic for which there exists a constant C_0 such that

$$|f(z)| \leq C_0 \frac{|\rho(z)|^\alpha}{[1+|\rho(z)|]^{2\alpha}},\ \forall z \in D.$$

By introducing the conformal map ϕ, and a "nullifier" function g the following theorem gives a formula for approximating the $m-$th derivatives of f on Γ. Let g be analytic function on D, and for $k \in \mathbb{Z}$, set

$$S_j(z) = g(z)\text{sinc}\left[\frac{\phi(z)-jh}{h}\right] = g(z)S(j,h)\circ\phi(z),\ z \in D$$

Theorem 2.1. [29, p. 208] *Let $\phi' f/g \in \mathbf{B}(D)$,*

$$\sup_{-\pi/h\leq t\leq\pi/h} \left|\left(\frac{d}{dx}\right)^n g(x)\exp(it\phi(x))\right| \leq C_1 h^{-n},\ x \in \Gamma,$$

for $n = 0,1,2,...,m$, with C_1 a constant depending only on m,ϕ and g. If $f/g \in \mathbf{L}_\alpha(D)$, α a positive constant, then taking $h = \sqrt{\pi d/(\alpha N)}$ it follows that

$$\sup_{x\in\Gamma}\left|f^{(n)}(x) - \sum_{j=-N}^{N} \frac{f(x_j)}{g(x_j)} S_j^{(n)}(x)\right| \leq C_2 N^{\frac{n+1}{2}} \exp(-\sqrt{\pi d\alpha N})$$

for $n = 0,1,...,m$, with C_2 a constant depending only on m,ϕ,g,d,α and f.

The approximation of the $m-$th derivative of f in Theorem 2.1 is simply an $m-$th derivative of of $C_N(f/g,h,x)$ in equation (4). The weight function g is chosen relative to the order of the derivative that is to be approximated. For instance, to approximate the $m-$th derivative, the choice $g(x) = 1/(\phi'(x))^m$ is often suffice. The Sinc method requires that the derivatives of Sinc functions be evaluated at the nodes. Technical calculations provide the following results that will be useful in formulating the discrete system [29, 27], and these quantities are delineated by

$$\delta_{jk}^{(q)} = h^q \frac{d^q}{d\phi^q}[S_j \circ \phi(x)]\Big|_{x=x_k}, q = 0,1,2.$$

In particular, the following convenient notation will be useful in formulating the discrete system

$$\delta_{jk}^{(0)} = [S(j,h)\circ\phi(x)]\Big|_{x=x_k} = \begin{cases} 1, & j=k \\ 0, & j\neq k, \end{cases}$$

$$\delta_{jk}^{(1)} = h\frac{d}{d\phi}[S(j,h)\circ\phi(x)]\Big|_{x=x_k} = \begin{cases} 0, & j=k \\ \frac{(-1)^{k-j}}{(k-j)}, & j\neq k \end{cases}$$

and,

$$\delta_{jk}^{(2)} = h^2\frac{d^2}{d\phi^2}[S(j,h)\circ\phi(x)]\Big|_{x=x_k} = \begin{cases} \frac{-\pi^2}{3}, & j=k \\ \frac{-2(-1)^{k-j}}{(k-j)^2}, & j\neq k \end{cases}$$

So the approximation of a function $f(x)$ by Sinc expansion is given by

$$f(x) \approx f_N(x) = \sum_{j=-N}^{N} \frac{f(x_j)}{g(x_j)} S_j(x). \tag{5}$$

To approximate the $k-$th derivative of $f(x)$, we solve the linear system of equations

$$\int_a^b f_N^{(k)}(x)\frac{S(k,h)\circ\phi(x)}{\phi'(x)}dx = 0,\ k=-N,...,N$$

Integration by parts to change integrals involving derivatives of f_N into integrals involving f_N, so the approximation of the first and second derivatives at the Sinc nodes x_k takes the form

$$f'(x_k) = \sum_{j=-N}^{N}\left\{\frac{\delta_{jk}^{(1)}}{h} + \delta_{jk}^{(0)}g'(x_j)\right\}\frac{f(x_j)}{g(x_j)} + E_1 \tag{6}$$

and,

$$f''(x_k) = \sum_{j=-N}^{N}\left\{\frac{\delta_{jk}^{(2)}}{h^2} + h\delta_{jk}^{(1)}\left[\frac{\phi''(x_k)}{(\phi'(x_k))^2}\right]\right\}\frac{f(x_j)}{g(x_j)} + E_2 \tag{7}$$

where $E_1 = O(N\exp(-\sqrt{\pi d\alpha N}))$ and $E_2 = O(N^{3/2}\exp(-\sqrt{\pi d\alpha N}))$. The approximations (6), (7) are more conveniently recorded by defining the vector $\vec{f} = (f_{-N},....,f_0,....,f_N)^T$. Then define the $m\times m, (m=2N+1)$ Toeplitz matrices $I_m^{(q)} = [\delta_{jk}^{(q)}], q=0,1,2.$ i.e., the matrix whose $jk-$ entry is given by $\delta_{jk}^{(q)}, q=0,1,2.$ Also define the diagonal matrix $D(g) = \text{diag}\,[g(x_{-N}),...,g(x_N)]$. Note that the matrix $I^{(2)}$ is a symmetric matrix, i.e., $I_{jk}^{(2)} = I_{kj}^{(2)}$. The matrix $I^{(1)}$ is skew-symmetric matrix, i.e., $I_{jk}^{(1)} = -I_{kj}^{(1)}$ and they take the form

$$I_m^{(2)} = \begin{pmatrix} \frac{-\pi^2}{3} & 2 & \vdots & \frac{(-1)^{m-1}}{m-1} \\ 2 & . & . & . \\ \vdots & . & . & \vdots \\ \frac{(-1)^{m-1}}{m-1} & \vdots & 2 & \frac{-\pi^2}{3} \end{pmatrix},\ I_m^{(1)} = \begin{pmatrix} 0 & -1 & \dots & \frac{(-1)^{m-1}}{m-1} \\ 1 & 0 & & \vdots \\ \vdots & & & \vdots \\ \frac{(-1)^{m-1}}{m-1} & \dots & 1 & 0 \end{pmatrix} \tag{8}$$

While the matrix $I^{(0)}$ is an identity matrix. For the present paper the interval Γ in Theorem 2.1 is (a,b). Therefore, to approximate the first derivative, we take

$\phi(x) = \ln(\frac{x-a}{b-x})$, and $g(x) = 1/\phi'(x)$. Then the approximation of the first derivative evaluated at the vector nodes x_j can be written as

$$\vec{f}'(x_j) \approx \Big[\frac{-1}{h} I_m^{(1)} D(\phi') + I_m^{(0)} D(\phi''/\phi')\Big]\vec{f}(x_j) \equiv A\vec{f}(x_j) \tag{9}$$

and for the second derivative takes the form

$$\vec{f}''(x_j) \approx \Big[\frac{1}{h^2} I_m^{(2)} + h I_m^{(1)} D(\phi'/\phi'^2)\Big]\vec{f}(x_j) \equiv B\vec{f}(x_j) \tag{10}$$

3. Basic Definition of Fractional Calculus

This section is devoted to a description of the operational properties of the purpose of acquainting with sufficient fractional calculus theory, to enable us to follow the solution of the fractional Burgers equation. Many definitions and studies of fractional calculus have been proposed in the last two centuries. These definitions include, Riemman-Liouville, Weyl, Reize, Campos, Caputa, and Nishimoto fractional operator. Mainly, in this paper, we will re-introduce section 2 of [1]. The Riemann-Liouville definition of fractional derivative operator J_a^α is defined as follows:

Definition 3.1. Let $\alpha \in \mathbb{R}_+$. The operator J^α, defined on the usual Lebesque space $L_1[a, b]$ by

$$J_a^\alpha f(x) = \tfrac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt$$

$$J_a^0 f(x) = f(x)$$

for $a \leq x \leq b$, is called the *Riemann-Liouville fractional integral operator of order* α.

Properties of the operator J^α can be found in [16], we mention the following: For $f \in L_1[a, b], \alpha, \beta \geq 0$ and $\gamma > -1$

(1) $J_a^\alpha f(x)$ exists for almost every $x \in [a, b]$

(2) $J_a^\alpha J_a^\beta f(x) = J_a^{\alpha+\beta} f(x)$

(3) $J_a^\alpha J_a^\beta f(x) = J_a^\beta J_a^\alpha f(x)$

(4) $J_a^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}(x-a)^{\alpha+\gamma}$.

As mentioned in [11], the Riemann-Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce now a modified fractional differentiation operator D^α proposed by Caputo in his work on the theory of viscoelasticity [23].

Definition 3.2. The fractional derivative of $f(x)$ in the Caputo sense is defined as

$$D^\alpha f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)dt, \tag{11}$$

$m - 1 < \alpha \leq m, m \in \mathbb{N}, x > 0$.

Also, we need here two of its basic properties.

Lemma 3.1. *If* $m - 1 < \alpha \leq m$, *and* $f \in L_1[a, b]$, *then* $D_a^\alpha J_a^\alpha f(x) = f(x)$, *and*

$$J_a^\alpha D_a^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^-) \frac{(x-a)^k}{k!},\ x > 0.$$

The Caputo fractional derivative is considered in the Caputo sense. The reason for adopting the Caputo definition is as follows [11]. To solve differential equations, we need to specify additional conditions in order to produce an unique solution. For the case of Caputo fractional differential equations, these additional conditions are just the traditional conditions, which are taken to those of classical differential equations, and are therefore familiar to us. In contrast, for Riemann-Liouville fractional differential equations, these additional conditions constitute certain fractional derivatives of the unknown solution at the initial point $x = 0$, which are functions of x. The unknown function $u = u(x,t)$ is assumed to be a causal function of time, i.e., vanishing for $t < 0$. Also, the initial conditions are not physical; furthermore, it is not clear how much quantities are to be measured from experiment, say, so that they can be appropriately assigned in an analysis. For more details on the geometric and physical interpretation for fractional derivatives of both Riemann-Liouville and Caputo types see [23, 11].

Definition 3.3. For m to be the smallest integer that exceeds α, the Caputo fractional derivatives of order $\alpha > 0$ are defined as

$$D^{\alpha}u(x,t) = \frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} = \begin{cases} \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1}\frac{\partial^m u(x,\tau)}{\partial \tau^m}d\tau, & m-1<\alpha<m \\ \frac{\partial^m u(x,t)}{\partial t^m}, & \alpha = m \in \mathbb{N} \end{cases}$$

For mathematical properties of fractional derivatives and integrals one can consult the mentioned references.

4. Shifted-Legendre Polynomials

Legendre polynomials $\ell_k(t), k = 0,1,2,...$ are the eigenfunctions for the Sturm-Liouville problem

$$\frac{d}{dt}[(1-t^2)\ell_k'(t)] + k(k+1)\ell_k(t) = 0,\ t \in [-1,1]$$

that are orthogonal in $[-1,1]$, and satisfy the orthogonality property

$$\int_{-1}^{1}\ell_i(t)\ell_j(t)dt = \frac{2}{2i+1}\delta_{ij} = \begin{cases} \frac{2}{2i+1}, & i=j \\ 0, & i \neq j, \end{cases}$$

and the difference equation

$$\ell_{i+1}(t) = \frac{2i+1}{i+1}t\ell_k(t) - \frac{i}{i+1}\ell_{i-1}(t),\ i \geq 1,$$

where $\ell_0(t) = 1$, and $\ell_1(t) = t$. In order to use Legendre polynomials in the interval $[0,T]$, we define the shifted Legendre polynomials, $P_i(z) = \ell(\frac{2t}{T} - 1)$, so that the new Legendre polynomials $P_i(z)$ satisfy the orthogonality condition

$$\int_0^T P_i(z)P_j(z)dz = \frac{T}{2i+1}\delta_{ij} = \begin{cases} \frac{T}{2i+1}, & i=j \\ 0, & i \neq j, \end{cases}$$

The analytic closed form of the shifted Legendre polynomials of degree i is given by

$$P_i(t) = \sum_{k=0}^{i}(-1)^{i+k}\frac{(i+k)!}{(i-k)!}\frac{t^k}{(k!)^2T^k}.$$

Note that $P_i(T) = 1$ and $P_i(0) = (-1)^i$. To approximate a function $u(t)$, that is square integrable in $[0,T]$, using the first $(i+1)-$terms shifted Legendre polynomials,

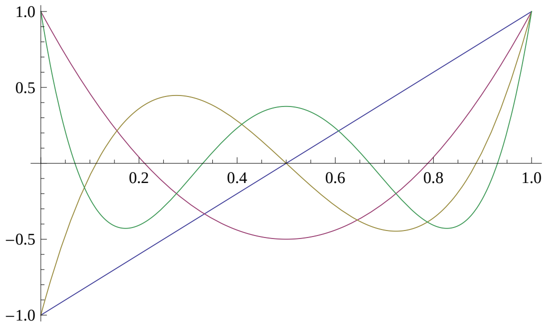

FIGURE 2. Shifted Lengendre Polynomials $P_1(t)$ till $P_4(t)$. when $T = 1$

we may use

$$u_i(t) = \sum_{j=0}^{i} c_j P_j(t) = C^T \Phi(t) \tag{12}$$

where the shifted Legendre coefficient vector C is given by $C^T = [c_0, c_1, ..., c_i]$. While the shifted Legendre vector $\Phi(t)$ is $\Phi(t) = [P_0(t), P_1(t), ..., P_i(t)]^T$. In equation (12), the coefficients c_j can be calculated by

$$c_j = \frac{2j+1}{T} \int_0^T u(t) P_j(t) dt, \; j = 1, 2, ...$$

One of the common and efficient methods for solving fractional partial differential equations of order $\alpha > 0$ is to use the shifted Legendre polynomials. The derivative of order $\alpha > 0$ is given by

$$D^\alpha u_i(t) = D^\alpha \sum_{j=0}^{i} c_j P_j(t)$$

Using the fact that Caputo's fractional differentiation is a linear operator, we get

$$D^\alpha u_i(t) = \sum_{j=0}^{i} c_j D^\alpha P_j(t) = \sum_{j=0}^{\lceil\alpha\rceil - 1} c_j D^\alpha P_j(t) + \sum_{j=\lceil\alpha\rceil}^{i} c_j D^\alpha P_j(t) \tag{13}$$

with $\lceil\alpha\rceil$ denoting the integer part of α. Recalling that, for $\alpha \in \mathbb{N}$, the Caputo differential operator coincides with the usual differential operator of integer order. Therefore, the first sum in equation (13) vanishes, while in the second sum, we use property 4 in Definition 3.1 to arrive to the formula that give us the derivative of order α in Caputo sense for $u_i(t)$ is

$$D^\alpha(u_i(t)) = \sum_{j=\lceil\alpha\rceil}^{i} \sum_{k=\lceil\alpha\rceil}^{j} c_j \; b_{j,k}^{(\alpha)} \; t^{k-\alpha}, \quad \text{where} \quad b_{j,k}^{(\alpha)} = \frac{(-1)^{j+k}(j+k)!}{(j-k)!(k!)\Gamma(k-\alpha+1)T^k}. \tag{14}$$

Example 4.1. Consider the case when $i = 2$, and $\alpha = 1/2$, for the function $u(t) = t^2$ Using the shifted Legendre series (12), we have $t^2 = \frac{1}{3}P_0(t) + \frac{1}{2}P_1(t) + \frac{1}{6}P_2(t)$. Using equation (14), we obtain

$$D^{\frac{1}{2}}(t^2) = \sum_{j=1}^{2} \sum_{k=1}^{j} c_j \; b_{j,k}^{(\frac{1}{2})} \; t^{k-\frac{1}{2}} = c_1 \; b_{1,1}^{(\frac{1}{2})} t^{\frac{1}{2}} + c_2 \; b_{2,1}^{(\frac{1}{2})} t^{\frac{1}{2}} + c_2 \; b_{2,2}^{(\frac{1}{2})} t^{\frac{3}{2}}$$

$$= \frac{1}{2}\frac{2}{\Gamma(\frac{3}{2})}t^{\frac{1}{2}} + \frac{1}{6}\frac{-6}{\Gamma(\frac{3}{2})}t^{\frac{1}{2}} + \frac{1}{6}\frac{12}{\Gamma(\frac{5}{2})}t^{\frac{3}{2}} = \frac{8}{3\sqrt{\pi}}t^{\frac{3}{2}}.$$

It has been proved in [21] that the fractional derivative for the shifted Legendre polynomials can be approximated by $D^{(\alpha)}\Phi(t)$, where $D^{(\alpha)}$ is the $(i+1)\times(i+1)$ matrix of fractional derivative of order $\alpha > 0$ in the Caputo sense, and is defined as

$$D^{(\alpha)} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \theta_{\lceil\alpha\rceil,0,k} & \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \theta_{\lceil\alpha\rceil,1,k} & \cdots & \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \theta_{\lceil\alpha\rceil,i,k} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=\lceil\alpha\rceil}^{m} \theta_{m,0,k} & \sum_{k=\lceil\alpha\rceil}^{m} \theta_{m,1,k} & \cdots & \sum_{k=\lceil\alpha\rceil}^{m} \theta_{m,i,k} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=\lceil\alpha\rceil}^{i} \theta_{i,0,k} & \sum_{k=\lceil\alpha\rceil}^{i} \theta_{i,1,k} & \cdots & \sum_{k=\lceil\alpha\rceil}^{i} \theta_{i,i,k} \end{pmatrix} \tag{15}$$

where $\theta_{m,j,k}$ is given by

$$\theta_{m,j,k} = (2j+1)\sum_{\ell=0}^{j} \frac{(-1)^{m+j+k+\ell}(m+k)!(\ell+j)!}{(m-k)!\Gamma(k-\alpha+1)(\ell!)^2(k+\ell-\alpha+1)}.$$

For example, if $i = 4$, the operational matrix derivative of order $\alpha = 1/2$ in the Caputo sense is given by

$$D^{(\frac{1}{2})} = \sqrt{\pi}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ -3+\frac{6}{\pi} & \frac{6}{\pi} & 0 & 0 & 0 \\ 11-\frac{30}{\pi} & \frac{15}{2}-\frac{30}{\pi} & \frac{5}{2} & 0 & 0 \\ -45+\frac{410}{3\pi} & \frac{-105}{2}+\frac{174}{\pi} & \frac{-35}{2}+\frac{140}{3\pi} & \frac{28}{3\pi} & 0 \end{pmatrix}$$

We may convert the fractional Burgers' equation to an integral equation. The following Theorem has been proved in [5], and will be used to solve the obtained integral equation.

Theorem 4.1. [5] *Let $\Phi(t)$ be a shifted Legendre polynomial, then $I^\nu\Phi(t) \approx A^\nu\Phi(t)$, where A^ν is the $(i+1)\times(i+1)$ operational matrix of integration of order ν Riemann-Liouville sense and is defined as follows*

$$A^\nu = \begin{pmatrix} \sum_{k=0}^{0}\theta_{0,0,k} & \sum_{k=0}^{0}\theta_{0,1,k} & \cdots & \sum_{k=0}^{0}\theta_{0,i,k} \\ \sum_{k=0}^{1}\theta_{1,0,k} & \sum_{k=0}^{1}\theta_{1,1,k} & \cdots & \sum_{k=0}^{1}\theta_{1,i,k} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=0}^{m}\theta_{m,0,k} & \sum_{k=0}^{m}\theta_{m,1,k} & \cdots & \sum_{k=0}^{m}\theta_{m,i,k} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=0}^{i}\theta_{i,0,k} & \sum_{k=0}^{i}\theta_{i,1,k} & \cdots & \sum_{k=0}^{i}\theta_{i,i,k} \end{pmatrix} \tag{16}$$

where $\theta_{m,j,k}$ is given by

$$\theta_{m,j,k} = (2j+1)\sum_{\ell=0}^{j}\frac{(-1)^{m+j+k+\ell}(m+k)!(\ell+j)!}{(m-k)!\;k!\;(k+\alpha+1)(j-\ell)!\;(\ell!)^2(k+\ell+\alpha+1)}.$$

5. Analysis of the Method

The concern of the existence and uniqueness of the solution to fractional Burgers' equations has been discussed in [17, 24] by using Banach fixed point theorem. To solve the problem in equation (1), we use our approximate solution

$$u_{N,n}(x,t) = \sum_{i=-N}^{N}\sum_{j=0}^{n} c_{ij}S_i(x)P_j(t), \tag{17}$$

where $S_i(x)$ is the Sinc basis function, and $P_j(t)$ are the $(n+1)$ shifted Legendre polynomials. The unknown coefficients $\{c_{ij}\}$ are determined by collocation scheme. For sake of simplicity, we assume that $\gamma(t) = \delta(t) = 0$ in equation (3). To solve equation (1), substituting equation (17) into equation (1), we obtain

$$\frac{\partial^\alpha u_{N,n}(x,t)}{\partial t^\alpha} + u_{N,n}(x,t)\frac{\partial u_{N,n}(x,t)}{\partial x} = \epsilon\frac{\partial^2 u_{N,n}(x,t)}{\partial x^2} \tag{18}$$

A collocation scheme can be defined by evaluating equation (18) at the Sinc nodes, $x_k = \phi^{-1}(kh),\ k = 0, \mp 1, \mp 2, ...$ and the shifted Legendre roots $t_\ell,\ \ell = 1, 2, ..., n+1$ of $P_{n+1}(t)$. So, for $0 < \alpha \leq 1$, we have

$$\frac{\partial^\alpha u_{N,n}(x_k,t)}{\partial t^\alpha} = \sum_{i=-N}^{N}\sum_{j=0}^{n} c_{ij}S_i(x_k)D^\alpha P_j(t)$$

Using equation (14), together with the fact $\delta_{ik}^{(0)} = 0$, if $i \neq k$, we arrive at

$$\frac{\partial^\alpha u_{N,n}(x_k,t)}{\partial t^\alpha} = \sum_{i=-N}^{N}\sum_{j=1}^{n}\sum_{s=1}^{j} c_{ij}\ b_{j,s}^{(\alpha)}S_i(x_k)t^{s-\alpha} = \sum_{i=-N}^{N}\sum_{j=1}^{n}\sum_{s=1}^{j} c_{ij}\ b_{j,s}^{(\alpha)}\delta_{ik}^{(0)}t^{s-\alpha} \tag{19}$$

$$= \sum_{j=1}^{n} \sum_{s=1}^{j} c_{kj}\, b_{j,s}^{(\alpha)} t^{s-\alpha}.$$

While,

$$\frac{\partial^2 u_{N,n}(x_k,t)}{\partial x^2} = \sum_{i=-N}^{N} \sum_{j=0}^{n} c_{ij}\, \frac{d^2 S_i(x)}{dx^2}\Big|_{x=x_k} P_j(t)$$

Employing equation (7), we get

$$\frac{\partial^2 u_{N,n}(x_k,t)}{\partial x^2} = \sum_{i=-N}^{N} \sum_{j=0}^{n} c_{ij} \left\{ \frac{\delta_{jk}^{(2)}}{h^2} + h\delta_{jk}^{(1)} \Big[\frac{\phi''(x_k)}{(\phi'(x_k))^2} \Big] \right\} \frac{1}{g(x_j)} P_j(t) \tag{20}$$

For the first derivative u_x, employing equation (6) we obtain

$$\frac{\partial u_{N,n}(x_k,t)}{\partial x} = -\sum_{i=-N}^{N} \sum_{j=0}^{n} c_{ij} \left\{ \frac{\delta_{jk}^{(1)}}{h} + \delta_{jk}^{(0)} g'(x_k) \right\} \frac{1}{g(x_k)}\ P_j(t) \tag{21}$$

Finally, for the non-linear term uu_x, we have

$$u_{N,n}(x_k,t) \frac{\partial u_{N,n}(x_k,t)}{\partial x} = \frac{1}{2} \sum_{i=-N}^{N} \sum_{j=0}^{n} c_{ij} \left\{ \frac{\delta_{jk}^{(1)}}{h} + \delta_{jk}^{(0)} g'(x_k) \right\}^2 \frac{1}{g(x_k)}\ P_j(t) \tag{22}$$

Also, from the initial condition in equation (2), we obtain

$$\sum_{i=-N}^{N} \sum_{j=0}^{n} c_{ij} S_i(x) P_j(0) = u_0(x). \tag{23}$$

To obtain a matrix representation of the above equations, and using the matrices in (9), (10) and (15), with the notation $U = [u(x_i, t_j)], U^0 = [u_0(x_i, 0)]$, we arrive at the following discrete system

$$D^{(\alpha)} U + U \circ AU = \epsilon BU \tag{24}$$

where the symbol "$\circ$" means the Hadamard matrix multiplication. Then the number of unknown coefficients c_{ij} is equal to $(n+1)(2m+1)$ and can be obtained by solving equations (24) and (23) by a fixed point iteration, consequently, the approximate solution $u_{N,n}(x,t)$ can be calculated. An alternative solution is to convert the fractional Burgers' equation to an integral equation by operating with J^α on both sides of equation (1), and using Lemma 3.1, we get

$$u(x,t) = \sum_{k=0}^{m-1} \frac{\partial^k u}{\partial t^k}(x, 0^+) \frac{t^k}{k!} + J^\alpha \Big(\epsilon \frac{\partial^2 u(x,t)}{\partial x^2} - u(x,t) \frac{\partial u(x,t)}{\partial x} \Big) \tag{25}$$

Since $0 < \alpha \le 1$, we choose $m = 1$ in equation (25). Using the operational matrix of integration for the shifted Legendre polynomials A^ν in equation (16), we arrive at the discrete system

$$U = [\epsilon U \circ B - \frac{1}{2} U \circ AU](A^\nu)^{-1} + U^0.$$

Note that the approximation for $u_x(x,t)$ and $u_{xx}(x,t)$ in matrix form has an exponential error E_1 and E_2 respectively. Thus the approximate solution will converge to the exact solution exponentially. The convergence proof of the solution for the discrete system can be done using fixed point Theory.

Remark 5.1. In equation (3), if $\gamma(t), \delta(t)$ are not both zero, then we may reformulate the problem to a homogeneous boundary conditions via the transformation $w(x,t) = u(x,t) + \frac{x-b}{b-a}\gamma(t) + \frac{a-x}{b-a}\delta(t)$.

6. Numerical Results

Here, we obtain some numerical results for the solutions of Burgers' equation. We use the parameters, $d = \pi/2, N = 8, n = 6$ to check the performance for the solution of the fractional Burgers' equation. The computations associated with the examples were performed using Mathematica and Maple.

Example 6.1. Consider the non-linear fractional Burgers' equation

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + u(x,t)\frac{\partial u(x,t)}{\partial x} = \epsilon \frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0 \le x \le 1,\ t > 0 \tag{26}$$

subject to the initial condition, $u(x,0) = c\Big[1 - \tanh(\frac{cx}{2\epsilon})\Big]$, and to the boundary conditions $u(0,t) = 0 = u(1,t)$. It is be noted that the value of $\alpha = 1$ is the only case for which we know the exact solution and has the closed form

$$u(x,t) = \frac{1}{10}\Big[1 - \tanh\Big(\frac{x - 0.1t}{20\epsilon}\Big)\Big] \tag{27}$$

In order to illustrate the approximate solution is efficient and accurate, we will give explicit values of the parameters t and α for fixed $x = 0.2$. The comparison of the

t	$\alpha = 1$	$\alpha = 0.99$	$\alpha = 0.1$	$\alpha = 0.001$
0.1	0.0905307	0.0905444	0.0941872	0.0949997
0.2	0.0910282	0.0910486	0.0944853	0.0950032
0.3	0.0915257	0.0915502	0.0946695	0.0950052
0.4	0.0920232	0.0920500	0.0948048	0.0950066
0.5	0.0925208	0.0925486	0.0949125	0.0950077

TABLE 1. The results obtained by equation (27) when $\alpha = 1$, and by equation (24) for various values of α when $x = 0.2, \epsilon = 0.1$.

t	$\alpha = 1$	$\alpha = 0.99$	$\alpha = 0.1$	$\alpha = 0.001$
0.1	0.0270809	0.0271703	0.0508952	0.0561874
0.2	0.0303211	0.0304540	0.0528370	0.0562098
0.3	0.0335614	0.0337205	0.0540368	0.0562230
0.4	0.0368017	0.0369760	0.0549181	0.0562325
0.5	0.0400419	0.0402232	0.0556194	0.0562395

TABLE 2. The results obtained by equation (27) when $\alpha = 1$, and by equation (24) for various values of α when $x = 0.2, \epsilon = 0.01$.

numerical solutions using the present method, and those obtained by using Equation (27) when $\alpha = 1$ are shown in Tables 1, 2, and also are depicted in Figures 3, 4. From the numerical solutions in Tables 1, 2, it can be seen that the exact solution ($\alpha = 1$) is quite close to the approximate solution when $\alpha = 0.99$. Also, it is observed that the values of the approximate solution at different α's have the same behavior as those obtained using equation (27) for which $\alpha = 1$. This shows the approximate solution is efficient. In the theory of fractional calculus, it is obvious that when the fractional derivative $\alpha(m - 1 < \alpha \leq m)$ tends to positive integer m, then the approximate solution *continuously* tends to the exact solution of the problem with derivative $m = 1$. A closer look at the values in Tables 1 and 2, we observe that our approach do have this characteristic. It can be seen from Figure 4 that the approximate solution when $\alpha = 0.999$ by the present method is nearly identical with the exact solution when $\alpha = 1$. In Figure 5, the comparison shows that as $\alpha \to 1$, the approximate solution tends to the exact solution in the case of $\alpha = 1$.

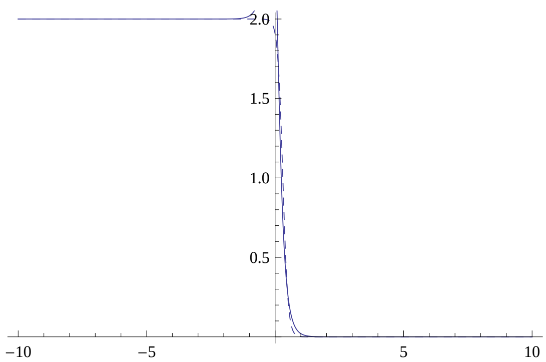

FIGURE 3. Dashed line $\alpha = 0.999$, Solid line $\alpha = 1$, and $\epsilon = 0.2$.

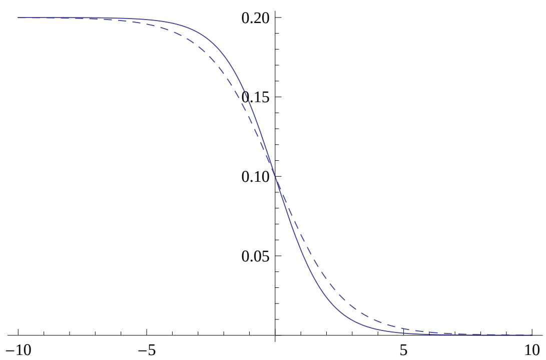

FIGURE 4. Dashed line $\alpha = 0.999$, Solid line $\alpha = 1$, and $\epsilon = 1$.

Figures 6 and 7 show the approximate solution for $\alpha = 0.5$ and $\alpha = 0.1$ respectively. Comparison of Figures 8 and 9 shows that the solution continuously depends on the fractional derivatives. Clear conclusion can be drawn from Figures $4 - 9$ that the solution tends to a finite number as $|x|$ approaches infinity for all values of α, which is in full agreement with the results in [11].

Example 6.2. As a second example, the classical Burgers equation often appears in traffic flow and gas dynamics [8]. The flow through porous media can be better described by fractional models than the classical ones, since they include inherently memory effects caused by obstacles in the structures. We consider once again the

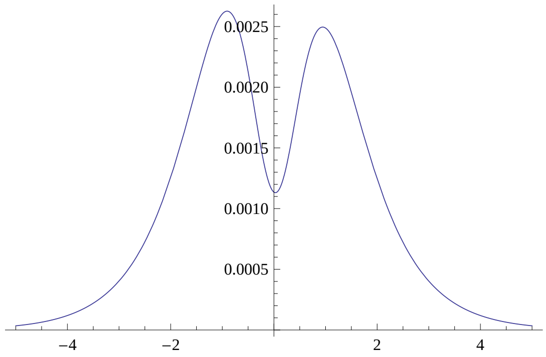

FIGURE 5. The difference between the exact solution ($\alpha = 1$), and the approximate solution for $t = 0.1, \epsilon = 0.1$, and $\alpha = 0.95$.

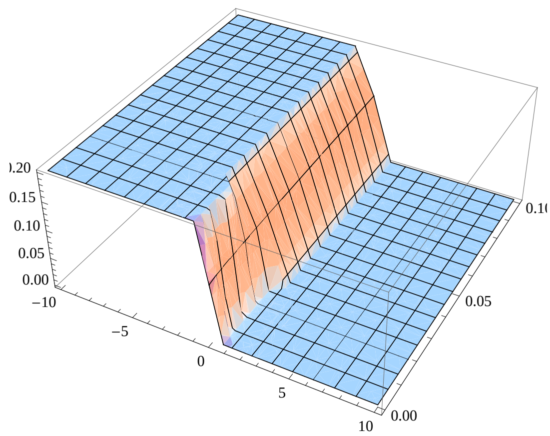

FIGURE 6. Approximate numerical solution $u(x,t)$ with $\alpha = 0.5$,and $\epsilon = 0.05$.

time-fractional Burgers equation (26). However, in this case we consider different initial conditions, where $u(x, 0) = \sin(6\pi x) + 2x(1 - x) - x\sin(6\pi)$, where $u(x,t)$ is the flow's velocity, and ϵ is the viscosity coefficient. It is revealed that the effect of the fractional derivative accumulates slowly to give rise to a significant dissipation. Numerical solutions for $u(x,t)$ are depicted in Figures 10 and 11 for two different values of α and $\epsilon = 0.3$. Comparing Figure 10 to Figure 11, we observe that as the fractional order increases, the rate of diffusion at the beginning time becomes slow.

Discussion and Conclusions

The Legendre Sinc-Collocation method appears to be very promising for solving the fractional Burgers' equation. An important advantage to be gained from the use of this method is the ability to produce very accurate results on a reasonably coarse mesh. For the assumptions considered, the resulting nonlinear system of algebraic equations was solved efficiently by fixed-point iteration. The example presented demonstrate the accuracy of the method, which is an improvement over current methods such as finite elements and finite difference methods. This feature shows the method to be an

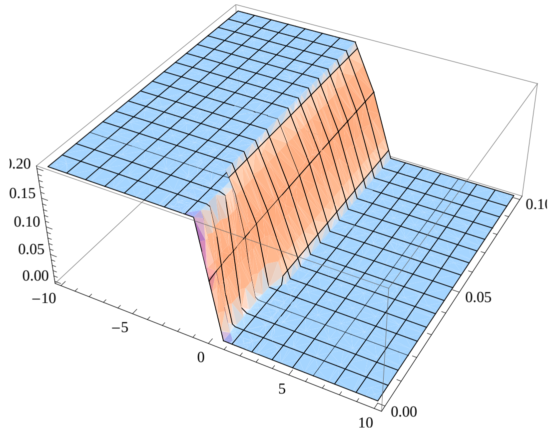

FIGURE 7. Approximate numerical solution $u(x,t)$ with $\alpha = 0.1$, and $\epsilon = 0.05$.

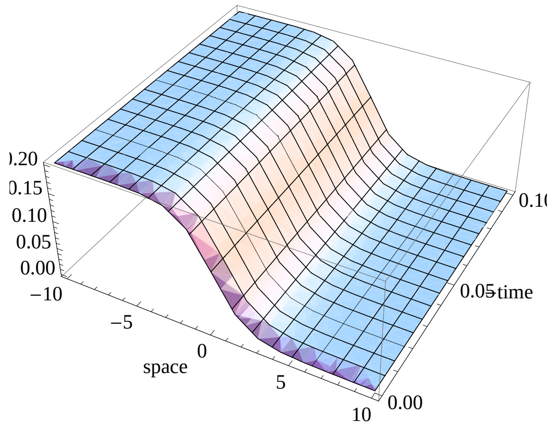

FIGURE 8. Approximate numerical solution $u(x,t)$ with $\alpha = 0.01$, and $\epsilon = 0.2$.

attractive for numerical solutions to the fractional Burgers' equation. We conclude, with confidence, that the collocation using Sinc basis can be considered as a beneficial method for solving a broad class of fractional nonlinear partial differential equations. The study of these equations will be the matter of furthers investigations.

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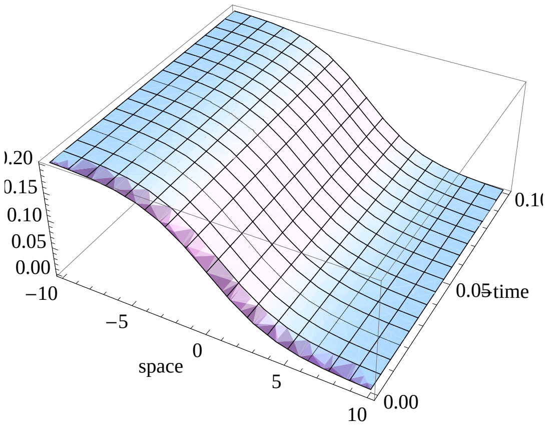

FIGURE 9. Approximate numerical solution $u(x,t)$ with $\alpha = 0.1$, and $\epsilon = 0.2$.

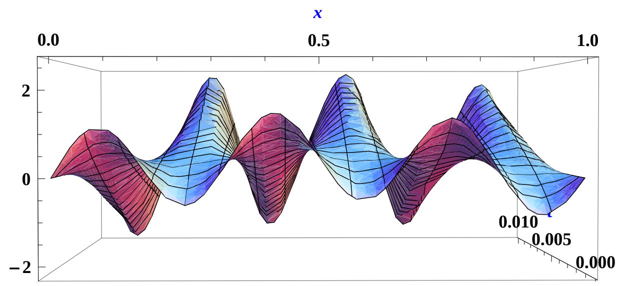

FIGURE 10. Approximate numerical solution $u(x,t)$ of Example 6.2 with $\alpha = 0.5$, and $\epsilon = 0.3$.

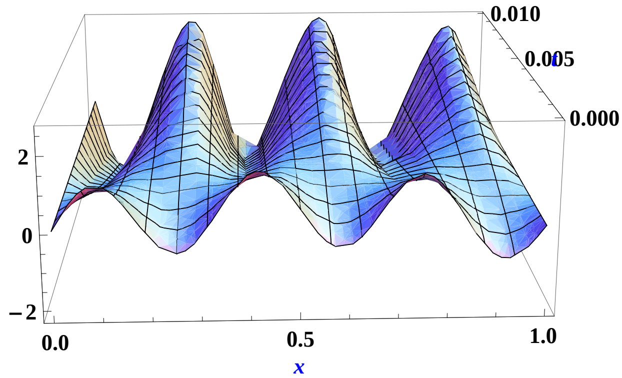

FIGURE 11. Approximate numerical solution $u(x,t)$ of Example 6.2 with $\alpha = 0.9$, and $\epsilon = 0.3$.

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