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Existence and multiplicity of solutions for a class of isotropic elliptic equations with variable exponent

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Abstract. The present paper deals with a nonhomogeneous problem involving weight and variable exponents on a bounded domain with smooth boundary in $\mathbb{R}^N$. Our approach relies on the variable exponent theory of generalized Lebesgue–Sobolev spaces and arguments based on variational methods and a variant of the mountain pass theorem.

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1. Introduction and preliminary results

In this paper we are concerned with the existence and multiplicity of solutions for the boundary eigenvalue problem

$$\begin{cases} -\text{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) + |u|^{p(x)-2}u = \lambda|u|^{q(x)-2}u - h(x)|u|^{r(x)-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary, λ is a positive real number, p, q, r are continuous functions on $\overline{\Omega}$ satisfying $2 \leq p(x) < q(x) < r(x) < p^*(x)$, where $p^*(x) = \frac{Np(x)}{N-p(x)}$ and $p(x) < N$ for all $x \in \overline{\Omega}$, while $h : \overline{\Omega} \to [0, \infty)$ is a continuous function such that the following two conditions hold true

$$\left(\frac{\lambda^{r(x)}}{h(x)^{q(x)}}\right)^{\frac{1}{r(x)-q(x)}} \in L^1(\Omega), \tag{2}$$

$$\left(\frac{\lambda^{r(x)-2}}{h(x)^{q(x)-2}}\right)^{\frac{1}{r(x)-q(x)}} \in L^{\frac{r(\cdot)}{r(\cdot)-2}}(\Omega). \tag{3}$$

We intend to prove that if λ is sufficiently small then we do not have any solution for problem (1), while if λ is large enough then there exist at least two nontrivial weak solutions for problem (1). We point out that similar results were obtained for Laplace equations in [1], [4] and [14]. As well, for p-Laplace equations we refer to [11], [12] and [16].

We highlight the presence of the $p(\cdot)$-Laplace operator $\Delta_{p(x)}u = \text{div}\left(|\nabla u|^{p(x)-2}u\right)$ in problem (1). This is a natural extension of the p-Laplace operator, where $p > 0$ is a constant. However, unlike the p-Laplace operator, $p(\cdot)$-Laplace operator is nonhomogeneous. The study of nonlinear elliptic equations involving quasilinear homogeneous type operators like the p-Laplace operator is based on the theory of standard Sobolev spaces in order to find weak solutions, while in the case of $p(\cdot)$-Laplace operators the natural setting for this approach is the use of the variable exponent Sobolev spaces.

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Due to the interest regarding the variable Sobolev spaces, motivated by their applicability to diverse fields, in the past decades appeared many papers which involve such spaces. The variable exponent Sobolev spaces are used to model various phenomena among which the image restoration, and for the modelling of electrorheological fluids (or smart fluids). The first major discovery on electrorheological fluids was in 1949 due to Willis Winslow. These fluids have the interesting property that their effective viscosity depends on the electric field in the fluid. They can raise the viscosity by as much as five orders of magnitude. This phenomenon is known as the Winslow effect. Electrorheological fluids have been used in robotics and space technology. The experimental research has been done mainly in the USA, for example in NASA laboratories.

We recall in what follows some definitions and basic properties of *Lebesgue–Sobolev spaces with variable exponent.*

We set

$$C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} p(x) > 1\}$$

and denote, for every $p \in C_+(\overline{\Omega})$,

$$p^+ = \sup_{x\in\Omega} p(x) \quad \text{and} \quad p^- = \inf_{x\in\Omega} p(x).$$

For any $p \in C_+(\overline{\Omega})$ we define the *variable exponent Lebesgue space*

$$L^{p(\cdot)}(\Omega) = \left\{u;\ u \text{ is a measurable real-valued function with } \int_\Omega |u|^{p(x)}dx < \infty\right\}.$$

On this space we define the so-called *Luxemburg norm*

$$|u|_{p(\cdot)} = \inf\left\{\mu > 0;\ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\right\}$$

and emphasize that $(L^{p(\cdot)}(\Omega), |\cdot|_{p(\cdot)})$ is a separable and reflexive Banach space. If $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents in $C_+(\overline{\Omega})$ satisfying $p_1(x) \le p_2(x)$ almost everywhere in Ω, then there exists the continuous embedding $L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$, whose norm does not exceed $|\Omega| + 1$.

We denote by $L^{p'(\cdot)}(\Omega)$ the conjugate space of $L^{p(\cdot)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$ the following Hölder-type inequality

$$\left|\int_\Omega uv\,dx\right| \le \left(\frac{1}{p^-} + \frac{1}{p'^-}\right)|u|_{p(\cdot)}|v|_{p'(\cdot)} \le 2|u|_{p(\cdot)}|v|_{p'(\cdot)} \tag{4}$$

holds true.

As well, $p(\cdot)$-*modular* of the $L^{p(\cdot)}(\Omega)$ space, which is the mapping $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(\cdot)}(u) = \int_\Omega |u|^{p(x)}dx$$

plays an essential role in handling the generalized Lebesgue spaces. If $(u_n), u \in L^{p(\cdot)}(\Omega)$, then the following relations hold true:

$$|u|_{p(\cdot)} < 1\ (= 1;\ > 1) \Leftrightarrow \rho_{p(\cdot)}(u) < 1\ (= 1;\ > 1), \tag{5}$$

$$|u|_{p(\cdot)} > 1 \Rightarrow |u|_{p(\cdot)}^{p^-} \le \rho_{p(\cdot)}(u) \le |u|_{p(\cdot)}^{p^+}, \tag{6}$$

$$|u|_{p(\cdot)} < 1 \Rightarrow |u|_{p(\cdot)}^{p^+} \le \rho_{p(\cdot)}(u) \le |u|_{p(\cdot)}^{p^-}, \tag{7}$$

$$|u_n - u|_{p(\cdot)} \to 0 \Leftrightarrow \rho_{p(\cdot)}(u_n - u) \to 0. \tag{8}$$

Look into [9] for more details of these facts and further properties of the variable exponent Lebesgue spaces.

Let $r:\overline{\Omega}\to(1,\infty)$ (such that $r^+<+\infty$), and $h:\overline{\Omega}\to[0,\infty)$ be continuous functions. We define the *weighted Lebesgue space*

$$L_h^{r(\cdot)}(\Omega)=\left\{u;\ u \text{ is a measurable real-valued function with } \int_\Omega h(x)|u|^{r(x)}dx<\infty\right\},$$

endowed with the norm

$$|u|_{h,r(\cdot)}=\inf\left\{\mu>0;\ \int_\Omega h(x)\left|\frac{u(x)}{\mu}\right|^{r(x)}dx\le 1\right\}.$$

If $h(x)\equiv 1$ on $\overline{\Omega}$, we can observe that the resulting norm is just $|\cdot|_{r(\cdot)}$.

Further, we denote by $W_0^{1,p(\cdot)}(\Omega)$ the *variable exponent Sobolev space* defined by

$$W_0^{1,p(\cdot)}(\Omega)=\left\{u;\ u_{|\partial\Omega}=0,\ u\in L^{p(\cdot)}(\Omega) \text{ and } |\nabla u|\in L^{p(\cdot)}(\Omega)\right\},$$

equipped with the equivalent norms

$$\|u\|_{p(\cdot)}=|u|_{p(\cdot)}+|\nabla u|_{p(\cdot)}$$

and

$$\|u\|=\inf\left\{\mu>0;\ \int_\Omega\left(\left|\frac{\nabla u(x)}{\mu}\right|^{p(x)}+\left|\frac{u(x)}{\mu}\right|^{p(x)}\right)dx\le 1\right\},$$

where, in the definition of $\|u\|_{p(\cdot)}$, $|\nabla u|_{p(\cdot)}$ is the Luxemburg norm of $|\nabla u|$. We recall that $W_0^{1,p(\cdot)}(\Omega)$ is a separable and reflexive Banach space. For the density of $C_0^\infty(\Omega)$ in $W_0^{1,p(\cdot)}(\Omega)$ we consider $p\in C_+(\overline{\Omega})$ to be logarithmic Hölder continuous, that is, there exists $M>0$ such that $|p(x)-p(y)|\le -M/\log(|x-y|)$ for any $x,y\in\Omega$ with $|x-y|\le 1/2$. Also, we note that if $s\in C_+(\overline{\Omega})$ and $s(x)<p^*(x)$ for all $x\in\overline{\Omega}$, then the embedding $W_0^{1,p(\cdot)}(\Omega)\hookrightarrow L^{s(\cdot)}(\Omega)$ is compact and continuous, where $p^*(x)=\frac{Np(x)}{N-p(x)}$ and $p(x)<N$.

We refer to [6]–[9] for more properties, details, extensions and further references.

Next, we define the *modular* of the $W_0^{1,p(\cdot)}(\Omega)$ space, which is the mapping $\varrho_{p(\cdot)}:W_0^{1,p(\cdot)}(\Omega)\to\mathbb{R}$ defined by

$$\varrho_{p(\cdot)}(u)=\int_\Omega\left(|\nabla u|^{p(x)}+|u|^{p(x)}\right)dx.$$

If $(u_n), u\in W_0^{1,p(\cdot)}(\Omega)$, then the following relations hold

$$\|u\|>1\Rightarrow\|u\|^{p^-}\le\varrho_{p(\cdot)}(u)\le\|u\|^{p^+},\tag{9}$$

$$\|u\|<1\Rightarrow\|u\|^{p^+}\le\varrho_{p(\cdot)}(u)\le\|u\|^{p^-},\tag{10}$$

$$\|u_n-u\|\to 0\Leftrightarrow\varrho_{p(\cdot)}(u_n-u)\to 0.\tag{11}$$

2. The main results

We seek weak solutions for problem (1) in a subspace of $W_0^{1,p(\cdot)}(\Omega)$, namely in the *weighted variable exponent Sobolev space* defined by

$$E=\left\{u\in W_0^{1,p(\cdot)}(\Omega);\ \int_\Omega h(x)|u|^{r(x)}dx<\infty\right\},$$

equipped with the norm

$$\|u\|_E=|u|_{p(\cdot)}+|\nabla u|_{p(\cdot)}+|u|_{h,r(\cdot)}.$$

We say that $u \in E$ is a *weak solution* of problem (1) if $u(x) = 0$ almost everywhere on $\partial\Omega$ and

$$\int_\Omega \big(|\nabla u|^{p(x)-2}\nabla u\nabla v+|u|^{p(x)-2}uv\big)dx-\lambda\int_\Omega |u|^{q(x)-2}uv\,dx+\int_\Omega h(x)|u|^{r(x)-2}uv\,dx=0 \tag{12}$$

for all $u, v \in E$.

Define the *energy functional* $\Phi : E \to \mathbb{R}$ by

$$\Phi(u)=\int_\Omega \frac{1}{p(x)}\big(|\nabla u|^{p(x)}+|u|^{p(x)}\big)dx-\lambda\int_\Omega\frac{1}{q(x)}|u|^{q(x)}dx+\int_\Omega\frac{h(x)}{r(x)}|u|^{r(x)}dx.$$

By standard arguments, $\Phi \in C^1(E,\mathbb{R})$ and the derivative is given by

$$\begin{aligned}\langle \Phi'(u),v\rangle &= \int_\Omega \big(|\nabla u|^{p(x)-2}\nabla u\nabla v+|u|^{p(x)-2}uv\big)dx-\lambda\int_\Omega |u|^{q(x)-2}uv\,dx\\ &\quad+\int_\Omega h(x)|u|^{r(x)-2}uv\,dx\end{aligned}$$

for any $u, v \in E$. Thus the weak solutions of (1) are exactly the critical points of Φ.

We intend to prove the two theorems from bellow.

Theorem 2.1. There exists $\lambda_0 > 0$ such that for $\lambda > \lambda_0$ problem (1) possesses at least two nontrivial weak solutions.

Theorem 2.2. There exists $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*]$ problem (1) does not have a nontrivial weak solution.

3. Proof of the main results

3.1. Proof of Theorem 2.1.

We split the proof in two steps.

Step 1. We shall prove the existence of a nontrivial solution for problem (1). Firstly, we will demonstrate the following two lemmas.

Lemma 3.1. The energy functional Φ is coercive on E.

Proof. We start by recalling the following inequality:

For any $a, b > 0$ and $0 < k < l$ we have

$$a|t|^k-b|t|^l\le C\cdot a\left(\frac{a}{b}\right)^{\frac{k}{l-k}},\quad \forall t\in\mathbb{R}, \tag{13}$$

where $C > 0$ is a constant depending on k and l (see [10]).

Choosing in (13) $k = q(x)$, $l = r(x)$, $a = \frac{\lambda}{q(x)}$ and $b = \frac{h(x)}{2r(x)}$ we find that

$$\frac{\lambda}{q(x)}|u|^{q(x)}-\frac{h(x)}{2r(x)}|u|^{r(x)}\le C\left(\frac{1}{q(x)}\right)^{\frac{r(x)}{r(x)-q(x)}}\big(2r(x)\big)^{\frac{q(x)}{r(x)-q(x)}}\left(\frac{\lambda^{r(x)}}{h(x)^{q(x)}}\right)^{\frac{1}{r(x)-q(x)}}.$$

Taking into account that $\left(\frac{1}{q(x)}\right)^{\frac{r(x)}{r(x)-q(x)}}\cdot(2r(x))^{\frac{q(x)}{r(x)-q(x)}}$ is bounded expression, we arrive at

$$\frac{\lambda}{q(x)}|u|^{q(x)}-\frac{h(x)}{2r(x)}|u|^{r(x)}\le k_1\left(\frac{\lambda^{r(x)}}{h(x)^{q(x)}}\right)^{\frac{1}{r(x)-q(x)}},$$

where $k_1 > 0$ is a constant. By (2) we deduce that there is a constant $k_2 > 0$ such that

$$\int_\Omega\left(\frac{\lambda}{q(x)}|u|^{q(x)}-\frac{h(x)}{2r(x)}|u|^{r(x)}\right)dx\le k_2.$$

Therefore, we have

$$\begin{aligned}\Phi(u) &= \int_\Omega \frac{1}{p(x)}\big(|\nabla u|^{p(x)} + |u|^{p(x)}\big)dx - \lambda\int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx + \int_\Omega \frac{h(x)}{r(x)}|u|^{r(x)}dx \\ &= \int_\Omega \frac{1}{p(x)}\big(|\nabla u|^{p(x)} + |u|^{p(x)}\big)dx - \int_\Omega \left(\frac{\lambda}{q(x)}|u|^{q(x)} - \frac{h(x)}{2r(x)}|u|^{r(x)}\right)dx \\ &\quad - \frac{1}{2}\int_\Omega \frac{h(x)}{r(x)}|u|^{r(x)}dx + \int_\Omega \frac{h(x)}{r(x)}|u|^{r(x)}dx \\ &\geq \int_\Omega \frac{1}{p(x)}\big(|\nabla u|^{p(x)} + |u|^{p(x)}\big)dx + \frac{1}{2}\int_\Omega \frac{h(x)}{r(x)}|u|^{r(x)}dx - k_2 \\ &\geq \frac{1}{p^+}\int_\Omega \big(|\nabla u|^{p(x)} + |u|^{p(x)}\big)dx + \frac{1}{2r^+}\int_\Omega h(x)|u|^{r(x)}dx - k_2.\end{aligned}$$

Consequently,

$$\Phi(u) \geq k_3\left(\int_\Omega \big(|\nabla u|^{p(x)} + |u|^{p(x)}\big)dx + \int_\Omega h(x)|u|^{r(x)}dx\right) - k_2, \tag{14}$$

where $k_3 = \frac{1}{2r^+}$. Now, let $v \in L^{r(\cdot)}(\Omega)$, $r \in C_+(\overline{\Omega})$, be such that $|v|_{r(\cdot)} > 1$. By (6) we get

$$|v|_{r(\cdot)}^{r^-} \leq \int_\Omega |v|^{r(x)}dx.$$

If we take $v(x) = h(x)^{\frac{1}{r(x)}}u(x)$ we obtain the following

$$|u|_{h,r(\cdot)}^{r^-} \leq \int_\Omega h(x)|u|^{r(x)}dx.$$

Also, let $u \in E$ be such that $\|u\| > 1$. By (9), (14) and the formerly relation we deduce

$$\Phi(u) \geq k_3\big(\|u\|^{p^-} + |u|_{h,r(\cdot)}^{r^-}\big) - k_2 \geq k_3\big(\|u\| + |u|_{h,r(\cdot)}\big) - k_2 \geq k_4\|u\|_E - k_2,$$

where k_4 is a positive constant. We infer that $\Phi(u) \to \infty$ as $\|u\|_E \to \infty$, namely Φ is coercive on E. $\square$

Lemma 3.2. Assume that (u_n) is a sequence in E such that $\Phi(u_n)$ is bounded. Then there exists a subsequence of (u_n), labeled again (u_n), which converges weakly in E to some $u_0 \in E$ and

$$\Phi(u_0) \leq \liminf_{n\to\infty} \Phi(u_n).$$

Proof. Let $v_n \in L^{r(\cdot)}(\Omega)$, $r \in C_+(\overline{\Omega})$, be such that $|v_n|_{r(\cdot)} < 1$. Then, by (7), we have

$$\int_\Omega |v_n|^{r(x)}dx \leq |v_n|_{r(\cdot)}^{r^-}.$$

Taking $v_n(x) = h(x)^{\frac{1}{r(x)}}u_n(x)$ we obtain the relation

$$\int_\Omega h(x)|u_n|^{r(x)}dx \leq |u_n|_{h,r(\cdot)}^{r^-} < 1.$$

Similarly, for $|v_n|_{r(\cdot)} > 1$ together with (6), we get

$$1 < |u_n|_{h,r(\cdot)}^{r^-} \leq \int_\Omega h(x)|u_n|^{r(x)}dx.$$

Since $\int_\Omega h(x)|u_n|^{r(x)}dx$ is bounded we deduce that $|u_n|_{h,r(\cdot)}$ is bounded.

On the other hand, by (14) we deduce that

$$\Phi(u_n) \geq k_3\bigg(\int_\Omega \big(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\big)dx + \int_\Omega h(x)|u_n|^{r(x)}dx\bigg) - k_2,$$

where k_2, k_3 are positive constants. Since $\Phi(u_n)$ is bounded, the previous inequality implies that $\int_\Omega \big(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\big)dx$ and $\int_\Omega h(x)|u_n|^{r(x)}dx$ are bounded.

Taking into account that $\int_\Omega \big(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\big)dx$ is bounded and using the properties (9) and (10) it is clear that $\|u_n\|$ is bounded. Therefore, $\|u_n\|_E$ is bounded, that is there exists a subsequence of (u_n), labeled again (u_n), which converges weakly in E to some $u_0 \in E$. In fact, there exists $u_0 \in E$ such that

$$u_n \text{ converges weakly to } u_0 \text{ in } W_0^{1,p(\cdot)}(\Omega),$$
$$u_n \text{ converges strongly to } u_0 \text{ in } L_h^{r(\cdot)}(\Omega).$$

We define

$$F(x,u) = \frac{\lambda}{q(x)}|u|^{q(x)} - \frac{h(x)}{r(x)}|u|^{r(x)}$$

and

$$f(x,u) = F_u(x,u) = \lambda|u|^{q(x)-2}u - h(x)|u|^{r(x)-2}u.$$

We observe that

$$f_u(x,u) = \lambda(q(x)-1)|u|^{q(x)-2} - h(x)(r(x)-1)|u|^{r(x)-2}.$$

Applying (13) for $a = \lambda(q(x)-1)$, $b = h(x)(r(x)-1)$, $k = q(x)-2$ and $l = r(x)-2$ we arrive at

$$f_u(x,u) \leq C\bigg(\frac{q(x)-1}{r(x)-1}\bigg)^{\frac{q(x)-2}{r(x)-q(x)}} \cdot (q(x)-1) \cdot \bigg(\frac{\lambda^{r(x)-2}}{h(x)^{q(x)-2}}\bigg)^{\frac{1}{r(x)-q(x)}}.$$

It is evident that $\Big(\frac{q(x)-1}{r(x)-1}\Big)^{\frac{q(x)-2}{r(x)-q(x)}} \cdot (q(x)-1)$ is a bounded expression. Then

$$f_u(x,u) \leq c_1\bigg(\frac{\lambda^{r(x)-2}}{h(x)^{q(x)-2}}\bigg)^{\frac{1}{r(x)-q(x)}}, \tag{15}$$

where c_1 is a positive constant. Using the definitions of Φ and F we can write that

$$\begin{aligned}\Phi(u_0) - \Phi(u_n) = &\int_\Omega \frac{1}{p(x)}\big(|\nabla u_0|^{p(x)} + |u_0|^{p(x)}\big)dx - \int_\Omega \frac{1}{p(x)}\big(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\big)dx \\ &+ \int_\Omega [F(x,u_n) - F(x,u_0)]dx.\end{aligned} \tag{16}$$

On the other hand, the following equalities are true

$$\begin{aligned}\int_0^s f_u(x, u_0 + t(u_n - u_0))dt &= \frac{f(x, u_0 + s(u_n - u_0)) - f(x,u_0)}{u_n - u_0} \\ &= \frac{F_u(x, u_0 + s(u_n - u_0)) - F_u(x,u_0)}{u_n - u_0}\end{aligned}$$

and integrating over $[0,1]$ we obtain

$$\begin{aligned}\int_0^1\int_0^s f_u(x,u_0+t(u_n-u_0))dt\,ds &= \frac{\displaystyle\int_0^1 [F_u(x,u_0+s(u_n-u_0))-F_u(x,u_0)]ds}{u_n-u_0}\\ &= \frac{F(x,u_n)-F(x,u_0)}{(u_n-u_0)^2}-\frac{f(x,u_0)}{u_n-u_0}.\end{aligned}$$

We infer that

$$F(x,u_n)-F(x,u_0)=(u_n-u_0)^2\int_0^1\int_0^s f_u(x,u_0+t(u_n-u_0))dt\,ds+(u_n-u_0)f(x,u_0). \tag{17}$$

The relations (15)–(17) yield

$$\begin{aligned}\Phi(u_0)-\Phi(u_n) &= \int_\Omega \frac{1}{p(x)}\big(|\nabla u_0|^{p(x)}+|u_0|^{p(x)}\big)dx-\int_\Omega \frac{1}{p(x)}\big(|\nabla u_n|^{p(x)}+|u_n|^{p(x)}\big)dx\\ &\quad+\int_\Omega (u_n-u_0)^2\int_0^1\int_0^s f_u(x,u_0+t(u_n-u_0))dt\,ds\,dx\\ &\quad+\int_\Omega (u_n-u_0)f(x,u_0)dx\\ &\le \int_\Omega \frac{1}{p(x)}\big(|\nabla u_0|^{p(x)}+|u_0|^{p(x)}\big)dx-\int_\Omega \frac{1}{p(x)}\big(|\nabla u_n|^{p(x)}+|u_n|^{p(x)}\big)dx\\ &\quad+c_2\int_\Omega (u_n-u_0)^2\left(\frac{\lambda^{r(x)-2}}{h(x)^{q(x)-2}}\right)^{\frac{1}{r(x)-q(x)}}dx\\ &\quad+\int_\Omega (u_n-u_0)f(x,u_0)dx,\end{aligned} \tag{18}$$

where c_2 is a positive constant. In what follows we will prove that the last two integrals converge to 0 as $n\to\infty$.

For this purpose, we define $I:E\to\mathbb{R}$ by

$$I(v)=\int_\Omega f(x,u_0)v\,dx.$$

It is easy to see that I is linear. Also, we need I to be continuous.

$$\begin{aligned}|I(v)| &\le \int_\Omega |f(x,u_0)v|dx=\int_\Omega \big|\lambda|u_0|^{q(x)-2}u_0-h(x)|u_0|^{r(x)-2}u_0\big|\cdot|v|dx\\ &\le \lambda\int_\Omega |u_0|^{q(x)-1}|v|dx+\int_\Omega h(x)|u_0|^{r(x)-1}|v|dx.\end{aligned} \tag{19}$$

Applying the Hölder-type inequality (4) we find that

$$\int_\Omega |u_0|^{q(x)-1}|v|dx\le 2\big||u_0|^{q(x)-1}\big|_{\frac{q(\cdot)}{q(\cdot)-1}}|v|_{q(\cdot)}.$$

We know that the embedding $W_0^{1,p(\cdot)}(\Omega)\hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous, that is, there is a constant $C_0>0$ such that

$$|v|_{q(\cdot)}\le C_0\|v\|,\quad \forall v\in W_0^{1,p(\cdot)}(\Omega).$$

On the other hand,

$$\|v\|\le\|v\|_E.$$

Now, combining the last three inequalities we get

$$\int_\Omega |u_0|^{q(x)-1}|v|dx \le C_1\|v\|_E, \tag{20}$$

where C_1 is a positive constant. Next, by Hölder-type inequality (4) we have

$$\begin{aligned}\int_\Omega h(x)|u_0|^{r(x)-1}|v|dx &= \int_\Omega \big(h(x)^{\frac{r(x)-1}{r(x)}}|u_0|^{r(x)-1}\big)\big(h(x)^{\frac{1}{r(x)}}|v|\big)dx \\ &\le 2\Big|h(x)^{\frac{r(x)-1}{r(x)}}|u_0|^{r(x)-1}\Big|_{\frac{r(\cdot)}{r(\cdot)-1}}\Big|h(x)^{\frac{1}{r(x)}}|v|\Big|_{r(\cdot)} \\ &\le C_2|v|_{h,r(\cdot)} \le C_2\|v\|_E, \end{aligned}\tag{21}$$

where $C_2 > 0$ is a constant.

Therefore, (19)–(21) shows that there is a constant C_3 such that

$$|I(v)| \le C_3\|v\|_E, \quad \forall v \in E,$$

in other words I is continuous. Bearing in mind that $u_n \rightharpoonup u_0$ in E and taking into account that I is linear and continuous we obtain that

$$I(u_n) \to I(u_0),$$

that is

$$\lim_{n\to\infty}\int_\Omega f(x,u_0)(u_n-u_0)dx = 0. \tag{22}$$

We have obtained above that $u_n \rightharpoonup u_0$ in $W_0^{1,p(\cdot)}(\Omega)$. On the other hand, since $r \in C_+(\overline{\Omega})$ and $r(x) < p^*(x)$ it follows that the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ is compact. Thus $u_n \to u_0$ in $L^{r(\cdot)}(\Omega)$, meaning that

$$\int_\Omega |u_n-u_0|^{r(x)}dx \to 0$$

or else

$$\int_\Omega \big(|u_n-u_0|^2\big)^{\frac{r(x)}{2}}dx \to 0.$$

Hence we conclude that

$$(u_n-u_0)^2 \in L^{\frac{r(\cdot)}{2}}(\Omega).$$

Keeping in mind the above condition, (3) and Hölder's inequality (4) we have

$$\int_\Omega (u_n-u_0)^2\left(\frac{\lambda^{r(x)-2}}{h(x)^{q(x)-2}}\right)^{\frac{1}{r(x)-q(x)}}dx \le 2\left|\left(\frac{\lambda^{r(x)-2}}{h(x)^{q(x)-2}}\right)^{\frac{1}{r(x)-q(x)}}\right|_{\frac{r(\cdot)}{r(\cdot)-2}}\big|(u_n-u_0)^2\big|_{\frac{r(\cdot)}{2}}.$$

But

$$\rho_{\frac{r(\cdot)}{2}}((u_n-u_0)^2) = \int_\Omega |(u_n-u_0)^2|^{\frac{r(x)}{2}}dx = \int_\Omega |u_n-u_0|^{r(x)}dx \to 0.$$

This fact combined with (8) yield that

$$\big|(u_n-u_0)^2\big|_{\frac{r(\cdot)}{2}} \to 0.$$

Then

$$\lim_{n\to\infty}\int_\Omega (u_n-u_0)^2\left(\frac{\lambda^{r(x)-2}}{h(x)^{q(x)-2}}\right)^{\frac{1}{r(x)-q(x)}}dx = 0. \tag{23}$$

We now propose to show that the functionals $\Phi_1 : W_0^{1,p(\cdot)}(\Omega) \to \mathbb{R}$,

$$\Phi_1(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx$$

and $\Phi_2 : W_0^{1,p(\cdot)}(\Omega) \to \mathbb{R}$,

$$\Phi_2(u) = \int_\Omega \frac{1}{p(x)} |u|^{p(x)} dx$$

are convex. Indeed, taking in consideration that the function

$$[0, \infty) \ni t \to t^\gamma$$

is convex for each $\gamma > 1$ it follows that for any $x \in \Omega$ fixed it the inequalities

$$\left| \frac{\xi + \psi}{2} \right|^{p(x)} \leq \left| \frac{|\xi| + |\psi|}{2} \right|^{p(x)} \leq \frac{1}{2} |\xi|^{p(x)} + \frac{1}{2} |\psi|^{p(x)}, \ \forall\ \xi, \psi \in \mathbb{R}^N, \tag{24}$$

$$\left| \frac{\alpha + \beta}{2} \right|^{p(x)} \leq \left| \frac{|\alpha| + |\beta|}{2} \right|^{p(x)} \leq \frac{1}{2} |\alpha|^{p(x)} + \frac{1}{2} |\beta|^{p(x)}, \ \forall\ \alpha, \beta \in \mathbb{R} \tag{25}$$

hold. Using (24) we obtain

$$\left| \frac{\nabla u + \nabla v}{2} \right|^{p(x)} \leq \frac{1}{2} |\nabla u|^{p(x)} + \frac{1}{2} |\nabla v|^{p(x)}, \ \forall\ u, v \in W_0^{1,p(\cdot)}(\Omega), \ x \in \Omega.$$

Multiplying by $\frac{1}{p(x)}$ and integrating over Ω we deduce that

$$\Phi_1 \left(\frac{u + v}{2} \right) \leq \frac{1}{2} \Phi_1(u) + \frac{1}{2} \Phi_1(v), \ \forall\ u, v \in W_0^{1,p(\cdot)}(\Omega).$$

Also, by (25) we obtain

$$\left| \frac{u + v}{2} \right|^{p(x)} \leq \frac{1}{2} |u|^{p(x)} + \frac{1}{2} |v|^{p(x)}, \ \forall\ u, v \in W_0^{1,p(\cdot)}(\Omega), \ x \in \Omega.$$

Multiplying by $\frac{1}{p(x)}$ and integrating over Ω we deduce that

$$\Phi_2 \left(\frac{u + v}{2} \right) \leq \frac{1}{2} \Phi_2(u) + \frac{1}{2} \Phi_2(v), \ \forall\ u, v \in W_0^{1,p(\cdot)}(\Omega).$$

Since Φ_1, Φ_2 are convex, obviously $\Phi_1 + \Phi_2$ is convex on $W_0^{1,p(\cdot)}(\Omega)$. To prove that the functional $\Phi_1 + \Phi_2$ is weakly lower semicontinuous on $W_0^{1,p(\cdot)}(\Omega)$, it is enough to show that this is lower semicontinuous on $W_0^{1,p(\cdot)}(\Omega)$ (see Corollary III.8 in [3]). Therefore, we fix $u \in W_0^{1,p(\cdot)}(\Omega)$ and $\varepsilon > 0$. Let $v \in W_0^{1,p(\cdot)}(\Omega)$ be arbitrary. Considering that $\Phi_1 + \Phi_2$ is convex and Hölder-type inequality (4) holds true, we have the following

$$
\begin{aligned}
\Phi_1(v)+\Phi_2(v) &\geq \Phi_1(u)+\Phi_2(u)+\langle \Phi_1'(u)+\Phi_2'(u), v-u\rangle \\
&= \Phi_1(u)+\Phi_2(u)+\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla(v-u)dx+\int_\Omega |u|^{p(x)-2}u(v-u)dx \\
&\geq \Phi_1(u)+\Phi_2(u)-\int_\Omega |\nabla u|^{p(x)-1}|\nabla(v-u)|dx-\int_\Omega |u|^{p(x)-1}|v-u|dx \\
&\geq \Phi_1(u)+\Phi_2(u) \\
&\quad - c\Big(\big||\nabla u|^{p(x)-1}\big|_{\frac{p(\cdot)}{p(\cdot)-1}}|\nabla(v-u)|_{p(\cdot)}+\big||u|^{p(x)-1}\big|_{\frac{p(\cdot)}{p(\cdot)-1}}|v-u|_{p(\cdot)}\Big) \\
&\geq \Phi_1(u)+\Phi_2(u) \\
&\quad - c\Big(\big||\nabla u|^{p(x)-1}\big|_{\frac{p(\cdot)}{p(\cdot)-1}}+\big||u|^{p(x)-1}\big|_{\frac{p(\cdot)}{p(\cdot)-1}}\Big)\big(|\nabla(v-u)|_{p(\cdot)}+|v-u|_{p(\cdot)}\big) \\
&\geq \Phi_1(u)+\Phi_2(u)-\tilde{c}\|v-u\| \\
&\geq \Phi_1(u)+\Phi_2(u)-\varepsilon
\end{aligned}
$$

for any $v \in W_0^{1,p(\cdot)}(\Omega)$ with $\|v-u\| < \varepsilon/\tilde{c}$, where c, $\tilde{c}$ are two positive constants. This means that $\Phi_1+\Phi_2$ is lower semicontinuous on $W_0^{1,p(\cdot)}(\Omega)$ and implicitly weakly lower semicontinuous, namely

$$\liminf_{n\to\infty}(\Phi_1+\Phi_2)(u_n) \geq (\Phi_1+\Phi_2)(u_0),$$

or else

$$\liminf_{n\to\infty}\int_\Omega \frac{1}{p(x)}\big(|\nabla u_n|^{p(x)}+|u_n|^{p(x)}\big)dx \geq \int_\Omega \frac{1}{p(x)}\big(|\nabla u_0|^{p(x)}+|u_0|^{p(x)}\big)dx.$$

Passing to the limit in (18) and using (22), (23) and the above inequality it follows that

$$\liminf_{n\to\infty}\Phi(u_n) \geq \Phi(u_0).$$

In other words Φ is weakly lower semicontinuous on E. □

Now, by Theorem 1.2 in [13], Lemma 3.1 and Lemma 3.2 we deduce that there is $u \in E$ a global minimizer of Φ, accordingly

$$\Phi(u) = \inf_{v\in E}\Phi(v).$$

Obviously, u is weak solution of problem (1), and we intend to show that $u \not\equiv 0$ in E. It is enough to show that $\inf_E \Phi < 0$ as long as λ is sufficiently large.

We establish

$$\widetilde{\lambda} = \inf\Big\{q^+\int_\Omega \frac{1}{p(x)}\big(|\nabla u|^{p(x)}+|u|^{p(x)}\big)dx+q^+\int_\Omega \frac{h(x)}{r(x)}|u|^{r(x)}dx;\ u\in E,\ \int_\Omega |u|^{q(x)}dx=1\Big\}.$$

For all $u \in E$ with $\int_\Omega |u|^{q(x)}dx = 1$, Hölder's inequality (4) leads us to

$$\lambda = \int_\Omega \frac{\lambda}{h(x)^{\frac{q(x)}{r(x)}}}h(x)^{\frac{q(x)}{r(x)}}|u|^{q(x)}dx \leq 2\left|\frac{\lambda}{h(x)^{\frac{q(x)}{r(x)}}}\right|_{\frac{r(\cdot)}{r(\cdot)-q(\cdot)}}\left|h(x)^{\frac{q(x)}{r(x)}}|u|^{q(x)}\right|_{\frac{r(\cdot)}{q(\cdot)}}. \tag{26}$$

Focusing our attention on the case when $u \in E$ satisfies

$$\left| h(x)^{\frac{q(x)}{r(x)}} |u|^{q(x)} \right|_{\frac{r(\cdot)}{q(\cdot)}} > 1,$$

by (6), (7) and (26) we arrive at

$$\lambda \leq 2 \left[\int_\Omega \left(\frac{\lambda^{r(x)}}{h(x)^{q(x)}} \right)^{\frac{1}{r(x)-q(x)}} dx \right]^{1/(\frac{r}{r-q})^\pm} \left(\int_\Omega h(x)|u|^{r(x)} dx \right)^{1/(\frac{r}{q})^-}.$$

Hence, we get

$$\int_\Omega h(x)|u|^{r(x)} dx \geq \left(\frac{\lambda}{2} \right)^{(\frac{r}{q})^-} \left[\int_\Omega \left(\frac{\lambda^{r(x)}}{h(x)^{q(x)}} \right)^{\frac{1}{r(x)-q(x)}} dx \right]^{-(\frac{r}{q})^- / (\frac{r}{r-q})^\pm}$$

and thereby

$$\widetilde{\lambda} \geq q^+ \int_\Omega \frac{h(x)}{r(x)} |u|^{r(x)} dx \geq \frac{q^+}{r^+} \cdot \left(\frac{\lambda}{2} \right)^{(\frac{r}{q})^-} \left[\int_\Omega \left(\frac{\lambda^{r(x)}}{h(x)^{q(x)}} \right)^{\frac{1}{r(x)-q(x)}} dx \right]^{-(\frac{r}{q})^- / (\frac{r}{r-q})^\pm}.$$

Thus, we obtain $\widetilde{\lambda} > 0$. Let $\lambda > \widetilde{\lambda}$. Then there is a function $\tilde{u} \in E$ with $\displaystyle\int_\Omega |\tilde{u}|^{q(x)} dx = 1$ such that

$$\lambda \int_\Omega |\tilde{u}|^{q(x)} dx = \lambda > q^+ \int_\Omega \frac{1}{p(x)} \left(|\nabla \tilde{u}|^{p(x)} + |\tilde{u}|^{p(x)} \right) dx + q^+ \int_\Omega \frac{h(x)}{r(x)} |\tilde{u}|^{r(x)} dx,$$

whence we obtain

$$\begin{aligned} \lambda \int_\Omega \frac{1}{q(x)} |\tilde{u}|^{q(x)} dx &\geq \frac{\lambda}{q^+} \int_\Omega |\tilde{u}|^{q(x)} dx \\ &> \int_\Omega \frac{1}{p(x)} \left(|\nabla \tilde{u}|^{p(x)} + |\tilde{u}|^{p(x)} \right) dx + \int_\Omega \frac{h(x)}{r(x)} |\tilde{u}|^{r(x)} dx. \end{aligned}$$

From here it is easy to see that

$$\Phi(\tilde{u}) = \int_\Omega \frac{1}{p(x)} \left(|\nabla \tilde{u}|^{p(x)} + |\tilde{u}|^{p(x)} \right) dx - \lambda \int_\Omega \frac{1}{q(x)} |\tilde{u}|^{q(x)} dx + \int_\Omega \frac{h(x)}{r(x)} |\tilde{u}|^{r(x)} dx < 0,$$

and thereby we obtain that $\inf_E \Phi < 0$. We can conclude that there is $\lambda_0 = \widetilde{\lambda}$ such that (1) has a nontrivial weak solution $\tilde{u} \in E$, for every $\lambda > \lambda_0$, satisfying $\Phi(\tilde{u}) < 0$. Since $\Phi(\tilde{u}) = \Phi(|\tilde{u}|)$ we may assume that $\tilde{u} \geq 0$ almost everywhere in Ω. In other words, u which we have found above is not identically 0.

Step 2. In what follows we handle to find a second nontrivial weak solution for problem (1). We fix $\lambda \geq \lambda_0$ and establish

$$g(x,t) = \begin{cases} 0, & \text{if } t < 0 \\ \lambda t^{q(x)-1} - h(x) t^{r(x)-1}, & \text{if } 0 \leq t \leq \tilde{u}(x) \\ \lambda \tilde{u}(x)^{q(x)-1} - h(x) \tilde{u}(x)^{r(x)-1}, & \text{if } t > \tilde{u}(x) \end{cases}$$

and

$$G(x,t) = \int_0^t g(x,s) ds.$$

We define the functional $\Lambda : E \to \mathbb{R}$ by

$$\Lambda(u) = \int_\Omega \frac{1}{p(x)}\big(|\nabla u|^{p(x)} + |u|^{p(x)}\big)dx - \int_\Omega G(x,u)dx.$$

By standard arguments, $\Lambda \in C^1(E,\mathbb{R})$ and the derivative is given by

$$\langle \Lambda'(u), v\rangle = \int_\Omega \big(|\nabla u|^{p(x)-2}\nabla u \nabla v + |u|^{p(x)-2}uv\big)dx - \int_\Omega g(x,u)v\,dx$$

for all $u, v \in E$. Clearly, if u is a critical point of Λ then $u \geq 0$ almost everywhere in Ω.

In the first instance, on the basis of mountain pass theorem, we intend to find a critical point $\hat{u} \in E$ of Λ such that $\Lambda(\hat{u}) > 0$. For this purpose, we prove the following two lemmas.

Lemma 3.3. There is $\rho \in (0, \|\tilde{u}\|)$ and $\delta > 0$ such that $\Lambda(u) \geq \delta$, for every $u \in E$ satisfying $\|u\| = \rho$.

Proof. Taking into account that the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous we deduce that there is a constant $M > 1$ such that

$$|u|_{q(\cdot)} \leq M \cdot \|u\|, \quad \forall u \in W_0^{1,p(\cdot)}(\Omega). \tag{27}$$

We fix $\rho \in (0,1)$ such that $\rho < \frac{1}{M}$. By virtue of (27) we have

$$|u|_{q(\cdot)} < 1 \text{ for all } u \in W_0^{1,p(\cdot)}(\Omega) \text{ satisfaying } \|u\| = \rho.$$

Relaying on (7) and (27) we get

$$\lambda \int_\Omega |u|^{q(x)}dx \leq M_1\|u\|^{q^-}, \tag{28}$$

where $M_1 = \lambda M^{q^-}$.

For any $u \in E$ we have

$$\begin{aligned}
\Lambda(u) &= \int_\Omega \frac{1}{p(x)}\big(|\nabla u|^{p(x)} + |u|^{p(x)}\big)dx - \int_{[u>\tilde{u}]} G(x,u)dx - \int_{[u\leq\tilde{u}]} G(x,u)dx \\
&= \int_\Omega \frac{1}{p(x)}\big(|\nabla u|^{p(x)} + |u|^{p(x)}\big)dx - \lambda\int_{[u>\tilde{u}]} \tilde{u}^{q(x)-1}u\,dx + \int_{[u>\tilde{u}]} h(x)\tilde{u}^{r(x)-1}u\,dx \\
&\quad - \lambda\int_{[u\leq\tilde{u}]} \frac{1}{q(x)}u^{q(x)}\,dx + \int_{[u\leq\tilde{u}]} \frac{h(x)}{r(x)}u^{r(x)}dx \\
&> \frac{1}{p^+}\int_\Omega \big(|\nabla u|^{p(x)} + |u|^{p(x)}\big)dx - \lambda\int_{[u>\tilde{u}]} u^{q(x)}dx - \frac{\lambda}{q^-}\int_{[u\leq\tilde{u}]} u^{q(x)}dx \\
&> \frac{1}{p^+}\int_\Omega \big(|\nabla u|^{p(x)} + |u|^{p(x)}\big)dx - \lambda\int_\Omega |u|^{q(x)}dx.
\end{aligned} \tag{29}$$

As consequence of (10), (27)–(29) we arrive at

$$\Lambda(u) > \|u\|^{p^+}\left(\frac{1}{p^+} - M_1\|u\|^{q^- - p^+}\right),$$

for any $u \in E$ satisfying $\|u\| = \rho$.

Now, we define the function $\varphi : [0,1] \to \mathbb{R}$ thus

$$\varphi(t) = \frac{1}{p^+} - M_1 t^{q^- - p^+}.$$

We can see that this is positive in a neighborhood of the origin, such that the choice of $\rho \in (0,1)$ is so small that $\delta = \rho^{p^+}\varphi(\rho) > 0$. This concludes our demonstration. $\square$

Lemma 3.4. The functional Λ is coercive on E.

Proof. The following are true for every $u \in E$

$$\begin{aligned}\Lambda(u) &= \int_\Omega \frac{1}{p(x)}\big(|\nabla u|^{p(x)}+|u|^{p(x)}\big)dx - \lambda\int_{[u>\tilde u]}\tilde u^{q(x)-1}u\,dx + \int_{[u>\tilde u]}h(x)\tilde u^{r(x)-1}u\,dx\\ &\quad -\lambda\int_{[u\le\tilde u]}\frac{1}{q(x)}u^{q(x)}\,dx + \int_{[u\le\tilde u]}\frac{h(x)}{r(x)}u^{r(x)}dx\\ &> \frac{1}{p^+}\int_\Omega\big(|\nabla u|^{p(x)}+|u|^{p(x)}\big)dx - \lambda\int_{[u>\tilde u]}\tilde u^{q(x)}dx - \frac{\lambda}{q^-}\int_{[u\le\tilde u]}\tilde u^{q(x)}dx\\ &> \frac{1}{p^+}\int_\Omega\big(|\nabla u|^{p(x)}+|u|^{p(x)}\big)dx - \lambda\int_\Omega\tilde u^{q(x)}dx\\ &= \frac{1}{p^+}\int_\Omega\big(|\nabla u|^{p(x)}+|u|^{p(x)}\big)dx - M_2,\end{aligned}$$

where $M_2 > 0$ is a constant. Let $u \in E$ be such that $\|u\| > 1$. The above and relation (9) imply

$$\Lambda(u) > \frac{1}{p^+}\|u\|^{p^-} - M_2.$$

From that reason Λ is coercive on E, and the proof of lemma is complete. $\square$

By Lemma 3.3 and the mountain pass theorem (see [2] with the variant given by Theorem 1.15 in [15]) we derive that there exists a sequence $(u_n) \subset E$ so that

$$\Lambda(u_n) \to c > 0 \quad\text{and}\quad \Lambda'(u_n) \to 0 \tag{30}$$

where

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\Lambda(\gamma(t))$$

and

$$\Gamma = \{\gamma \in C([0,1],E);\ \gamma(0)=0,\ \gamma(1)=\tilde u\}.$$

By mens of (30) and Lemma 3.4 we can see that (u_n) is bounded and, as a result, there exists $\hat u \in E$ so that , up to a subsequence, (u_n) converges weakly to $\hat u$ in E. Standard arguments, based on the Sobolev embeddings, shows that

$$\lim_{n\to\infty}\langle\Lambda'(u_n),v\rangle = \langle\Lambda'(\hat u),v\rangle,$$

for any $v \in C_0^\infty(\Omega)$. From that fact together with $E \subset W_0^{1,p(\cdot)}(\Omega)$ and $C_0^\infty(\Omega)$ is dense in $W_0^{1,p(\cdot)}(\Omega)$ lead us to $\Lambda'(\hat u) = 0$, that is, $\hat u$ is a critical point of Λ.

Further, we want to prove that $\hat u$ is a critical point for Φ as well. First of all, we need to prove that $\hat u \le \tilde u$. We recall that the positive part of v is denoted by $v^+(x) = \max\{v(x),0\}$. By Theorem 7.6 in [5] we know that if $v \in E$ then $v^+ \in E$. We have the following

$$
\begin{aligned}
0 &= \langle \Lambda'(\hat{u}), (\hat{u}-\tilde{u})^+\rangle - \langle \Phi'(\tilde{u}), (\hat{u}-\tilde{u})^+\rangle \\
&= \int_\Omega \big(|\nabla \hat{u}|^{p(x)-2}\nabla \hat{u} - |\nabla \tilde{u}|^{p(x)-2}\nabla \tilde{u}\big)\nabla(\hat{u}-\tilde{u})^+ dx \\
&\quad + \int_\Omega \big(|\hat{u}|^{p(x)-2}\hat{u} - |\tilde{u}|^{p(x)-2}\tilde{u}\big)(\hat{u}-\tilde{u})^+ dx \\
&\quad - \int_\Omega \big[g(x,\hat{u}) - \lambda \tilde{u}^{q(x)-1} + h(x)\tilde{u}^{r(x)-1}\big](\hat{u}-\tilde{u})^+ dx \\
&= \int_{[\hat{u}>\tilde{u}]} \big(|\nabla \hat{u}|^{p(x)-2}\nabla \hat{u} - |\nabla \tilde{u}|^{p(x)-2}\nabla \tilde{u}\big)(\nabla \hat{u} - \nabla \tilde{u}) dx \\
&\quad + \int_{[\hat{u}>\tilde{u}]} \big(|\hat{u}|^{p(x)-2}\hat{u} - |\tilde{u}|^{p(x)-2}\tilde{u}\big)(\hat{u}-\tilde{u}) dx \\
&\geq \int_{[\hat{u}>\tilde{u}]} \big(|\nabla \hat{u}|^{p(x)-1} - |\nabla \tilde{u}|^{p(x)-1}\big)(|\nabla \hat{u}| - |\nabla \tilde{u}|) dx \\
&\quad + \int_{[\hat{u}>\tilde{u}]} \big(|\hat{u}|^{p(x)-1} - |\tilde{u}|^{p(x)-1}\big)(|\hat{u}| - |\tilde{u}|) dx \geq 0.
\end{aligned}
$$

From here we deduce that $0 \leq \hat{u} \leq \tilde{u}$ in Ω. Therefore, we have

$$g(x,\hat{u}) = \lambda \hat{u}^{q(x)-1} - h(x)\hat{u}^{r(x)-1}$$

and

$$G(x,\hat{u}) = \frac{\lambda}{q(x)}\hat{u}^{q(x)} - \frac{h(x)}{r(x)}\hat{u}^{r(x)},$$

wherefrom

$$\Lambda(\hat{u}) = \Phi(\hat{u})$$

and

$$\Lambda'(\hat{u}) = \Phi'(\hat{u}).$$

Therefore, we can write

$$\Phi(\hat{u}) = \Lambda(\hat{u}) > 0 = \Phi(0) > \Phi(\tilde{u})$$

and

$$0 = \Lambda'(\hat{u}) = \Phi'(\hat{u}),$$

which means that $\hat{u}$ is a weak solution of (1) such that $0 \leq \hat{u} \leq \tilde{u}$, $\hat{u} \neq 0$ and $\hat{u} \neq \tilde{u}$. Consequently, problem (1) has at least two nontrivial weak solutions.

3.2. Proof of Theorem 2.2. In order to prove our assertion, we assume by contradiction that $u \in E$ is a weak solution of problem (1). If we take $v = u$ in (12) we obtain

$$\int_\Omega \big(|\nabla u|^{p(x)} + |u|^{p(x)}\big)dx + \int_\Omega h(x)|u|^{r(x)}dx = \lambda \int_\Omega |u|^{q(x)}dx. \tag{31}$$

Obviously $\int_\Omega h(x)|u|^{r(x)}dx \geq 0$, so

$$\int_\Omega \big(|\nabla u|^{p(x)} + |u|^{p(x)}\big)dx \leq \lambda \int_\Omega |u|^{q(x)}dx. \tag{32}$$

Since $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$ it follows that $W_0^{1,p(\cdot)}(\Omega)$ is continuously embedded in $L^{q(\cdot)}(\Omega)$. Thus, there exists a constant $C > 1$ such that

$$|u|_{q(\cdot)} \leq C\|u\|, \quad \forall u \in W_0^{1,p(\cdot)}(\Omega). \tag{33}$$

Let $u \in W_0^{1,p(\cdot)}(\Omega)$ be such that $\|u\| < \frac{1}{C}$. Then the previous relation implies

$$|u|_{q(\cdot)} < 1 \text{ for all } u \in W_0^{1,p(\cdot)}(\Omega) \text{ with } \|u\| < \frac{1}{C}.$$

Hence, by property (7) we have

$$\int_\Omega |u|^{q(x)} dx \leq |u|_{q(\cdot)}^{q^-}. \tag{34}$$

From (32)–(34) we infer

$$\int_\Omega \left(|\nabla u|^{p(x)} + |u|^{p(x)}\right) dx \leq \lambda C^{q^-} \|u\|^{q^-}.$$

Also, since $\|u\| < 1$, by property (10) we have

$$\|u\|^{p^+} \leq \int_\Omega \left(|\nabla u|^{p(x)} + |u|^{p(x)}\right) dx.$$

Combining the last two inequalities we get

$$\frac{1}{(\lambda C^{q^-})^{\frac{p^+}{q^- - p^+}}} \leq \int_\Omega \left(|\nabla u|^{p(x)} + |u|^{p(x)}\right) dx. \tag{35}$$

Now, applying the Young's inequality

$$ab \leq \frac{a^\alpha}{\alpha} + \frac{b^\beta}{\beta}, \quad \forall a, b > 0, \text{ where } \alpha, \beta > 1 \text{ satisfies } \frac{1}{\alpha} + \frac{1}{\beta} = 1$$

for $a = h(x)^{\frac{q(x)}{r(x)}} |u|^{q(x)}$, $b = \frac{\lambda}{h(x)^{\frac{q(x)}{r(x)}}}$, $\alpha = \frac{r(x)}{q(x)}$ and $\beta = \frac{r(x)}{r(x)-q(x)}$ we get

$$\lambda |u|^{q(x)} \leq \frac{q(x)}{r(x)} h(x) |u|^{r(x)} + \frac{r(x) - q(x)}{r(x)} \cdot \left(\frac{\lambda^{r(x)}}{h(x)^{q(x)}}\right)^{\frac{1}{r(x)-q(x)}}.$$

Evidently $\frac{q(x)}{r(x)} < 1$ and $\frac{r(x)-q(x)}{r(x)} < 1$, therefore, integrating over Ω we arrive at

$$\lambda \int_\Omega |u|^{q(x)} dx < \int_\Omega h(x)|u|^{r(x)} dx + \int_\Omega \left(\frac{\lambda^{r(x)}}{h(x)^{q(x)}}\right)^{\frac{1}{r(x)-q(x)}} dx.$$

By the above inequality and (31) we can see that

$$\int_\Omega \left(|\nabla u|^{p(x)} + |u|^{p(x)}\right) dx < \int_\Omega \left(\frac{\lambda^{r(x)}}{h(x)^{q(x)}}\right)^{\frac{1}{r(x)-q(x)}} dx.$$

Consequently, from the last inequality and (35) we arrive at

$$\lambda > \left[C^{\frac{q^- p^+}{q^- - p^+}} \int_\Omega \left(\frac{\lambda^{r(x)}}{h(x)^{q(x)}}\right)^{\frac{1}{r(x)-q(x)}} dx\right]^{-\frac{q^- - p^+}{p^+}}$$

for any $u \in E$ with $\|u\| < \frac{1}{C}$. Denoting the term in the right-hand side by λ^*, we conclude that Theorem 2.2 holds true.

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