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Existence of entropy solutions for some strongly nonlinear $p(x)$-parabolic problems with L^1-data

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Abstract. This paper is devoted to study the following strongly nonlinear $p(x)$-parabolic problem

$$\begin{cases} \frac{\partial u}{\partial t} - \text{div}\, a(x,t,\nabla u) + g(x,t,u,\nabla u) + \delta|u|^{p(x)-2}u = f - \text{div}\, \phi(u) & \text{in } Q_T, \\ u = 0 & \text{on } \Sigma_T, \\ u(x,0) = u_0 & \text{in } \Omega, \end{cases} \tag{1}$$

with $f \in L^1(Q_T)$, $\phi(\cdot) \in C^0(I\!R, I\!R^N)$, $u_0 \in L^1(\Omega)$ and $\delta > 0$. We prove the existence of entropy solutions for this problem in the parabolic space with variable exponent V.

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1. Introduction

An intensive study of variable exponent Lebesgue and Sobolev spaces has been undertaken during the last years by several authors, inspired primarily by the work of Kovácik and Rákosník [12] from 1991. This impulse comes from their physical applications, e.g., processes of image restoration, flows of electro-rheological fluids, thermistor problem, filtration through inhomogeneous media.

Let Ω be a bounded open subset of $I\!R^N$ ($N \geq 2$). For $T > 0$, we denote by Q_T the cylinder $\Omega \times (0,T)$ and by Σ_T the lateral surface $\partial\Omega \times (0,T)$. Boccardo, Gallouët and Vazquez [7] have studied the nonlinear parabolic problem

$$\begin{cases} u_t + Au + \alpha_0|u|^{s-1}u = f & \text{in } Q_T, \\ u = 0 & \text{on } \Sigma_T, \\ u(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{2}$$

with $f \in L^1(Q_T)$, they have proved the solutions existence and some important regularity results, (see also [1, 6, 14]). Recently, Bendahmane, Wittbold and Zimmermann [5] have treated the parabolic problem

$$\begin{cases} u_t - \text{div}\, (|\nabla u|^{p(x)-2}\nabla u) = f & \text{in } Q_T, \\ u = 0 & \text{on } \Sigma_T, \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{3}$$

they proved the existence and uniqueness of renormalized solutions for this nonlinear parabolic problem. Moreover, they proved some regularity of the solutions, (see [2, 3] for more interesting results).

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In this paper, we establish the existence of entropy solutions for the following strongly nonlinear initial-boundary $p(x)$-parabolic problem

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div} a(x,t,\nabla u) + g(x,t,u,\nabla u) + \delta |u|^{p(x)-2}u = f - \operatorname{div} \phi(u) & \text{in } Q_T, \\ u = 0 & \text{on } \Sigma_T, \\ u(x,0) = u_0 & \text{in } \Omega, \end{cases} \tag{4}$$

where $f \in L^1(Q_T)$, $\phi(\cdot) \in C^0(I\!R, I\!R^N)$ $u_0 \in L^1(\Omega)$ and $\delta > 0$. Under our assumptions, it is more suitable to use the notion of entropy solution, introduced for the first time by Bénilan et al. [6]. Also, it is reasonable to study our problem in the framework of variable exponent space

$$V = \Big\{ u \in L^{p_-}(0,T; W_0^{1,p(x)}(\Omega)) \text{ such that } u \in L^{p(x)}(Q_T) \quad \text{and} \quad |\nabla u| \in L^{p(x)}(Q_T) \Big\}.$$

introduced by Bendahmane et al. in [5].

This paper is organized as follows: In the section 2 we recall some important definitions and results regarding variable exponent Lebesgue and Sobolev spaces. The section 3 is devoted to give some property concerning the time mollification in the space of variable exponent. We introduce in the section 4 some assumptions on $a(x,t,\xi)$ and $g(x,t,s,\xi)$ for which our problem has at least one solution. The section 5 contains some important lemmas useful to prove our main result. The section 6 will be devoted to show the existence of entropy solutions for our problem (4).

2. Preliminaries

Let Ω be a bounded open subset of $I\!R^N$ $(N \geq 2)$, we denote

$$C_+(\overline{\Omega}) = \big\{ \text{measurable function} \quad p(\cdot) : \overline{\Omega} \longmapsto I\!R \quad \text{such that} \quad 1 < p_- \leq p_+ < \infty \big\},$$

where

$$p_- = ess\inf\{p(x) \,/\, x \in \overline{\Omega}\} \qquad \text{and} \qquad p_+ = ess\sup\{p(x) \,/\, x \in \overline{\Omega}\}.$$

We define the variable exponent Lebesgue space for $p(\cdot) \in C_+(\overline{\Omega})$ by

$$L^{p(x)}(\Omega) = \{ u : \Omega \longmapsto I\!R \quad \text{measurable} \;/\; \int_\Omega |u(x)|^{p(x)}\, dx < \infty \},$$

the space $L^{p(x)}(\Omega)$ under the norm

$$\|u\|_{p(x)} = \inf \left\{ \lambda > 0, \quad \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}$$

is a uniformly convex Banach space, and therefore reflexive. We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ (see [9], [16]).

Proposition 2.1. (see [9], [16]) (Generalized Hölder inequality)
(i) *For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have*

$$\left| \int_\Omega u\, v\, dx \right| \leq \Big(\frac{1}{p_-} + \frac{1}{p'_-} \Big) \|u\|_{p(x)}\, \|v\|_{p'(x)} .$$

(ii) *For all $p_1, p_2 \in C_+(\overline{\Omega})$ such that $p_1(x) \leq p_2(x)$ a.e in Ω, then $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ and the embedding is continuous.*

Proposition 2.2. (see [9], [16]) *We denote the modular*

$$\rho(u) = \int_{\Omega} |u|^{p(x)}\, dx \qquad \forall u \in L^{p(x)}(\Omega),$$

then, the following assertions holds

(i): $\|u\|_{p(x)} < 1 \quad (resp, = 1, > 1) \iff \rho(u) < 1 \quad (resp, = 1, > 1)$,

(ii): $\|u\|_{p(x)} > 1 \implies \|u\|^{p_-}_{p(x)} \le \rho(u) \le \|u\|^{p_+}_{p(x)}$ *and* $\|u\|_{p(x)} < 1 \implies \|u\|^{p_+}_{p(x)} \le \rho(u) \le \|u\|^{p_-}_{p(x)}$,

(iii): $\|u_n\|_{p(x)} \to 0 \iff \rho(u_n) \to 0,$ *and* $\|u_n\|_{p(x)} \to \infty \iff \rho(u_n) \to \infty.$

Which implies that the norm convergence and the modular convergence are equivalent.

Now, we define the variable exponent Sobolev space by

$$W^{1,p(x)}(\Omega) = \{\, u \in L^{p(x)}(\Omega) \quad \text{and} \quad |\nabla u| \in L^{p(x)}(\Omega)\},$$

normed by

$$\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)} \qquad \forall u \in W^{1,p(x)}(\Omega).$$

We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$, and we define the Sobolev exponent by $p^*(x) = \frac{Np(x)}{N-p(x)}$ for $p(x) < N$.

Proposition 2.3. (see [9])

(i): *Assuming* $1 < p_- \le p_+ < \infty$, *the spaces* $W^{1,p(x)}(\Omega)$ *and* $W_0^{1,p(x)}(\Omega)$ *are separable and reflexive Banach spaces.*

(ii): *If* $q(\cdot) \in C_+(\bar{\Omega})$ *and* $q(x) < p^*(x)$ *for any* $x \in \Omega$, *then the embedding* $W_0^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}(\Omega)$ *is continuous and compact.*

Remark 2.1. Recall that the definition of these spaces requires only the measurability of $p(x)$. In this work we do not need to use Sobolev and Poincaré inequality. Note that the sharp Sobolev inequality is proved for $p(x)$ log-Hölder continuous, while the Poincaré inequality requires only the continuity of $p(x)$, (see [8, 10]).

Lemma 2.4. *Let* Ω *be a bounded open subset of* $I\!R^N (N \ge 2)$, $T > 0$ *and* $p(\cdot) \in \mathcal{C}_+(\overline{\Omega})$, *then we have the following continuous dense embedding*

$$L^{p_+}(0,T; L^{p(x)}(\Omega)) \hookrightarrow L^{p(x)}(Q_T) \hookrightarrow L^{p_-}(0,T; L^{p(x)}(\Omega)) \tag{5}$$

Definition 2.1. Let $p(\cdot) \in \mathcal{C}_+(\overline{\Omega})$ and $T > 0$, we define the space V by

$$V = \Big\{ u \in L^{p_-}(0,T; W_0^{1,p(x)}(\Omega)) \text{ such that } u \in L^{p(x)}(Q_T) \text{ and } |\nabla u| \in L^{p(x)}(Q_T) \Big\}.$$

We denote the modular $\rho_{1,p(x)}(\cdot)$ for any $u \in V$ by

$$\rho_{1,p(x)}(u) = \int_{Q_T} |u|^{p(x)}\, dx\, dt + \int_{Q_T} |\nabla u|^{p(x)}\, dx\, dt.$$

The space V endowed by the norm

$$\|u\|_V = \|u\|_{L^{p(x)}(Q_T)} + \|\nabla u\|_{L^{p(x)}(Q_T)}$$

is a separable and reflexive Banach space, (for more details see [5]).

Definition 2.2. We denote the dual of the space of V by $V^\star$, and for any $F \in V^\star$ there exists $(F_0, F_1, \ldots, F_N) \in (L^{p'(x)}(Q_T))^{N+1}$, such that

$$F = F_0 - \sum_{i=1}^{N} \frac{\partial F_i}{\partial x_i}.$$

Moreover, for all $u \in V$ we have

$$\langle F, u\rangle = \int_{Q_T} F_0 u \, dx\, dt + \sum_{i=1}^{N} \int_{Q_T} F_i \frac{\partial u}{\partial x_i} \, dx\, dt,$$

and we define a norm on the dual space by

$$\|F\|_{V^\star} = \sum_{i=0}^{N} \|F_i\|_{L^{p'(x)}(Q_T)}.$$

It's clear that $V^\star \subset L^{p'_-}(0,T; W^{-1,p'(x)}(\Omega))$.

Lemma 2.5. *Let B_0, B and B_1 be a Banach spaces with $B_0 \subset B \subset B_1$. Let us set*

$$Y = \{u \ : \ u \in L^{p_0}(0,T;B_0) \quad and \quad u_t \in L^{p_1}(0,T;B_1)\}$$

where $p_0 > 1$ and $p_1 > 1$ are reals numbers.
Assuming that the embedding $B_0 \hookrightarrow\hookrightarrow B$ is compact, then

$$Y \hookrightarrow\hookrightarrow L^{p_0}(0,T;B)$$

and this embedding is compact.

Remark 2.2. Let $p_- > \dfrac{2N}{N+2}$, then $W_0^{1,p(x)}(\Omega) \subset L^2(\Omega) \subset W^{-1,p'(x)}(\Omega)$. We set

$$B_0 = W_0^{1,p(x)}(\Omega), \qquad B = L^2(\Omega) \qquad \text{and} \qquad B_1 = W^{-1,p'(x)}(\Omega),$$

with $p_0 = p_-$ and $p_1 = p'_-$. In view of the Lemma 2.5, we obtain

$$\{u \ : \ u \in V \quad \text{and} \quad u_t \in V^\star\} \subseteq Y \hookrightarrow\hookrightarrow L^1(Q_T). \tag{6}$$

Moreover, in view of [5], we have

$$\{u \ : \ u \in V \quad \text{and} \quad u_t \in V^\star\} \subseteq C([0,T]; L^1(\Omega)). \tag{7}$$

3. The time mollification of a function u in V.

Let $\mu \geq 0$, we introduce the time mollification u_μ of a function $u \in V$, by

$$u_\mu(x,t) = \mu \int_{-\infty}^{t} \overline{u}(x,s)\exp(\mu(s-t))\, ds \qquad \text{where} \quad \overline{u}(x,s) = u(x,s)\chi_{(0,T)}(s).$$

Proposition 3.1. *If $u \in L^{p(x)}(Q_T)$, then u_μ is measurable in Q_T, $\dfrac{\partial u_\mu}{\partial t} = \mu(u - u_\mu)$ and*

$$\int_{Q_T} |u_\mu|^{p(x)} \, dx\, dt \leq \int_{Q_T} |u|^{p(x)} \, dx\, dt.$$

Proof. The applications $(x,s,t) \mapsto u(x,s)\exp(\mu(s-t))$ is measurable in $\Omega \times [0,T] \times [0,T]$, using Fubini's theorem we deduce that u_μ is measurable.
In view of Jensen's integral inequality and since $\displaystyle\int_{-\infty}^{0} \mu \exp(\mu s)\, ds = 1$, we obtain

$$\begin{aligned} \Big| \int_{-\infty}^{t} \mu \exp(\mu(s-t))\, \overline{u}(x,s)\, ds\Big|^{p(x)} &= \Big| \int_{-\infty}^{0} \mu \exp(\mu s)\, \overline{u}(x,s+t)\, ds\Big|^{p(x)} \\ &\leq \int_{-\infty}^{0} \mu \exp(\mu s)\, |\overline{u}(x,s+t)|^{p(x)}\, ds, \end{aligned}$$

it follows that,

$$\begin{aligned}\int_{Q_T} |u_\mu(x,t)|^{p(x)}\,dx\,dt &\le \int_{-\infty}^{+\infty}\int_\Omega\Big(\int_{-\infty}^{0}\mu\,\exp(\mu s)\,|\overline{u}(x,s+t)|^{p(x)}\,ds\Big)\,dx\,dt\\ &= \int_{-\infty}^{0}\mu\,\exp(\mu s)\Big(\int_{-\infty}^{+\infty}\int_\Omega|\overline{u}(x,s+t)|^{p(x)}\,dx\,dt\Big)\,ds\\ &= \int_{-\infty}^{0}\mu\,\exp(\mu s)\Big(\int_{Q_T}|u(x,t)|^{p(x)}\,dx\,dt\Big)\,ds\\ &= \int_{Q_T}|u(x,t)|^{p(x)}\,dx\,dt.\end{aligned}$$

Furthermore,

$$\begin{aligned}\frac{\partial u_\mu}{\partial t} &= \lim_{r\to0}\frac{u_\mu(x,t+r)-u_\mu(x,t)}{r}\\ &= \lim_{r\to0}\frac{\mu}{r}\int_t^{t+r}\overline{u}(x,s)\exp(\mu(s-t-r))\,ds + u_\mu(x,t)\lim_{r\to0}\frac{\exp(-\mu r)-1}{r}\\ &= \mu u(x,t)-\mu u_\mu(x,t).\end{aligned}$$

□

Proposition 3.2. *If* $u\in V$, *then* $u_\mu\to u$ *in* V *as* $\mu\to+\infty$.

Proof. Firstly, since $\dfrac{\partial u_\mu}{\partial x_i}=\Big(\dfrac{\partial u}{\partial x_i}\Big)_\mu$ and in view of Proposition 3.1, we can easily see that $u_\mu\in V$.
Now, we are in a position to prove that $u_\mu\to u$ in V as $\mu\to+\infty$.
Let $(\psi_k)_k\subset D(Q_T)$ such that $\psi_k\to u$ in V. We have

$$\big|(\psi_k)_\mu(x,t)-\psi_k(x,t)\big| = \frac{1}{\mu}\Big|\frac{\partial(\psi_k)_\mu}{\partial t}(x,t)\Big| \le \frac{1}{\mu}\Big\|\frac{\partial\psi_k}{\partial t}\Big\|_{L^\infty(Q_T)}. \tag{8}$$

On the one hand, since $u_\mu-(\psi_k)_\mu=(u-\psi_k)_\mu$ and thanks to the Proposition 3.1 and (8), we obtain

$$\begin{aligned}\int_{Q_T}|u_\mu-u|^{p(x)}\,dx\,dt &\le \int_{Q_T}|(u-\psi_k)_\mu|^{p(x)}\,dx\,dt+\int_{Q_T}|(\psi_k)_\mu-\psi_k|^{p(x)}\,dx\,dt\\ &\quad+\int_{Q_T}|\psi_k-u|^{p(x)}\,dx\,dt\\ &\le 2\int_{Q_T}|u-\psi_k|^{p(x)}\,dx\,dt+\frac{\text{meas}(Q_T)}{\mu^{p_-}}\Big(\Big\|\frac{\partial\psi_k}{\partial t}\Big\|_{L^\infty(Q_T)}+1\Big)^{p_+}.\end{aligned}$$

Let $\varepsilon>0$, there exists $k_0(\varepsilon)>0$ such that

$$\int_{Q_T}|u-\psi_k|^{p(x)}\,dx\,dt\le\frac{\varepsilon}{3}\qquad\text{for all } k\ge k_0,$$

and there exists $\mu_0(\varepsilon)>0$ such that

$$\frac{\text{meas}(Q_T)}{\mu^{p_-}}\Big(\Big\|\frac{\partial\psi_k}{\partial t}\Big\|_\infty+1\Big)^{p_+}\le\frac{\varepsilon}{3}\qquad\text{for all}\quad \mu\ge\mu_0(\varepsilon).$$

Hence

$$\int_{Q_T}|u_\mu-u|^{p(x)}\,dx\,dt\le\varepsilon\qquad\text{for all}\quad\mu\ge\mu_0(\varepsilon).$$

which implies that $u_\mu \to u$ in $L^{p(x)}(Q_T)$.
On the other hand, since $\dfrac{\partial u_\mu}{\partial x_i} = \Big(\dfrac{\partial u}{\partial x_i}\Big)_\mu$. Following the same reasoning as above for each $\dfrac{\partial u}{\partial x_i}$, we can conclude the desired result. □

Proposition 3.3. *If* $u_n \to u$ *in* V, *then* $(u_n)_\mu \to u_\mu$ *in* V.

Proof. Assuming $u_n \to u$ in V. It's clear that

$$(u_n)_\mu - u_\mu = (u_n - u)_\mu \qquad \text{and} \qquad \frac{\partial (u_n)_\mu}{\partial x_i} - \frac{\partial u_\mu}{\partial x_i} = \Big(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i}\Big)_\mu$$

Using the Proposition 8, we deduce that

$$\begin{aligned} \rho_{1,p(x)}((u_n)_\mu - u_\mu) &= \int_{Q_T} |(u_n-u)_\mu|^{p(x)}\,dx\,dt + \sum_{i=1}^N \int_{Q_T} \Big|\Big(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i}\Big)_\mu\Big|^{p(x)}\,dx\,dt \\ &\le \int_{Q_T} |u_n-u|^{p(x)}\,dx\,dt + \sum_{i=1}^N \int_{Q_T} \Big|\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i}\Big|^{p(x)}\,dx\,dt \\ &= \rho_{1,p(x)}(u_n-u). \end{aligned}$$

It follows that $(u_n)_\mu \to u_\mu$ in V. □

Remark 3.1. We have $|(T_k(u))_\mu| \le k$ for all $u \in V$.

Indeed,

$$|(T_k(u))_\mu| = \Big|\int_{-\infty}^t \mu\,\exp(\mu(s-t))\,\overline{T_k(u(x,s))}\,ds\Big| \le k\int_{-\infty}^t \mu\,\exp(\mu(s-t))\,ds = k,$$

with $\overline{T_k(u(x,s))} = T_k(u(x,s))\cdot\chi_{(0,T)}(s)$.

4. Essential Assumptions

Let Ω be a bounded open subset of $I\!R^N\,(N \ge 2)$, $0 < T < \infty$ and taking $p(\cdot) \in C_+(\overline{\Omega})$ such that $p_- > \frac{2N}{N+2}$.
We consider a Leray-Lions operator A from V into its dual $V^\star$, defined by the formula

$$Au = -\text{div}\;a(x,t,\nabla u) + \delta|u|^{p(x)-2}u \tag{9}$$

where $\delta > 0$ and $a : Q_T \times I\!R^N \mapsto I\!R^N$ is a Carathéodory function (measurable with respect to (x,t) in Q_T for every ξ in $I\!R^N$, and continuous with respect to ξ in $I\!R^N$ for almost every (x,t) in Q_T) which satisfies the following conditions

$$|a(x,t,\xi)| \le \beta\,(K(x,t) + |\xi|^{p(x)-1}), \tag{10}$$

$$a(x,t,\xi)\cdot\xi \ge \alpha|\xi|^{p(x)}, \tag{11}$$

$$\big(a(x,t,\xi) - a(x,t,\overline{\xi})\big)\cdot(\xi - \overline{\xi}) > 0 \quad \text{for all } \xi \ne \overline{\xi} \text{ in } I\!R^N, \tag{12}$$

for a.e. $(x,t) \in Q_T$ and all $\xi \in I\!R^N$, where $K(x,t)$ is a positive function lying in $L^{p'(x)}(Q_T)$ and $\alpha, \beta > 0$.
The nonlinear term $g(x,t,s,\xi)$ is a Carathéodory function which satisfies

$$g(x,t,s,\xi)s \;\ge\; 0, \tag{13}$$

$$|g(x,t,s,\xi)| \le b(|s|)(c(x,t) + |\xi|^{p(x)}), \tag{14}$$

where $b(\cdot) : I\!R^+ \mapsto I\!R^+$ is a continuous, nondecreasing function, and $c(\cdot,\cdot) : \Omega \times (0,T) \mapsto I\!R^+$ with $c(\cdot,\cdot) \in L^1(Q_T)$.
We consider the strongly nonlinear $p(x)$-parabolic problem

$$\begin{cases} u_t - \operatorname{div} a(x,t,\nabla u) + g(x,t,u,\nabla u) + \delta |u|^{p(x)-2}u = f - \operatorname{div} \phi(u) & \text{in } Q_T, \\ u = 0 & \text{on } \Sigma_T, \\ u(x,0) = u_0 & \text{in } \Omega, \end{cases} \tag{15}$$

with $\phi(\cdot) \in C^0(I\!R, I\!R^N)$, $f \in L^1(Q_T)$, $u_0 \in L^1(\Omega)$ and $\delta > 0$.

5. Some technical Lemmas

Lemma 5.1. (see [11], Theorem 13.47) *Let $(u_n)_n$ be a sequence in $L^1(\Omega)$ and $u \in L^1(\Omega)$ such that $u_n \to u$ a.e. in Ω, $u_n,\ u \geq 0$ a.e. and $\int_\Omega u_n\,dx \to \int_\Omega u\,dx$, then $u_n \to u$ in $L^1(\Omega)$.*

Lemma 5.2. (see [4]) *Let $p(\cdot) \in C_+(\bar\Omega)$, $g \in L^{p(x)}(\Omega)$ and $g_n \in L^{p(x)}(\Omega)$ with $\|g_n\|_{p(x)} \leq C$.
If $g_n(x) \to g(x)$ a.e. on Ω, then $g_n \rightharpoonup g$ in $L^{p(x)}(\Omega)$.*

Lemma 5.3. *Let $u \in V$ then $T_k(u) \in V$ with $k > 0$. Moreover, we have*

$$T_k(u) \longrightarrow u \quad in \quad V \qquad as \quad k \to \infty.$$

The proof of this Lemma is the same as in the case of constant exponent p.

Lemma 5.4. *Assuming that (10) – (12) holds, and let $(u_n)_n$ be a sequence in V such that $u_n \rightharpoonup u$ in V and*

$$\begin{aligned} &\int_{Q_T} \Big(a(x,t,\nabla u_n) - a(x,t,\nabla u)\Big)\cdot(\nabla u_n - \nabla u)\,dx\,dt \\ &\quad + \int_{Q_T} \Big(|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u\Big)(u_n - u)\,dx\,dt \longrightarrow 0 \qquad for \quad n \to \infty, \end{aligned} \tag{16}$$

then $u_n \to u$ in V for a subsequence.

Proof. Let

$$D_n = \big(a(x,t,\nabla u_n) - a(x,t,\nabla u)\big)\cdot(\nabla u_n - \nabla u) + \big(|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u\big)(u_n - u),$$

thanks to (12) we have D_n is a positive function, and in view of (16), we get $D_n \to 0$ in $L^1(Q_T)$ *as* $n \to \infty$.
We have $u_n \rightharpoonup u$ in V, and in view of the compact embedding (6) we obtain $u_n \to u$ strongly in $L^2(Q_T)$, it follows that $u_n \to u$ a.e in Q_T, and since $D_n \to 0$ a.e in Q_T, there exists a subset B in Q_T with measure zero such that $\forall (x,t) \in Q_T \backslash B$

$$|u_n(x,t)| < \infty,\ |\nabla u_n(x,t)| < \infty,\ |K(x,t)| < \infty,\ u_n(x,t) \to u(x,t)$$

and $D_n \to 0$ a.e. in Ω. We have

$$\begin{aligned} D_n&(x,t) \\ &= \big(a(x,t,\nabla u_n) - a(x,t,\nabla u)\big)\cdot(\nabla u_n - \nabla u) + \big(|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u\big)(u_n - u) \\ &\geq \alpha|\nabla u_n|^{p(x)} + \alpha|\nabla u|^{p(x)} + |u_n|^{p(x)} + |u|^{p(x)} - \beta\big(K(x,t) + |\nabla u_n|^{p(x)-1}\big)|\nabla u| \\ &\qquad - \beta\big(K(x,t) + |\nabla u|^{p(x)-1}\big)|\nabla u_n| - |u_n|^{p(x)-1}|u| - |u|^{p(x)-1}|u_n| \\ &\geq \alpha|\nabla u_n|^{p(x)} - C_{x,t}\Big(1 + |\nabla u_n|^{p(x)-1} + |\nabla u_n|\Big), \end{aligned}$$

where $C_{x,t}$ depending on (x,t), without dependence on n. (since $u_n(x,t) \to u(x,t)$ then $(u_n)_n$ is bounded), we obtain

$$D_n(x) \geq |\nabla u_n|^{p(x)} \Big(\alpha - \frac{C_{x,t}}{|\nabla u_n|^{p(x)}} - \frac{C_{x,t}}{|\nabla u_n|} - \frac{C_{x,t}}{|\nabla u_n|^{p(x)-1}} \Big),$$

by the standard argument $(\nabla u_n)_n$ is bounded almost everywhere in Q_T, (Indeed, if $|\nabla u_n| \to \infty$ in a measurable subset $E \in Q_T$ then

$$\begin{aligned} &\lim_{n\to\infty} \int_{Q_T} D_n(x)\, dx\, dt \\ &\quad \geq \lim_{n\to\infty} \int_E |\nabla u_n|^{p(x)} \big(\alpha - \frac{C_{x,t}}{|\nabla u_n|^{p(x)}} - \frac{C_{x,t}}{|\nabla u_n|} - \frac{C_{x,t}}{|\nabla u_n|^{p(x)-1}} \big)\, dx\, dt = \infty, \end{aligned}$$

which is absurd since $D_n \to 0$ in $L^1(Q_T)$).

Let ξ^* an accumulation point of $(\nabla u_n)_n$, we have $|\xi^*| < \infty$ and by the continuity of the Carathéodory function $a(x,t,\cdot)$, we obtain

$$\Big(a(x,t,\xi^*) - a(x,t,\nabla u) \Big) \cdot (\xi^* - \nabla u) = 0,$$

thanks to (12) we have $\xi^* = \nabla u$, the uniqueness of the accumulation point implies that $\nabla u_n(x,t) \to \nabla u(x,t)$ a.e in Q_T.
since $(a(x,t,\nabla u_n))_n$ is bounded in $(L^{p'(x)}(Q_T))^N$ and $a(x,t,\nabla u_n) \to a(x,t,\nabla u)$ a.e in Q_T, in view of the Lemma 5.2, we can establish that

$$a(x,t,\nabla u_n) \rightharpoonup a(x,t,\nabla u) \quad in \quad (L^{p'(x)}(Q_T))^N.$$

Using (16) and the Lemma 5.1, we deduce that

$$|u_n|^{p(x)} \longrightarrow |u|^{p(x)} \quad \text{in} \quad L^1(Q_T), \tag{17}$$

and

$$a(x,t,\nabla u_n) \cdot \nabla u_n \longrightarrow a(x,t,\nabla u) \cdot \nabla u \quad \text{in} \quad L^1(Q_T). \tag{18}$$

According to the condition (11) we have

$$\alpha |\nabla u_n|^{p(x)} \leq a(x,t,\nabla u_n) \cdot \nabla u_n,$$

Let $z_n = \nabla u_n$, $z = \nabla u$ and $y_n = \dfrac{a(x,t,\nabla u_n) \cdot \nabla u_n}{\alpha}$, $y = \dfrac{a(x,t,\nabla u) \cdot \nabla u}{\alpha}$, in view of the *Fatou Lemma*, we get

$$\int_{Q_T} 2y\, dx\, dt \leq \liminf_{n\to\infty} \int_{Q_T} (y_n + y - \frac{1}{2^{p_+ -1}} |z_n - z|^{p(x)})\, dx\, dt,$$

then $0 \leq -\limsup_{n\to\infty} \displaystyle\int_{Q_T} |z_n - z|^{p(x)}\, dx\, dt$, and since

$$0 \leq \liminf_{n\to\infty} \int_{Q_T} |z_n - z|^{p(x)}\, dx\, dt \leq \limsup_{n\to\infty} \int_{Q_T} |z_n - z|^{p(x)}\, dx\, dt \leq 0,$$

it follows that $\displaystyle\int_{Q_T} |\nabla u_n - \nabla u|^{p(x)}\, dx\, dt \longrightarrow 0$ as $n \to \infty$, and we get

$$\nabla u_n \longrightarrow \nabla u \quad in \quad (L^{p(x)}(Q_T))^N,$$

thanks to (17), we deduce that

$$u_n \longrightarrow u \quad in \quad V,$$

which conclude the proof the Lemma 5.4. □

6. Main results

Let $T_k(s) = \max(-k, \min(s, k))$, we set

$$\varphi_k(r) = \int_0^r T_k(s)\,ds = \begin{cases} \frac{r^2}{2} & \text{if} \quad |r| \le k, \\ k|r| - \frac{k^2}{2} & \text{if} \quad |r| > k. \end{cases}$$

Definition 6.1. A measurable function u is an entropy solution of the strongly nonlinear parabolic problem (15) if

$$T_k(u) \in V \quad \forall k > 0, \qquad g(x,t,u,\nabla u) \in L^1(Q_T), \qquad |u|^{p(x)-2}u \in L^1(Q_T),$$

$$\begin{cases} \int_\Omega \varphi_k(u-\psi)(T)\,dx - \int_\Omega \varphi_k(u-\psi)(0)\,dx + \int_{Q_T} \frac{\partial \psi}{\partial t} T_k(u-\psi)\,dx\,dt \\ + \int_{Q_T} a(x,t,\nabla u)\cdot \nabla T_k(u-\psi)\,dx\,dt + \int_{Q_T} g(x,t,u,\nabla u) T_k(u-\psi)\,dx\,dt \\ +\delta \int_{Q_T} |u|^{p(x)-2}u\; T_k(u-\psi)\,dx\,dt \le \int_{Q_T} f T_k(u-\psi)\,dx\,dt + \int_{Q_T} \phi(u)\cdot \nabla T_k(u-\psi)\,dx\,dt, \end{cases} \tag{19}$$

for all $\psi \in V \cap L^\infty(Q_T)$ with $\dfrac{\partial \psi}{\partial t} \in V^\star + L^1(Q_T)$.

Theorem 6.1. *Assuming that* $(10)-(14)$ *holds, with* $f \in L^1(Q_T)$, $u_0 \in L^1(Q_T)$ *and* $\phi(\cdot) \in C^0(I\!R, I\!R^N)$. *Then the problem* (15) *has at least one entropy solution.*

Proof of the Theorem 6.1.

Step 1 : Approximate problems.
Let $(f_n)_n$ be a sequence in $V^\star \cap L^1(Q_T)$ such that $f_n \to f$ in $L^1(Q_T)$ with $|f_n| \le |f|$, and let $(u_{0,n})_n$ be a sequence in $C_0^\infty(\Omega)$ such that $u_{0,n} \to u_0$ in $L^1(\Omega)$ and $|u_{0,n}| \le |u_0|$. We consider the approximate problem

$$\begin{cases} (u_n)_t - \operatorname{div} a(x,t,\nabla u_n) + g_n(x,t,u_n,\nabla u_n) + \delta |u_n|^{p(x)-2}u_n = f_n - \operatorname{div} \phi_n(u_n) & \text{in} \quad Q_T, \\ u_n(x,t) = \; 0 & \text{on} \quad \Sigma_T, \\ u_n(x,0) = \; u_{0,n} & \text{in} \quad \Omega, \end{cases} \tag{20}$$

with $\phi_n(s) = \phi(T_n(s))$ and $g_n(x,t,s,\xi) = \dfrac{g(x,t,s,\xi)}{1 + \frac{1}{n}|g(x,t,s,\xi)|}$, note that

$$g_n(x,t,s,\xi)s \ge 0, \quad |g_n(x,t,s,\xi)| \le |g(x,t,s,\xi)| \quad \text{and} \quad |g_n(x,t,s,\xi)| \le n \quad \forall n \in I\!N^*.$$

We define the operator $G_n : V \longmapsto V^\star$, by

$$\int_0^T \langle G_n u, v\rangle\,dt = \int_{Q_T} g_n(x,t,u,\nabla u) v\,dx\,dt \qquad \forall v \in V.$$

Thanks to the Hölder type inequality, we have for all $u,\ v \in V$

$$\begin{aligned} \Big|\int_{Q_T} g_n(x,t,u,\nabla u) v\,dx\,dt\Big| &\le \Big(\frac{1}{p_-} + \frac{1}{p'_-}\Big)\, \big\|g_n(x,t,u,\nabla u)\big\|_{L^{p'(x)}(Q_T)} \big\|v\big\|_{L^{p(x)}(Q_T)} \\ &\le 2\,\Big(\int_{Q_T} |g_n(x,t,u,\nabla u)|^{p'(x)}\,dx\,dt + 1\Big)^{\frac{1}{p'_-}} \|v\|_V \\ &\le 2\,\big(n^{p'_+}.\mathrm{meas}(Q_T) + 1\big)^{\frac{1}{p'_-}} \|v\|_V \\ &\le C_0 \|v\|_V\ , \end{aligned} \tag{21}$$

and we define the operator $R_n = \text{div}\, \phi_n \,:\, V \longmapsto V^\star,$ such that

$$\int_0^T \langle R_n(u), v\rangle \, dt = -\int_{Q_T} \phi_n(u)\cdot \nabla v \, dx\, dt \qquad \forall u, v \in V,$$

we have

$$\begin{aligned} |\int_{Q_T} \phi_n(u)\cdot \nabla v \, dx\, dt| &\leq 2\, \big\|\phi_n(u)\big\|_{L^{p'(x)}(Q_T)} \big\|\nabla v\big\|_{L^{p(x)}(Q_T)} \\ &\leq 2\,\big(\int_{Q_T} |\phi_n(u)|^{p'(x)}\, dx\, dt + 1\big)^{\frac{1}{p'_-}} \|v\|_V \\ &\leq 2\,\big(\sup_{|s|\leq n} (|\phi(s)|+1)^{p'_+}.meas(Q_T) + 1\big)^{\frac{1}{p'_-}} \|v\|_V \\ &\leq C_1 \|v\|_V. \end{aligned} \tag{22}$$

In view of the Lemma 7.1 (see Appendix) and (7), there exists at least one weak solution $u_n \in V$ of the problem (20), (cf. [13]).

Step 2 : Weak convergence of truncations.
Let n large enough. By taking $T_k(u_n)$ as a test function in (20), we obtain

$$\begin{aligned} &\int_0^T \langle \frac{\partial u_n}{\partial t}, T_k(u_n)\rangle\, dt + \int_{Q_T} a(x,t,\nabla u_n)\cdot \nabla T_k(u_n)\, dx\, dt + \int_{Q_T} g_n(x,t,u_n,\nabla u_n) T_k(u_n)\, dx\, dt \\ &\quad + \delta \int_{Q_T} |u_n|^{p(x)-2} u_n T_k(u_n)\, dx\, dt = \int_{Q_T} f_n T_k(u_n)\, dx\, dt + \int_{Q_T} \phi_n(u_n)\cdot \nabla T_k(u_n)\, dx\, dt. \end{aligned} \tag{23}$$

We have $\varphi_k(r) = \displaystyle\int_0^r T_k(s)\, ds$ and since $|\varphi_k(r)| \leq k|r|,$ then

$$\begin{aligned} \int_0^T \langle \frac{\partial u_n}{\partial t}, T_k(u_n)\rangle\, dt &= \int_\Omega \int_0^T \frac{\partial u_n}{\partial t} T_k(u_n)\, dt\, dx \\ &= \int_\Omega \int_0^T \frac{\partial \varphi_k(u_n)}{\partial t}\, dt\, dx \\ &= \int_\Omega \varphi_k(u_n(T))\, dx - \int_\Omega \varphi_k(u_{0,n})\, dx \\ &\geq \int_\Omega \varphi_k(u_n(T))\, dx - k\|u_0\|_{L^1(\Omega)}. \end{aligned} \tag{24}$$

The third term on the left-hand side of (23) is positif due to the sign condition. For the second and fourth terms on the left-hand side of (23), we have

$$\int_{Q_T} a(x,t,\nabla u_n)\cdot \nabla T_k(u_n)\, dx\, dt \geq \alpha \int_{Q_T} |\nabla T_k(u_n)|^{p(x)}\, dx\, dt, \tag{25}$$

and

$$\int_{Q_T} |u_n|^{p(x)-2} u_n T_k(u_n)\, dx\, dt \geq \int_{Q_T} |T_k(u_n)|^{p(x)}\, dx\, dt. \tag{26}$$

Concerning the two terms on the right-hand side of (23), we have

$$\int_{Q_T} f_n\, T_k(u_n)\, dx\, dt \leq k \int_{Q_T} |f_n|\, dx\, dt \leq k\|f\|_{L^1(Q_T)}. \tag{27}$$

Taking $\Phi_n(t) = \int_0^t \phi_n(\tau)\, d\tau$, then $\Phi_n(0) = 0_{I\!R^N}$ and $\Phi_n(\cdot) \in C^1(I\!R, I\!R^N)$, by the Divergence Theorem, we obtain

$$\begin{aligned}\int_{Q_T} \phi_n(u_n)\cdot\nabla T_k(u_n)\,dx\,dt &= \int_0^T\int_\Omega \phi_n(T_k(u_n))\cdot\nabla T_k(u_n)\,dx\,dt\\ &= \int_0^T\int_\Omega \text{div }\Phi_n(T_k(u_n))\,dx\,dt\\ &= \int_0^T\int_{\partial\Omega} \Phi_n(T_k(u_n))\cdot\vec{n}\,d\sigma\,dt\\ &= \sum_{i=1}^N\int_0^T\int_{\partial\Omega} \Phi_{n,i}(T_k(u_n)).n_i\,d\sigma\,dt = 0,\end{aligned} \tag{28}$$

since $u_n = 0$ on $\partial\Omega\times[0,T]$, with $\Phi_n = (\Phi_{n,1},\ldots,\Phi_{n,N})$ and $\vec{n} = (n_1, n_2, \ldots, n_N)$ The exterior normal vector on the boundary $\partial\Omega\times[0,T]$.

By combining $(23)-(28)$, we deduce that

$$\int_\Omega\varphi_k(u_n(T))dx+\alpha\int_{Q_T}|\nabla T_k(u_n)|^{p(x)}dxdt+\delta\int_{Q_T}|T_k(u_n)|^{p(x)}dxdt \le k(\|u_0\|_{L^1(\Omega)}+\|f\|_{L^1(Q_T)}), \tag{29}$$

since $\varphi_k(u_n(T)) \ge 0$, there exists a constant C_2 that does not depend on n and k, such that

$$\begin{aligned}&\|\nabla T_k(u_n)\|^{p_-}_{L^{p(x)}(Q_T)} + \|T_k(u_n)\|^{p_-}_{L^{p(x)}(Q_T)}\\ &\quad\le \int_{Q_T}|\nabla T_k(u_n)|^{p(x)}\,dx\,dt + \int_{Q_T}|T_k(u_n)|^{p(x)}\,dx\,dt + 2\\ &\quad\le kC_2,\end{aligned}$$

then

$$\|T_k(u_n)\|_V \le C_3 k^{\frac{1}{p_-}} \qquad \text{for} \quad k\ge 1. \tag{30}$$

Let $k \ge 1$, we have

$$\begin{aligned}k\ \text{meas}\{|u_n|>k\} &= \int_{\{|u_n|>k\}}|T_k(u_n)|\,dx\,dt\\ &\le \int_{Q_T}|T_k(u_n)|\,dx\,dt\\ &\le 2(meas(Q_T)+1)^{\frac{1}{p'_-}}\|T_k(u_n)\|_V\\ &\le C_4 k^{\frac{1}{p_-}},\end{aligned}$$

which implies that

$$\text{meas}\{|u_n|>k\} \le C_4\frac{1}{k^{1-\frac{1}{p_-}}} \to 0 \quad \text{as} \quad k\to\infty, \tag{31}$$

For all $\lambda > 0$, we have

$$\begin{aligned}\text{meas}\{|u_n-u_m|>\lambda\} &\le \text{meas}\{|u_n|>k\} + \text{meas}\{|u_m|>k\}\\ &\quad+\text{meas}\{|T_k(u_n)-T_k(u_m)|>\lambda\}.\end{aligned} \tag{32}$$

Using (31) we get that for all $\varepsilon > 0$, there exists $k_0 > 0$ such that

$$\text{meas}\{|u_n|>k\} \le \frac{\varepsilon}{3} \qquad \text{and} \qquad \text{meas}\{|u_m|>k\} \le \frac{\varepsilon}{3} \qquad \forall k \ge k_0(\varepsilon). \tag{33}$$

On the other hand, we have $(T_k(u_n))_n$ is bounded in V, then there exists a sequence still denoted $(T_k(u_n))_n$ such that

$$T_k(u_n) \rightharpoonup \eta_k \quad \text{in} \quad V \quad \text{as} \quad n\to\infty,$$

and by the compact embedding (6) we obtain

$$T_k(u_n) \to \eta_k \quad \text{in} \quad L^1(Q_T) \quad \text{and a.e in} \quad Q_T.$$

Thus we can assume that $(T_k(u_n))_n$ is a Cauchy sequence in measure in Q_T, then for all $k > 0$ and $\lambda,\ \varepsilon > 0$ there exists $n_0 = n_0(k, \lambda, \varepsilon)$ such that

$$\text{meas}\big\{|T_k(u_n) - T_k(u_m)| > \lambda\big\} \le \frac{\varepsilon}{3} \qquad \forall n, m \ge n_0. \tag{34}$$

By combining $(32) - (34)$, we deduce that for all $\varepsilon, \lambda > 0$, there exists $n_0 = n_0(\lambda, \varepsilon)$ such that

$$\text{meas}\{|u_n - u_m| > \lambda\} \le \varepsilon \qquad \forall n, m \ge n_0. \tag{35}$$

If follows that $(u_n)_n$ is a Cauchy sequence in measure, then there exists a subsequence still denoted $(u_n)_n$ such that

$$u_n \to u \qquad \text{a.e in} \quad Q_T.$$

We deduce that

$$T_k(u_n) \rightharpoonup T_k(u) \qquad \text{in} \quad V, \tag{36}$$

and in the view of the Lebesgue dominated convergence theorem

$$T_k(u_n) \to T_k(u) \qquad \text{in} \quad L^{p(x)}(Q_T). \tag{37}$$

Step 3 : A priori estimates.
Let $h > 0$, by taking $T_{h+1}(u_n) - T_h(u_n)$ as a test function in (20), we obtain

$$\begin{aligned}
&\int_0^T \langle \frac{\partial u_n}{\partial t}, T_{h+1}(u_n) - T_h(u_n)\rangle\, dt + \int_{Q_T} a(x,t,\nabla u_n)\cdot(\nabla T_{h+1}(u_n) - \nabla T_h(u_n))\, dx\, dt \\
&+ \int_{Q_T} g_n(x,t,u_n,\nabla u_n)(T_{h+1}(u_n) - T_h(u_n))\, dx\, dt + \delta \int_{Q_T} |u_n|^{p(x)-2} u_n (T_{h+1}(u_n) - T_h(u_n))\, dx\, dt \\
&= \int_{Q_T} f_n\cdot(T_{h+1}(u_n) - T_h(u_n))\, dx\, dt + \int_{Q_T} \phi_n(u_n)\cdot(\nabla T_{h+1}(u_n) - \nabla T_h(u_n))\, dx\, dt.
\end{aligned}$$

We have

$$\begin{aligned}
\int_0^T \langle \frac{\partial u_n}{\partial t}, T_{h+1}(u_n) - T_h(u_n)\rangle\, dt &= \int_\Omega \int_0^T \frac{\partial \varphi_{h+1}(u_n)}{\partial t}\, dt\, dx - \int_\Omega \int_0^T \frac{\partial \varphi_h(u_n)}{\partial t}\, dt\, dx \\
&= \int_\Omega \varphi_{h+1}(u_n(T)) - \varphi_{h+1}(u_{0,n})\, dx \\
&\quad - \int_\Omega \varphi_h(u_n(T)) - \varphi_h(u_{0,n})\, dx
\end{aligned}$$

with

$$\begin{aligned}
\int_\Omega \varphi_{h+1}(u_n(T))\, dx - \int_\Omega \varphi_h(u_n(T))\, dx &= \int_{\{h \le |u_n(T)| < h+1\}} (\frac{u_n^2(T)}{2} - h|u_n(T)| + \frac{h^2}{2})\, dx \\
&\quad + \int_{\{h+1 \le |u_n(T)|\}} (|u_n(T)| - h - \frac{1}{2})\, dx \ge 0.
\end{aligned} \tag{38}$$

It's clear that $T_{h+1}(u_n)-T_h(u_n)$ have the same sign as u_n, and $|T_{h+1}(u_n)-T_h(u_n)|\leq 1$, using (11) and (13) we obtain

$$\begin{aligned}
&\alpha\int_{\{h\leq|u_n|<h+1\}}|\nabla u_n|^{p(x)}\,dx\,dt+\delta\int_{\{h+1\leq|u_n|\}}|u_n|^{p(x)-1}\,dx\,dt\\
&\leq\int_{\{h\leq|u_n|<h+1\}}a(x,t,\nabla u_n)\cdot\nabla u_n\,dx\,dt+\delta\int_{\{h\leq|u_n|\}}|u_n|^{p(x)-2}u_n(T_{h+1}(u_n)-T_h(u_n))\,dx\,dt\\
&\leq\int_{Q_T}f_n\cdot(T_{h+1}(u_n)-T_h(u_n))\,dx\,dt+\int_{Q_T}\phi_n(u_n)\cdot(\nabla T_{h+1}(u_n)-\nabla T_h(u_n))\,dx\,dt\\
&\quad+\int_\Omega\varphi_{h+1}(u_{0,n})\,dx-\int_\Omega\varphi_h(u_{0,n})\,dx.
\end{aligned}\tag{39}$$

Concerning the terms on the right-hand side of (39), we have

$$\Big|\int_{Q_T}f_n\cdot(T_{h+1}(u_n)-T_h(u_n))\,dx\,dt\Big|\leq\int_{\{|u_n|\geq h\}}|f|\,dx\,dt\longrightarrow 0\qquad\text{as}\quad h\to\infty.\tag{40}$$

Similarly as (28), we obtain

$$\begin{aligned}
&\int_{Q_T}\phi_n(u_n)\cdot(\nabla T_{h+1}(u_n)-\nabla T_h(u_n))\,dx\,dt\\
&=\int_0^T\int_\Omega\phi_n(T_{h+1}(u_n))\cdot\nabla T_{h+1}(u_n)\,dx\,dt-\int_0^T\int_\Omega\phi_n(T_h(u_n))\cdot\nabla T_h(u_n)\,dx\,dt\\
&=\int_0^T\int_\Omega\operatorname{div}\,\Phi_n(T_{h+1}(u_n))\,dx\,dt-\int_0^T\int_\Omega\operatorname{div}\,\Phi_n(T_h(u_n))\,dx\,dt=0,
\end{aligned}\tag{41}$$

and since $u_{0,n}\to u_0$ in $L^1(\Omega)$, then

$$\begin{aligned}
\int_\Omega\varphi_{h+1}(u_{0,n})\,dx-\int_\Omega\varphi_h(u_{0,n})\,dx&=\int_{\{h\leq|u_{0,n}|<h+1\}}(\frac{|u_{0,n}|^2}{2}-h|u_{0,n}|+\frac{h^2}{2})\,dx\\
&\quad+\int_{\{h+1\leq|u_{0,n}|\}}(|u_{0,n}|-h-\frac{1}{2})\,dx\\
&\leq\int_{\{h\leq|u_{0,n}|<h+1\}}\frac{1}{2}\,dx+\int_{\{h+1\leq|u_{0,n}|\}}|u_0|\,dx\longrightarrow 0\quad\text{as}\quad h\to\infty,
\end{aligned}\tag{42}$$

By combining (39) – (42), we deduce that

$$\int_{\{h\leq|u_n|<h+1\}}|\nabla u_n|^{p(x)}\,dx\,dt\longrightarrow 0\qquad\text{as}\qquad h\to\infty,\tag{43}$$

and

$$\int_{\{h+1\leq|u_n|\}}|u_n|^{p(x)-1}\,dx\,dt\longrightarrow 0\qquad\text{as}\qquad h\to\infty.\tag{44}$$

Step 4 : Convergence of the gradient.

In the sequel, we denote by $\varepsilon_i(n)$ $i=1,\,2,\ldots$ a various functions of real numbers which converges to 0 as n tends to infinity (respectively for $\varepsilon_i(n,\mu)$ and $\varepsilon_i(n,\mu,h)$).

Let $\xi_k(s)=s.\exp(\gamma s^2)$ where $\gamma=\big(\frac{b(k)}{2\alpha}\big)^2$, it is obvious that

$$\xi_k'(s)-\frac{b(k)}{\alpha}|\xi_k(s)|\geq\frac{1}{2}\qquad\forall s\in I\!R,$$

Let $\omega_{n,\mu}=T_k(u_n)-(T_k(u))_\mu$, where $(T_k(u))_\mu$ is the mollification, with respect to time, of $T_k(u)$.

Taking $S_h(\cdot)\in C^2(I\!R)$ an increasing function, such that $S_h(r)=r$ for $|r|\leq h$ and $\operatorname{supp}(S_h')\subset[-h-1,h+1]$, then $\operatorname{supp}(S_h'')\subset[-h-1,-h]\cup[h,h+1]$.

It's clear that $T_k(u_n)-(T_k(u))_\mu$ have the same sign as u_n on the set $\{|u_n|>k\}$. By using $\xi_k(\omega_{n,\mu})S'_h(u_n)$ as a test function in (20), we obtain

$$\mathcal{J}^1_{n,\mu,h}+\mathcal{J}^2_{n,\mu,h}+\mathcal{J}^3_{n,\mu,h}+\mathcal{J}^4_{n,\mu,h}+\delta\ \mathcal{J}^5_{n,\mu,h}\leq \mathcal{J}^6_{n,\mu,h}+\mathcal{J}^7_{n,\mu,h}+\mathcal{J}^8_{n,\mu,h} \tag{45}$$

where,

$$\begin{aligned}
\mathcal{J}^1_{n,\mu,h} &= \int_0^T \langle \frac{\partial u_n}{\partial t}, \xi_k(\omega_{n,\mu})S'_h(u_n)\rangle\, dt,\\
\mathcal{J}^2_{n,\mu,h} &= \int_{Q_T} S'_h(u_n)a(x,t,\nabla u_n)\cdot(\nabla T_k(u_n)-\nabla(T_k(u))_\mu)\xi'_k(\omega_{n,\mu})\,dx\,dt,\\
\mathcal{J}^3_{n,\mu,h} &= \int_{Q_T} \xi_k(\omega_{n,\mu})S''_h(u_n)a(x,t,\nabla u_n)\cdot\nabla u_n\,dx\,dt,\\
\mathcal{J}^4_{n,\mu,h} &= \int_{\{|u_n|\leq k\}} g_n(x,t,u_n,\nabla u_n)S'_h(u_n)\xi_k(\omega_{n,\mu})\,dx\,dt,\\
\mathcal{J}^5_{n,\mu,h} &= \int_{\{|u_n|\leq k\}} |u_n|^{p(x)-2}u_nS'_h(u_n)\xi_k(\omega_{n,\mu})\,dx\,dt,\\
\mathcal{J}^6_{n,\mu,h} &= \int_{Q_T} f_nS'_h(u_n)\xi_k(\omega_{n,\mu})\,dx\,dt,\\
\mathcal{J}^7_{n,\mu,h} &= \int_{Q_T} S'_h(u_n)\phi_n(u_n)\cdot(\nabla T_k(u_n)-\nabla(T_k(u))_\mu)\xi'_k(\omega_{n,\mu})\,dx\,dt,\\
\mathcal{J}^8_{n,\mu,h} &= \int_{Q_T} \phi_n(u_n)\cdot\nabla u_nS''_h(u_n)\xi_k(\omega_{n,\mu})\,dx\,dt.
\end{aligned} \tag{46}$$

The first term

We have

$$\begin{aligned}
\mathcal{J}^1_{n,\mu,h} &= \int_{Q_T}\frac{\partial S_h(u_n)}{\partial t}\xi_k(T_k(u_n)-(T_k(u))_\mu)\,dx\,dt\\
&= \int_{Q_T}\frac{\partial(S_h(u_n)-T_k(u_n))}{\partial t}\xi_k(T_k(u_n)-(T_k(u))_\mu)\,dx\,dt\\
&\quad+\int_{Q_T}\frac{\partial T_k(u_n)}{\partial t}\xi_k(T_k(u_n)-(T_k(u))_\mu)\,dx\,dt\\
&= \Big[\int_\Omega (S_h(u_n)-T_k(u_n))\xi_k(T_k(u_n)-(T_k(u))_\mu)\,dx\Big]_0^T\\
&\quad-\int_{Q_T}(S_h(u_n)-T_k(u_n))\xi'_k(T_k(u_n)-(T_k(u))_\mu)\Big(\frac{\partial T_k(u_n)}{\partial t}-\frac{\partial(T_k(u))_\mu}{\partial t}\Big)\,dx\,dt\\
&\quad+\int_{Q_T}\frac{\partial T_k(u_n)}{\partial t}\xi_k(T_k(u_n)-(T_k(u))_\mu)\,dx\,dt\\
&= I_1+I_2+I_3.
\end{aligned} \tag{47}$$

Concerning the first term on right-hand side of (47), we have $S_h(u_n)=T_k(u_n)=u_n$ on $\{|u_n|\leq k\}$, and $|S_h(u_n)|\geq|T_k(u_n)|$ on the set $\{|u_n|>k\}$, since $S_h(u_n)$ and $T_k(u_n)$ have the same sign of u_n, we obtain

$$\begin{aligned}
I_1 &= \Big[\int_{\{|u_n|>k\}}(S_h(u_n)-T_k(u_n))\xi_k(T_k(u_n)-(T_k(u))_\mu)\,dx\Big]_0^T\\
&\geq -\int_{\{|u_{0,n}|>k\}}(S_h(u_{0,n})-T_k(u_{0,n}))\xi_k(T_k(u_{0,n})-(T_k(u_0))_\mu)\,dx,
\end{aligned}$$

also, it's clear that $(T_k(u_0))_\mu=T_k(u_0)$, then

$$-\int_{\{|u_{0,n}|>k\}}(S_h(u_{0,n})-T_k(u_{0,n}))\xi_k(T_k(u_{0,n})-T_k(u_0))\,dx\longrightarrow 0 \qquad \text{as}\quad n\to\infty,$$

we deduce that

$$I_1 \geq \varepsilon_1(n).$$

For the second term on right-hand side of (47), we have $(S_h(u_n) - T_k(u_n))\dfrac{\partial T_k(u_n)}{\partial t} = 0$ then

$$\begin{aligned}
I_2 &= \int_{\{|u_n|>k\}} (S_h(u_n) - T_k(u_n))\xi_k'(T_k(u_n) - (T_k(u))_\mu)\frac{\partial (T_k(u))_\mu}{\partial t}\,dx\,dt \\
&= \mu \int_{\{|u_n|>k\}} (S_h(u_n) - T_k(u_n))\xi_k'(T_k(u_n) - (T_k(u))_\mu)(T_k(u) - (T_k(u))_\mu)\,dx\,dt \\
&= \mu \int_{\{|u_n|>k\}} (S_h(u_n) - T_k(u_n))\xi_k'(T_k(u_n) - (T_k(u))_\mu)(T_k(u) - T_k(u_n))\,dx\,dt \\
&\quad + \mu \int_{\{|u_n|>k\}} (S_h(u_n) - T_k(u_n))\xi_k'(T_k(u_n) - (T_k(u))_\mu)(T_k(u_n) - (T_k(u))_\mu)\,dx\,dt \\
&\geq \mu \int_{\{|u_n|>k\}} (S_h(u_n) - T_k(u_n))\xi_k'(T_k(u_n) - (T_k(u))_\mu)(T_k(u) - T_k(u_n))\,dx\,dt,
\end{aligned}$$

it follows that

$$I_2 \geq \varepsilon_2(n).$$

Concerning the last term I_3, Let $\Psi(s) = \dfrac{1}{2\gamma}(\exp(\gamma s^2) - 1)$ then $\Psi'(s) = \xi_k(s)$, and we obtain

$$\begin{aligned}
I_3 &= \int_{Q_T} \frac{\partial (T_k(u_n) - (T_k(u))_\mu)}{\partial t}\xi_k(T_k(u_n) - (T_k(u))_\mu)\,dx\,dt \\
&\quad + \int_{Q_T} \frac{\partial (T_k(u))_\mu}{\partial t}\xi_k(T_k(u_n) - (T_k(u))_\mu)\,dx\,dt \\
&= \Big[\int_\Omega \Psi(T_k(u_n) - (T_k(u))_\mu)\,dx\Big]_0^T + \mu \int_{Q_T} (T_k(u) - (T_k(u))_\mu)\xi_k(T_k(u_n) - (T_k(u))_\mu)\,dx\,dt \\
&\geq -\int_\Omega \Psi(T_k(u_{0,n}) - T_k(u_0))\,dx + \mu \int_{Q_T} (T_k(u) - (T_k(u))_\mu)\xi_k(T_k(u_n) - (T_k(u))_\mu)\,dx\,dt \\
&\geq \varepsilon_3(n) + \mu \int_{Q_T} (T_k(u) - (T_k(u))_\mu)\xi_k(T_k(u) - (T_k(u))_\mu)\,dx\,dt \\
&\geq \varepsilon_3(n).
\end{aligned}$$

By combining these estimates, we conclude that

$$\mathcal{J}^1_{n,\mu,h} \geq \varepsilon_4(n). \tag{48}$$

The second term

Concerning the second term of (46), we have $S_h'(s) \geq 0$ and $S_h'(s) = 1$ for $|s| \leq k$, with $\operatorname{supp}(S_h') \subset [-h-1, h+1]$, then

$$\begin{aligned}
\mathcal{J}^2_{n,\mu,h} &= \int_{\{|u_n|\leq k\}} a(x,t,\nabla T_k(u_n)) \cdot (\nabla T_k(u_n) - \nabla (T_k(u))_\mu)\xi_k'(\omega_{n,\mu})\,dx\,dt \\
&\quad - \int_{\{k\leq |u_n|<h+1\}} S_h'(u_n) a(x,t,\nabla T_{h+1}(u_n)) \cdot \nabla (T_k(u))_\mu \xi_k'(\omega_{n,\mu})\,dx\,dt
\end{aligned}$$

$$
\begin{aligned}
&\geq \int_{Q_T} \Big(a(x,t,\nabla T_k(u_n)) - a(x,t,\nabla T_k(u))\Big)\cdot(\nabla T_k(u_n)-\nabla T_k(u))\xi_k'(\omega_{n,\mu})\,dx\,dt\\
&\quad - \xi_k'(2k)\int_{Q_T}|a(x,t,\nabla T_k(u))|\,|\nabla T_k(u_n)-\nabla T_k(u)|\,dx\,dt\\
&\quad - \xi_k'(2k)\int_{\{|u_n|\geq k\}}|a(x,t,\nabla T_k(u_n))|\,|\nabla (T_k(u))_\mu|\,dx\,dt\\
&\quad - \xi_k'(2k)\int_{Q_T}|a(x,t,\nabla T_k(u_n))|\,|\nabla T_k(u)-\nabla (T_k(u))_\mu|\,dx\,dt\\
&\quad - \xi_k'(2k)\|S_h'(\cdot)\|_{L^\infty(I\!R)}\int_{\{k\leq|u_n|<h+1\}}|a(x,t,\nabla T_{h+1}(u_n))|\,|\nabla (T_k(u))_\mu|\,dx\,dt.
\end{aligned}
\tag{49}
$$

We have $|a(x,t,\nabla T_k(u))|\in L^{p'(x)}(Q_T)$ and $\nabla T_k(u_n)\rightharpoonup\nabla T_k(u)$ in $(L^{p(x)}(Q_T))^N$, then

$$
\int_{Q_T}|a(x,t,\nabla T_k(u))|\,|\nabla T_k(u_n)-\nabla T_k(u)|\,dx\,dt\longrightarrow 0 \quad\text{as}\quad n\to\infty.
$$

For the three last terms on the right-hand side of (49), we have $|a(x,t,\nabla T_k(u_n))|$ is bounded in $L^{p'(x)}(Q_T)$, then there exists $\vartheta\in L^{p'(x)}(Q_T)$ such that $|a(x,t,\nabla T_k(u_n))|\rightharpoonup\vartheta$ in $L^{p'(x)}(Q_T)$, and since $\nabla(T_k(u))_\mu\to\nabla T_k(u)$ in $(L^{p(x)}(Q_T))^N$ it follows that

$$
\int_{\{|u_n|\geq k\}}|a(x,t,\nabla T_k(u_n))|\,|\nabla(T_k(u))_\mu|dxdt\longrightarrow\int_{\{|u|\geq k\}}\vartheta\,|\nabla T_k(u)|dxdt=0 \text{ as } \mu \text{ and } n\to\infty.
\tag{50}
$$

Similarly, we can prove that

$$
\int_{Q_T}|a(x,t,\nabla T_k(u_n))|\,|\nabla T_k(u)-\nabla(T_k(u))_\mu|\,dx\,dt\longrightarrow 0 \qquad\text{as}\quad \mu \quad\text{and}\quad n\to\infty.
\tag{51}
$$

and

$$
\int_{\{k<|u_n|\leq h+1\}}|a(x,t,\nabla T_{h+1}(u_n))|\,|\nabla(T_k(u))_\mu|dx\,dt\longrightarrow 0 \qquad\text{as}\quad \mu\quad\text{and}\quad n\to\infty.
\tag{52}
$$

By combining (49) – (52), we deduce that

$$
\begin{aligned}
\mathcal{J}^2_{n,\mu,h}\quad &\geq \int_{Q_T}\Big(a(x,t,\nabla T_k(u_n))-a(x,t,\nabla T_k(u))\Big)\cdot(\nabla T_k(u_n)-\nabla T_k(u))\xi_k'(\omega_{n,\mu})\,dx\,dt\\
&\quad+\varepsilon_5(\mu,n).
\end{aligned}
\tag{53}
$$

The third term

We have $\operatorname{supp}(S_h'')\subset[-h-1,-h]\cup[h,h+1]$, and in view of Young's inequality, we obtain

$$
\begin{aligned}
\big|\mathcal{J}^3_{n,\mu,h}\big| &\leq \|S_h''(\cdot)\|_{L^\infty(I\!R)}\int_{\{h\leq|u_n|<h+1\}}|a(x,t,\nabla T_{h+1}(u_n))|\,|\xi_k(\omega_{n,\mu})|\,|\nabla T_{h+1}(u_n)|\,dx\,dt\\
&\leq \beta\|S_h''(\cdot)\|_{L^\infty(I\!R)}\int_{\{h\leq|u_n|<h+1\}}(K(x,t)+|\nabla T_{h+1}(u_n)|^{p(x)-1})|\xi_k(\omega_{n,\mu})|\,|\nabla T_{h+1}(u_n)|\,dx\,dt\\
&\leq \beta\|S_h''(\cdot)\|_{L^\infty(I\!R)}\int_{\{h\leq|u_n|<h+1\}}|\xi_k(\omega_{n,\mu})|\,\frac{|K(x,t)|^{p'(x)}}{p'(x)}\,dx\,dt\\
&\qquad+\beta\xi_k(2k)\,\|S_h''(\cdot)\|_{L^\infty(I\!R)}\int_{\{h\leq|u_n|<h+1\}}(\frac{1}{p(x)}+1)|\nabla T_{h+1}(u_n)|^{p(x)}\,dx\,dt,
\end{aligned}
$$

since $\omega_{n,\mu}=T_k(u_n)-(T_k(u))_\mu\rightharpoonup 0$ weak-$\star$ in $L^\infty(Q_T)$ then

$$
\int_{\{h\leq|u_n|<h+1\}}|\xi_k(\omega_{n,\mu})|\,\frac{|K(x,t)|^{p'(x)}}{p'(x)}\,dx\,dt\longrightarrow 0\quad\text{as}\quad\mu,\ n\to\infty,
$$

and thanks to (43), we obtain

$$\int_{\{h\leq|u_n|<h+1\}}(\frac{1}{p(x)}+1)|\nabla T_{h+1}(u_n)|^{p(x)}\,dx\,dt\longrightarrow 0 \qquad \text{as} \quad h\to\infty,$$

it follows that

$$\mathcal{J}^3_{n,\mu,h}\longrightarrow 0 \qquad \text{as} \quad \mu,\ n \ \text{ then } h\to\infty. \tag{54}$$

The fourth term

We have

$$\begin{aligned}
&\left|\mathcal{J}^4_{n,\mu,h}\right| \leq \int_{\{|u_n|\leq k\}}|g_n(x,t,T_k(u_n),\nabla T_k(u_n))|\ |\xi_k(\omega_{n,\mu})|\,dx\,dt\\
&\leq b(k)\int_{\{|u_n|\leq k\}}(c(x,t)+|\nabla T_k(u_n)|^{p(x)})|\xi_k(\omega_{n,\mu})|\,dx\,dt\\
&\leq b(k)\int_{\{|u_n|\leq k\}}c(x,t)|\xi_k(\omega_{n,\mu})|\,dx\,dt+\frac{b(k)}{\alpha}\int_{Q_T}a(x,t,\nabla T_k(u_n))\cdot\nabla T_k(u_n)|\xi_k(\omega_{n,\mu})|\,dx\,dt\\
&=b(k)\int_{\{|u_n|\leq k\}}c(x,t)|\xi_k(\omega_{n,\mu})|\,dx\,dt\\
&\quad+\frac{b(k)}{\alpha}\int_{Q_T}(a(x,t,\nabla T_k(u_n))-a(x,t,\nabla T_k(u)))\cdot(\nabla T_k(u_n)-\nabla T_k(u))\ |\xi_k(\omega_{n,\mu})|\,dx\,dt\\
&\quad+\frac{b(k)}{\alpha}\int_{Q_T}a(x,t,\nabla T_k(u))\cdot(\nabla T_k(u_n)-\nabla T_k(u))\ |\xi_k(\omega_{n,\mu})|\,dx\,dt\\
&\quad+\frac{b(k)}{\alpha}\int_{Q_T}a(x,t,\nabla T_k(u_n))\cdot\nabla T_k(u)\ |\xi_k(\omega_{n,\mu})|\,dx\,dt.
\end{aligned} \tag{55}$$

We have $\xi_k(\omega_{n,\mu})\rightharpoonup 0$ weak-$\star$ in $L^\infty(Q_T)$, then

$$\int_{\{|u_n|\leq k\}}c(x,t)|\xi_k(\omega_{n,\mu})|\,dx\,dt\longrightarrow 0 \quad \text{as} \quad n \text{ and } \mu\to\infty. \tag{56}$$

Concerning the third and last terms on the right-hand side of (55), we have

$$\begin{aligned}
&\Big|\int_{Q_T}a(x,t,\nabla T_k(u))\cdot(\nabla T_k(u_n)-\nabla T_k(u))|\xi_k(\omega_{n,\mu})|\,dx\,dt\Big|\\
&\qquad\leq \xi_k(2k)\int_{Q_T}|a(x,t,\nabla T_k(u))|\ |\nabla T_k(u_n)-\nabla T_k(u)|\,dx\,dt\longrightarrow 0 \qquad \text{as} \quad n\to\infty,
\end{aligned} \tag{57}$$

and

$$\int_{Q_T}a(x,t,\nabla T_k(u_n))\cdot\nabla T_k(u)\ |\xi_k(\omega_{n,\mu})|\,dx\,dt\longrightarrow 0 \qquad \text{as} \quad n,\mu\to\infty. \tag{58}$$

By combining (55) – (58), we obtain

$$\begin{aligned}
\left|\mathcal{J}^4_{n,\mu,h}\right| \leq \frac{b(k)}{\alpha}&\int_{Q_T}(a(x,t,\nabla T_k(u_n))-a(x,t,\nabla T_k(u)))\cdot(\nabla T_k(u_n)-\nabla T_k(u))\ |\xi_k(\omega_{n,\mu})|\,dx\,dt\\
&+\varepsilon_6(n,\mu).
\end{aligned} \tag{59}$$

The fifth term

Since $S'_h(u_n)=1$ on the set $\{|u_n|\leq k\}$, we have

$$\mathcal{J}^5_{n,\mu,h} \ =\int_{\{|u_n|\leq k\}}|T_k(u_n)|^{p(x)-2}T_k(u_n)(T_k(u_n)-(T_k(u))_\mu)\exp(\gamma\omega^2_{n,\mu})\,dx\,dt$$

$$
\begin{aligned}
&= \int_{Q_T} \big(|T_k(u_n)|^{p(x)-2}T_k(u_n) - |T_k(u)|^{p(x)-2}T_k(u)\big)(T_k(u_n) - T_k(u)) \exp(\gamma\omega_{n,\mu}^2)\, dx\, dt \\
&\quad + \int_{Q_T} |T_k(u)|^{p(x)-2}T_k(u)(T_k(u_n) - T_k(u)) \exp(\gamma\omega_{n,\mu}^2)\, dx\, dt \\
&\quad + \int_{\{|u_n|\le k\}} |T_k(u_n)|^{p(x)-2}T_k(u_n)(T_k(u) - (T_k(u))_\mu) \exp(\gamma\omega_{n,\mu}^2)\, dx\, dt \\
&\quad - \int_{\{|u_n|> k\}} |T_k(u_n)|^{p(x)-2}T_k(u_n)(T_k(u_n) - T_k(u)) \exp(\gamma\omega_{n,\mu}^2)\, dx\, dt.
\end{aligned}
$$

Since $T_k(u_n) - T_k(u) \to 0$ and $T_k(u) - (T_k(u))_\mu \to 0$ strongly in $L^{p(x)}(Q_T)$, then the three last terms on the right-hand side of the previous equality tends to 0, and we obtain

$$
\begin{aligned}
\mathcal{J}_{n,\mu,h}^5 \quad &\ge \int_{Q_T} (|T_k(u_n)|^{p(x)-2}T_k(u_n) - |T_k(u)|^{p(x)-2}T_k(u))(T_k(u_n) - T_k(u))\, dx\, dt \\
&\quad + \varepsilon_7(n,\mu).
\end{aligned} \tag{60}
$$

The sixth term

We have $f_n \to f$ in $L^1(Q_T)$, and since $T_k(u_n) - (T_k(u))_\mu \rightharpoonup 0$ weak-$\star$ in $L^\infty(Q_T)$, then

$$
\left|\mathcal{J}_{n,\mu,h}^6\right| \le \|S_h'(\cdot)\|_{L^\infty(I\!R)} \int_{Q_T} |f_n|\, |\xi_k(T_k(u_n) - (T_k(u))_\mu)|\, dx\, dt \to 0 \quad \text{as } n,\ \mu \to \infty. \tag{61}
$$

The seventh and last terms

Let n large enough, it's clear that $\phi_n(T_{h+1}(u_n)) = \phi(T_{h+1}(u_n)) \to \phi(T_{h+1}(u))$ in $(L^{p'(x)}(Q_T))^N$, and since $\nabla T_k(u_n) - \nabla (T_k(u))_\mu \rightharpoonup 0$ in $(L^{p(x)}(Q_T))^N$, we conclude that

$$
\begin{aligned}
\left|\mathcal{J}_{n,\mu,h}^7\right| \quad &= \Big| \int_{\{|u_n|\le h+1\}} \phi_n(T_{h+1}(u_n)) \cdot (\nabla T_k(u_n) - \nabla (T_k(u))_\mu) S_h'(u_n)\xi_k'(\omega_{n,\mu})\, dx\, dt \Big| \\
&\le \xi_k(2k)\|S_h'(\cdot)\|_\infty \int_{\{|u_n|\le h+1\}} |\phi(T_{h+1}(u_n))|\ |\nabla T_k(u_n) - \nabla (T_k(u))_\mu|\, dx\, dt \to 0 \text{ as } n,\mu\to\infty.
\end{aligned} \tag{62}
$$

Concerning the last term, in view of Young's inequality and (43) we obtain

$$
\begin{aligned}
\left|\mathcal{J}_{n,\mu,h}^8\right| \quad &= \Big| \int_{\{h<|u_n|\le h+1\}} \phi_n(T_{h+1}(u_n)) \cdot \nabla u_n\ S_h''(u_n)\ \xi_k'(\omega_{n,\mu})\, dx\, dt \Big| \\
&\le \|S_h''(\cdot)\|_{L^\infty(I\!R)} \int_{\{h<|u_n|\le h+1\}} |\phi_n(T_{h+1}(u_n))|\ |\nabla u_n|\ |\xi_k'(\omega_{n,\mu})|\, dx\, dt \\
&\le \|S_h''(\cdot)\|_{L^\infty(I\!R)} \int_{\{h<|u_n|\le h+1\}} \frac{|\phi_n(T_{h+1}(u_n))|^{p'(x)}}{p'(x)} |\xi_k'(\omega_{n,\mu})|\, dx\, dt \\
&\quad + \xi_k(2k)\|S_h''\|_{L^\infty(I\!R)} \int_{\{h<|u_n|\le h+1\}} \frac{|\nabla u_n|^{p(x)}}{p(x)}\, dx\, dt \longrightarrow 0 \quad \text{as } n,\ \mu \text{ then } h\to\infty.
\end{aligned} \tag{63}
$$

Combining (48), (53), (54) and (60) – (63) we deduce that

$$
\begin{aligned}
&\int_{Q_T} \Big(a(x,t,\nabla T_k(u_n)) - a(x,t,\nabla T_k(u))\Big)(\nabla T_k(u_n) - \nabla T_k(u))\Big(\xi_k'(\omega_{n,\mu}) - \frac{b(k)}{\alpha}|\xi_k(\omega_{n,\mu})|\Big) dx\, dt \\
&\quad + \delta \int_{Q_T} (|T_k(u_n)|^{p(x)-2}T_k(u_n) - |T_k(u)|^{p(x)-2}T_k(u))(T_k(u_n) - T_k(u))\, dx\, dt
\end{aligned}
$$

$$
\begin{aligned}
&\leq \int_{Q_T} S'_h(u_n) a(x,t,\nabla u_n)\cdot(\nabla T_k(u_n)-\nabla(T_k(u))_\mu)\xi'_k(\omega_{n,\mu})\,dx\,dt\\
&\quad -\Big|\int_{\{|u_n|\leq k\}} g_n(x,t,u_n,\nabla u_n)S'_h(u_n)\xi_k(\omega_{n,\mu})\,dx\,dt\Big|\\
&\quad +\delta\int_{\{|u_n|\leq k\}} |u_n|^{p(x)-2}u_n S'_h(u_n)\xi_k(\omega_{n,\mu})\,dx\,dt+\varepsilon_8(n,\mu,h)\\
&\leq \varepsilon_9(n,\mu,h).
\end{aligned}
$$

By letting n, μ and h tend to infinity in the above inequality, we get

$$
\begin{aligned}
&\lim_{n\to\infty}\int_{Q_T}\Big(a(x,t,\nabla T_k(u_n))-a(x,t,\nabla T_k(u))\Big)(\nabla T_k(u_n)-\nabla T_k(u))\,dx\,dt\\
&\quad+\int_{Q_T}(|T_k(u_n)|^{p(x)-2}T_k(u_n)-|T_k(u)|^{p(x)-2}T_k(u))(T_k(u_n)-T_k(u))\,dx\,dt=0.
\end{aligned}
$$

In view of the Lemma 5.4, we deduce that

$$
T_k(u_n)\to T_k(u)\quad\text{in}\quad V\qquad\text{then}\qquad \nabla u_n\to\nabla u\quad\text{a.e in } Q_T. \tag{64}
$$

Step 5 : The equi-integrability of $g_n(x,t,u_n,\nabla u_n)$ *and* $|u_n|^{p(x)-2}u_n$.
To prove that

$$
g_n(x,t,u_n,\nabla u_n)\longrightarrow g(x,t,u,\nabla u)\text{ and }|u_n|^{p(x)-2}u_n\longrightarrow |u|^{p(x)-2}u\quad\text{strongly in } L^1(Q_T),
$$

using Vitali's theorem, it is sufficient to prove that $g_n(x,t,u_n,\nabla u_n)$ and $|u_n|^{p(x)-2}u_n$ are uniformly equi-integrable.
Indeed, taking $T_{h+1}(u_n)-T_h(u_n)$ as a test function in (20), we obtain

$$
\begin{aligned}
&\int_{Q_T}\frac{\partial\varphi_{h+1}(u_n)}{\partial t}-\frac{\partial\varphi_h(u_n)}{\partial t}\,dt\,dx+\int_{\{h\leq|u_n|<h+1\}}a(x,t,\nabla u_n)\cdot\nabla u_n\,dx\,dt\\
&+\int_{\{h\leq|u_n|\}}g_n(x,t,u_n,\nabla u_n)(T_{h+1}(u_n)-T_h(u_n))\,dx\,dt\\
&+\delta\int_{\{h\leq|u_n|\}}|u_n|^{p(x)-2}u_n(T_{h+1}(u_n)-T_h(u_n))\,dx\,dt\\
&=\int_{\{h\leq|u_n|\}}f_n\cdot(T_{h+1}(u_n)-T_h(u_n))\,dx\,dt+\int_{Q_T}\phi_n(u_n)\cdot(\nabla T_{h+1}(u_n)-\nabla T_h(u_n))\,dx\,dt.
\end{aligned}
\tag{65}
$$

We have

$$
\begin{aligned}
\int_{Q_T}\frac{\partial\varphi_{h+1}(u_n)}{\partial t}-\frac{\partial\varphi_h(u_n)}{\partial t}\,dt\,dx &=\int_\Omega\varphi_{h+1}(u_n(T))\,dx-\int_\Omega\varphi_{h+1}(u_{0,n})\,dx\\
&\quad-\int_\Omega\varphi_h(u_n(T))\,dx+\int_\Omega\varphi_h(u_{0,n})\,dx
\end{aligned}
$$

having in mind (38), then

$$
\int_{Q_T}\frac{\partial\varphi_{h+1}(u_n)}{\partial t}-\frac{\partial\varphi_h(u_n)}{\partial t}\,dt\,dx\geq-\int_\Omega\varphi_{h+1}(u_{0,n})\,dx+\int_\Omega\varphi_h(u_{0,n})\,dx,
$$

and since the second term on the left-hand side of (65) is positive, we obtain

$$\begin{aligned}
&\int_{\{h+1\le|u_n|\}} |g_n(x,t,u_n,\nabla u_n)|\,dx\,dt + \delta\int_{\{h+1\le|u_n|\}} |u_n|^{p(x)-1}\,dx\,dt \\
&\quad\le \int_{\{h\le|u_n|\}} g_n(x,t,u_n,\nabla u_n)(T_{h+1}(u_n)-T_h(u_n))\,dx\,dt \\
&\qquad +\delta\int_{\{h\le|u_n|\}} |u_n|^{p(x)-2}u_n(T_{h+1}(u_n)-T_h(u_n))\,dx\,dt \\
&\quad\le \int_{\{h\le|u_n|\}} |f|\,dx\,dt + \int_{Q_T} \phi_n(u_n)\cdot(\nabla T_{h+1}(u_n)-\nabla T_h(u_n))\,dx\,dt \\
&\qquad + \int_\Omega \varphi_{h+1}(u_{0,n})\,dx - \int_\Omega \varphi_h(u_{0,n})\,dx.
\end{aligned} \tag{66}$$

In view of $(40)-(42)$, we deduce that : for all $\eta>0$, there exists $h(\eta)>0$ such that

$$\int_{\{h(\eta)\le|u_n|\}} |g_n(x,t,u_n,\nabla u_n)|\,dx\,dt + \delta\int_{\{h(\eta)\le|u_n|\}} |u_n|^{p(x)-1}\,dx\,dt \le \frac{\eta}{2}. \tag{67}$$

On the other hand, for any measurable subset $E\subset Q_T$, we have

$$\begin{aligned}
&\int_E |g_n(x,t,u_n,\nabla u_n)|\,dx\,dt + \delta\int_E |u_n|^{p(x)-1}\,dx\,dt \\
&\quad\le \int_E b(h(\eta))(c(x,t)+|\nabla T_{h(\eta)}(u_n)|^{p(x)})\,dx\,dt + \delta\int_E |T_{h(\eta)}(u_n)|^{p(x)-1}\,dx\,dt \\
&\qquad + \int_{\{h(\eta)\le|u_n|\}} |g_n(x,t,u_n,\nabla u_n)|\,dx\,dt + \delta\int_{\{h(\eta)\le|u_n|\}} |u_n|^{p(x)-1}\,dx\,dt,
\end{aligned} \tag{68}$$

thanks to (64), there exists $\beta(\eta)>0$ such that

$$\int_E b(h(\eta))(c(x,t)+|\nabla T_{h(\eta)}(u_n)|^{p(x)})dxdt+\delta\int_E |T_{h(\eta)}(u_n)|^{p(x)-1}dxdt \le \frac{\eta}{2} \text{ for } meas(E)\le\beta(\eta). \tag{69}$$

Finally, by combining (67), (68) and (69), we obtain

$$\int_E |g_n(x,t,u_n,\nabla u_n)|\,dx\,dt + \delta\int_E |u_n|^{p(x)-1}\,dx\,dt \le \eta, \quad \text{with} \quad meas(E)\le\beta(\eta), \tag{70}$$

then $(g_n(x,t,u_n,\nabla u_n))_n$ and $(|u_n|^{p(x)-2}u_n)_n$ are equi-integrable. In view of Vitali's Theorem, we conclude that

$$g_n(x,t,u_n,\nabla u_n) \longrightarrow g(x,t,u,\nabla u) \quad \text{and} \quad |u_n|^{p(x)-2}u_n \longrightarrow |u|^{p(x)-2}u \quad \text{in} \quad L^1(Q_T). \tag{71}$$

Step 6 : The convergence of u_n *in* $C([0,T];L^1(\Omega))$..
Let $\psi\in V\cap L^\infty(Q_T)$ and m, n be two integers, then u_n and u_m verifies

$$\begin{aligned}
&\int_0^T \langle \frac{\partial u_n}{\partial t} - \frac{\partial u_m}{\partial t}, \psi\rangle\,dt + \int_{Q_T} \big(a(x,t,\nabla u_n)-a(x,t,\nabla u_m)\big)\cdot\nabla\psi\,dx\,dt \\
&\qquad + \int_{Q_T} \big(g_n(x,t,u_n,\nabla u_n)-g_m(x,t,u_m,\nabla u_m)\big)\psi\,dx\,dt \\
&\qquad +\delta\int_{Q_T} \big(|u_n|^{p(x)-2}u_n - |u_m|^{p(x)-2}u_m\big)\psi\,dx\,dt \\
&\quad = \int_{Q_T} (f_n-f_m)\psi\,dx\,dt + \int_{Q_T} \big(\phi_n(u_n)-\phi_m(u_m)\big)\cdot\nabla\psi\,dx\,dt,
\end{aligned}$$

By taking $\psi = T_1(u_n - u_m)\cdot\chi_{[0,s]}$ for $0 < s \le T$, we obtain

$$\begin{aligned}
&\int_\Omega\int_0^s \frac{\partial\varphi_1(u_n-u_m)}{\partial t}\,dt\,dx \\
&\quad+\int_0^s\int_\Omega \big(a(x,t,\nabla u_n)-a(x,t,\nabla u_m)\big)\cdot\nabla T_1(u_n-u_m)\,dx\,dt \\
&\quad+\int_0^s\int_\Omega \big(g_n(x,t,u_n,\nabla u_n)-g_m(x,t,u_m,\nabla u_m)\big)T_1(u_n-u_m)\,dx\,dt \\
&\quad+\delta\int_0^s\int_\Omega \big(|u_n|^{p(x)-2}u_n-|u_m|^{p(x)-2}u_m\big)T_1(u_n-u_m)\,dx\,dt \\
&=\int_0^s\int_\Omega (f_n-f_m)T_1(u_n-u_m)\,dx\,dt \\
&\quad+\int_0^s\int_\Omega \big(\phi_n(u_n)-\phi_m(u_m)\big)\cdot\nabla T_1(u_n-u_m)\,dx\,dt.
\end{aligned}\tag{72}$$

We have

$$\int_\Omega\int_0^s \frac{\partial\varphi_1(u_n-u_m)}{\partial t}\,dt\,dx=\int_\Omega \varphi_1(u_n(s)-u_m(s))\,dx-\int_\Omega \varphi_1(u_{0,n}-u_{0,m})\,dx.$$

Concerning the terms on the left-hand side of (72), since $\nabla T_1(u_n-u_m) = (\nabla u_n - \nabla u_m).\chi_{\{|u_n-u_m|\le 1\}}$, then

$$\int_0^s\int_\Omega \Big(a(x,t,\nabla u_n)-a(x,t,\nabla u_m)\Big)\cdot(\nabla u_n-\nabla u_m).\chi_{\{|u_n-u_m|\le 1\}}\,dx\,dt\ge 0,\tag{73}$$

also, we have

$$\int_0^s\int_\Omega \Big(|u_n|^{p(x)-2}u_n-|u_m|^{p(x)-2}u_m\Big)T_1(u_n-u_m)\,dx\,dt\ge 0,\tag{74}$$

and in view of (71), we obtain

$$\begin{aligned}
&\Big|\int_0^s\int_\Omega (g_n(x,t,u_n,\nabla u_n)-g_m(x,t,u_m,\nabla u_m))T_1(u_n-u_m)\,dx\,dt\Big| \\
&\quad\le\int_{Q_T} |g_n(x,t,u_n,\nabla u_n)-g_m(x,t,u_m,\nabla u_m)|\,dx\,dt\longrightarrow 0 \quad\text{as}\quad n,m\to\infty.
\end{aligned}\tag{75}$$

For the terms on the right-hand side of (72), since

$$\Big|\int_0^s\int_\Omega (f_n-f_m)\cdot T_1(u_n-u_m)\,dx\,dt\Big|\le\int_{Q_T}|f_n-f_m|\,dx\,dt\longrightarrow 0\quad\text{as}\quad n,m\to\infty.\tag{76}$$

and since (see Appendix), we have

$$\int_0^s\int_\Omega \big(\phi_n(u_n)-\phi_m(u_m)\big)\cdot\nabla T_1(u_n-u_m)\,dx\,dt\longrightarrow 0\quad\text{as}\quad n,m\to\infty.\tag{77}$$

By combining (72) – (77), and the fact that $|u_{0,n}-u_{0,m}|\to 0$ in $L^1(\Omega)$, we conclude that

$$\int_\Omega \varphi_1(u_n(s)-u_m(s))\,dx\longrightarrow 0\quad\text{as}\quad n,m\to\infty\qquad\text{for any}\quad 0<s\le T.\tag{78}$$

On the other hand, we have

$$\begin{aligned}
&\int_{\{|u_n-u_m|\le 1\}}|u_n(s)-u_m(s)|^2\,dx+\int_{\{|u_n-u_m|>1\}}|u_n(s)-u_m(s)|\,dx \\
&\le 2\int_\Omega \varphi_1(u_n(s)-u_m(s))\,dx,
\end{aligned}\tag{79}$$

and

$$\begin{aligned}\int_\Omega |u_n(s)-u_m(s)|\,dx &= \int_{\{|u_n-u_m|\le 1\}} |u_n(s)-u_m(s)|\,dx \\ &\quad + \int_{\{|u_n-u_m|>1\}} |u_n(s)-u_m(s)|\,dx \\ &\le \Big(\int_{\{|u_n-u_m|\le 1\}} |u_n(s)-u_m(s)|^2\,dx\Big)^{\frac{1}{2}} \cdot (\mathrm{meas}(\Omega))^{\frac{1}{2}} \\ &\quad + \int_{\{|u_n-u_m|>1\}} |u_n(s)-u_m(s)|\,dx,\end{aligned} \tag{80}$$

In view of (78) – (80), we deduce that

$$\int_\Omega |u_n(s)-u_m(s)|\,dx \longrightarrow 0 \quad \text{as} \quad m,n\to\infty. \tag{81}$$

Hence (u_n) is a Cauchy sequence in $C([0,T];L^1(\Omega))$, thus $u\in C([0,T];L^1(\Omega))$ and we have $u_n(s)\to u(s)$ in $L^1(\Omega)$ for any $0<s\le T$.

Step 7 : Passage to the limit.
Let $\psi\in V\cap L^\infty(Q_T)$ with $\frac{\partial\psi}{\partial t}\in V^\star+L^1(Q_T)$. By taking $T_k(u_n-\psi)$ as a test function in (20), we obtain

$$\begin{aligned}&\int_0^T \langle\frac{\partial u_n}{\partial t}, T_k(u_n-\psi)\rangle\,dt + \int_{Q_T} a(x,t,\nabla u_n)\cdot\nabla T_k(u_n-\psi)\,dx\,dt \\ &\quad + \int_{Q_T} g_n(x,t,u_n,\nabla u_n)T_k(u_n-\psi)\,dx\,dt + \delta\int_{Q_T}|u_n|^{p(x)-2}u_nT_k(u_n-\psi)\,dx\,dt \\ &= \int_{Q_T} f_nT_k(u_n-\psi)\,dx\,dt + \int_{Q_T}\phi_n(u_n)\cdot\nabla T_k(u_n-\psi)\,dx\,dt\end{aligned} \tag{82}$$

Taking $M=k+\|\psi\|_{L^\infty(Q_T)}$. If $|u_n|>M$ then $|u_n-\psi|\ge|u_n|-\|\psi\|_\infty>k$, therefore $\{|u_n-\psi|\le k\}\subseteq\{|u_n|\le M\}$, which implies that

$$\begin{aligned}&\int_{Q_T} a(x,t,\nabla u_n)\cdot\nabla T_k(u_n-\psi)\,dx\,dt \\ &\quad = \int_{\{|u_n-\psi|\le k\}} a(x,t,\nabla T_M(u_n))\cdot(\nabla T_M(u_n)-\nabla\psi)\,dx\,dt \\ &\quad = \int_{\{|u_n-\psi|\le k\}} (a(x,t,\nabla T_M(u_n))-a(x,t,\nabla\psi))\cdot(\nabla T_M(u_n)-\nabla\psi)\,dx\,dt \\ &\qquad + \int_{\{|u_n-\psi|\le k\}} a(x,t,\nabla\psi)\cdot(\nabla T_M(u_n)-\nabla\psi)\,dx\,dt\end{aligned}$$

since $\nabla T_M(u_n)\to\nabla T_M(u)$ in $(L^{p(x)}(Q_T))^N$, and in view of Fatou's Lemma, we obtain

$$\begin{aligned}&\liminf_{n\to+\infty}\int_{Q_T} a(x,t,\nabla u_n)\cdot\nabla T_k(u_n-\psi)\,dx\,dt \\ &\quad \ge \int_{\{|u-\psi|\le k\}} (a(x,t,\nabla T_M(u))-a(x,t,\nabla\psi))\cdot(\nabla T_M(u)-\nabla\psi)\,dx\,dt \\ &\qquad + \int_{\{|u-\psi|\le k\}} a(x,t,\nabla\psi)\cdot(\nabla T_M(u)-\nabla\psi)\,dx\,dt \\ &\quad = \int_{Q_T} a(x,t,\nabla u)\cdot\nabla T_k(u-\psi)\,dx\,dt.\end{aligned} \tag{83}$$

On the other hand, for the first term on the left-hand side of (82), we have $\frac{\partial u_n}{\partial t} = \frac{\partial (u_n - \psi)}{\partial t} + \frac{\partial \psi}{\partial t}$, then

$$\begin{aligned}\int_0^T \langle \frac{\partial u_n}{\partial t}, T_k(u_n - \psi)\rangle \, dt &= \int_0^T \langle \frac{\partial (u_n - \psi)}{\partial t}, T_k(u_n - \psi)\rangle \, dt + \int_0^T \langle \frac{\partial \psi}{\partial t}, T_k(u_n - \psi)\rangle \, dt \\ &= \int_\Omega \varphi_k(u_n - \psi)(T) \, dx - \int_\Omega \varphi_k(u_{0,n} - \psi(0)) \, dx \\ &\quad + \int_{Q_T} \frac{\partial \psi}{\partial t} T_k(u_n - \psi) \, dx \, dt,\end{aligned}$$

since $u_n \to u$ in $C([0,T]; L^1(\Omega))$ then $u_n(T) \to u(T)$ in $L^1(\Omega)$, it follows that

$$\int_\Omega \varphi_k(u_{0,n} - \psi(0)) \, dx \longrightarrow \int_\Omega \varphi_k(u_0 - \psi(0)) \, dx, \tag{84}$$

and

$$\int_\Omega \varphi_k(u_n - \psi)(T) \, dx \longrightarrow \int_\Omega \varphi_k(u - \psi)(T) \, dx. \tag{85}$$

We have $\frac{\partial \psi}{\partial t} \in V^\star + L^1(Q_T)$, since $T_k(u_n - \psi) \rightharpoonup T_k(u - \psi)$ in V and weak$-\star$ in $L^\infty(Q_T)$, then

$$\int_{Q_T} \frac{\partial \psi}{\partial t} T_k(u_n - \psi) \, dx \, dt \longrightarrow \int_{Q_T} \frac{\partial \psi}{\partial t} T_k(u - \psi) \, dx \, dt, \tag{86}$$

and

$$\int_{Q_T} f_n T_k(u_n - \psi) \, dx \, dt \longrightarrow \int_{Q_T} f T_k(u - \psi) \, dx \, dt. \tag{87}$$

in view of (71), we deduce that

$$\int_{Q_T} |u_n|^{p(x)-2} u_n T_k(u_n - \psi) \, dx \, dt \longrightarrow \int_{Q_T} |u|^{p(x)-2} u \; T_k(u - \psi) \, dx \, dt, \tag{88}$$

and

$$\int_{Q_T} g_n(x,t,u_n,\nabla u_n) T_k(u_n - \psi) \, dx \, dt \longrightarrow \int_{Q_T} g(x,t,u,\nabla u) T_k(u - \psi) \, dx \, dt. \tag{89}$$

On the other hand, since $\phi_n(T_M(u_n)) = \phi(T_M(u_n))$ for n large enough $(n \geq M)$, then

$$\begin{aligned}\lim_{n\to\infty} \int_{Q_T} \phi_n(u_n) \cdot \nabla T_k(u_n - \psi) \, dx \, dt &= \lim_{n\to\infty} \int_{\{|u_n - \psi| \leq k\}} \phi(T_M(u_n)) \cdot (\nabla T_M(u_n) - \nabla \psi) \, dx \, dt \\ &= \int_{\{|u - \psi| \leq k\}} \phi(T_M(u)) \cdot (\nabla T_M(u) - \nabla \psi) \, dx \, dt \\ &= \int_{Q_T} \phi(u) \cdot \nabla T_k(u - \psi) \, dx \, dt.\end{aligned} \tag{90}$$

By combining (82) – (90), we deduce that

$$\begin{aligned}&\int_\Omega \varphi_k(u - \psi)(T) \, dx - \int_\Omega \varphi_k(u - \psi)(0) \, dx + \int_{Q_T} \frac{\partial \psi}{\partial t} T_k(u - \psi) \, dx \, dt \\ &+ \int_{Q_T} a(x,t,\nabla u) \cdot \nabla T_k(u - \psi) \, dx \, dt + \int_{Q_T} g(x,t,u,\nabla u) T_k(u - \psi) \, dx \, dt \\ &+ \delta \int_{Q_T} |u|^{p(x)-2} u \; T_k(u - \psi) \, dx \, dt \leq \int_{Q_T} f T_k(u - \psi) \, dx \, dt + \int_{Q_T} \phi(u) \cdot \nabla T_k(u - \psi) \, dx \, dt,\end{aligned}$$

which conclude the proof of the Theorem 6.1.

7. Appendix

Lemma 7.1. *The operator* $B_n = A + G_n + R_n$ *is pseudo-monotone from* V *into* $V^\star$. *Moreover,* B_n *is coercive in the following sense*

$$\frac{\int_0^T \langle B_n v, v\rangle \, dt}{\|v\|_V} \longrightarrow +\infty \qquad \text{as} \quad \|v\|_V \longrightarrow +\infty \qquad \text{for} \quad v \in V.$$

Proof of Lemma 7.1. Using the Hölder's type inequality and the growth condition (10) we can show that the operator A is bounded, and by using (21) and (22), we conclude that B_n bounded. For the coercivity, thanks to (22) we have for any $u \in V$,

$$\begin{aligned}
\langle B_n u, u\rangle &= \langle Au, u\rangle + \langle G_n u, u\rangle + \langle R_n u, u\rangle \\
&= \int_{Q_T} a(x,t,\nabla u)\cdot\nabla u \, dx\, dt + \int_{Q_T} g_n(x,t,u,\nabla u) u \, dx\, dt + \delta \int_{Q_T} |u|^{p(x)} \, dx\, dt \\
&\quad - \int_{Q_T} \phi_n(u)\cdot\nabla u \, dx\, dt \\
&\geq \alpha \int_{Q_T} |\nabla u|^{p(x)} \, dx\, dt + \delta \int_{Q_T} |u|^{p(x)} \, dx\, dt - 2\|\phi_n(u)\|_{L^{p'(x)}(Q_T)} \|\nabla u\|_{L^{p(x)}(Q_T)} \\
&\geq \alpha(\|\nabla u\|^{p_-}_{L^{p(x)}(Q_T)} - 1) + \delta(\|u\|^{p_-}_{L^{p(x)}(Q_T)} - 1) - C_1.\|u\|_V \\
&\geq \frac{\min(\alpha,\delta)}{2^{\underline{p}} - 1} \|u\|_V^{p_-} - \alpha - \delta - C_1.\|u\|_V,
\end{aligned}$$

we deduce that

$$\frac{\int_0^T \langle B_n u, u\rangle \, dt}{\|u\|_V} \longrightarrow +\infty \qquad \text{as} \quad \|u\|_V \to +\infty.$$

It remain to show that B_n is pseudo-monotone. Let $(u_k)_k$ by a sequence in V such that

$$\begin{cases} u_k \rightharpoonup u & \text{in} \quad V, \\ B_n u_k \rightharpoonup \chi_n & \text{in} \quad V^\star, \\ \limsup\limits_{k\to\infty} \langle B_n u_k, u_k\rangle \leq \langle \chi_n, u\rangle. \end{cases} \tag{91}$$

We will prove that

$$\chi_n = B_n u \quad \text{and} \quad \langle B_n u_k, u_k\rangle \longrightarrow \langle \chi_n, u\rangle \ \text{ as } \ k \to +\infty.$$

Using the compact embedding (6), we have $u_k \to u$ in $L^1(Q_T)$ for a subsequence still denoted $(u_k)_k$.

By the growth condition, it's clear that $(a(x,t,\nabla u_k))_k$ is bounded in $(L^{p'(x)}(Q_T))^N$, therefore, there exists a function $\vartheta \in (L^{p'(x)}(Q_T))^N$ such that

$$a(x,t,\nabla u_k) \rightharpoonup \vartheta \ \text{ in } \ (L^{p'(x)}(Q_T))^N \ \text{ as } \ k \to \infty. \tag{92}$$

and

$$|u_k|^{p(x)-2} u_k \rightharpoonup |u|^{p(x)-2} u \qquad \text{in} \qquad L^{p'(x)}(Q_T). \tag{93}$$

Similarly, we have $(g_n(x,t,u_k,\nabla u_k))_k$ is bounded in $L^{p'(x)}(Q_T)$, then there exists a function $\psi_n \in L^{p'(x)}(Q_T)$ such that

$$g_n(x,t,u_k,\nabla u_k) \rightharpoonup \psi_n \ \text{ in } \ L^{p'(x)}(Q_T) \ \text{ as } \ k \to \infty, \tag{94}$$

and since $\phi_n(\cdot) = \phi \circ T_n(\cdot)$ is a bounded continuous function, using the Lebesgue dominated convergence theorem we have $T_n(u_k) \to T_n(u)$ in $L^{p(x)}(Q_T)$, then

$$\phi_n(u_k) \longrightarrow \phi_n(u) \ \text{ in } \ (L^{p'(x)}(Q_T))^N \ \text{ as } \ k \to \infty. \tag{95}$$

For all $v \in V$, we have

$$\begin{aligned}\langle \chi_n, v\rangle &= \lim_{k\to\infty} \langle B_n u_k, v\rangle \\ &= \lim_{k\to\infty} \int_{Q_T} a(x,t,\nabla u_k)\cdot \nabla v\,dx\,dt + \lim_{k\to\infty}\int_{Q_T} g_n(x,t,u_k,\nabla u_k)v\,dx\,dt \\ &\quad + \lim_{k\to\infty} \delta \int_{Q_T} |u_k|^{p(x)-2}u_k v\,dx\,dt - \lim_{k\to\infty}\int_{Q_T} \phi_n(u_k)\cdot\nabla v\,dx\,dt \\ &= \int_{Q_T} \vartheta\cdot\nabla v\,dx\,dt + \int_{Q_T}\psi_n v\,dx\,dt + \delta\int_{Q_T}|u|^{p(x)-2}uv\,dx\,dt - \int_{Q_T}\phi_n(u)\cdot\nabla v\,dx\,dt.\end{aligned} \tag{96}$$

By using (91) and (96), we obtain

$$\begin{aligned}\limsup_{k\to\infty}\langle B_n(u_k),u_k\rangle &= \limsup_{k\to\infty}\Big\{\int_{Q_T} a(x,t,\nabla u_k)\cdot\nabla u_k\,dx\,dt + \int_{Q_T} g_n(x,t,u_k,\nabla u_k)u_k\,dx\,dt \\ &\quad + \delta\int_{Q_T}|u_k|^{p(x)}\,dx\,dt - \int_{Q_T}\phi_n(u_k)\cdot\nabla u_k\,dx\,dt\Big\} \\ &\le \int_{Q_T}\vartheta\cdot\nabla u\,dx\,dt + \int_{Q_T}\psi_n u\,dx\,dt + \delta\int_{Q_T}|u|^{p(x)}\,dx\,dt - \int_{Q_T}\phi_n(u)\cdot\nabla u\,dx\,dt.\end{aligned} \tag{97}$$

Thanks to (94) and (95), we have

$$\int_{Q_T} g_n(x,t,u_k,\nabla u_k)u_k dxdt \to \int_{Q_T}\psi_n u dxdt \text{ and } \int_{Q_T}\phi_n(u_k)\cdot\nabla u_k dxdt \to \int_{Q_T}\phi_n(u)\cdot\nabla u dxdt, \tag{98}$$

therefore

$$\limsup_{k\to\infty}\Big\{\int_{Q_T} a(x,t,\nabla u_k)\cdot\nabla u_k dxdt + \delta\int_{Q_T}|u_k|^{p(x)}dxdt\Big\} \le \int_{Q_T}\vartheta\cdot\nabla u dxdt + \delta\int_{Q_T}|u|^{p(x)}dxdt. \tag{99}$$

On the other hand, using (12) we have

$$\begin{aligned}&\int_{Q_T}(a(x,t,\nabla u_k) - a(x,t,\nabla u))\cdot(\nabla u_k - \nabla u)\,dx\,dt \\ &\qquad + \delta\int_{Q_T}(|u_k|^{p(x)-2}u_k - |u|^{p(x)-2}u)(u_k-u)\,dx\,dt \ge 0,\end{aligned} \tag{100}$$

then

$$\begin{aligned}&\int_{Q_T} a(x,t,\nabla u_k)\cdot\nabla u_k\,dx\,dt + \delta\int_{Q_T}|u_k|^{p(x)}dx\,dt \\ &\quad \ge \int_{Q_T} a(x,t,\nabla u_k)\cdot\nabla u\,dx\,dt + \delta\int_{Q_T}|u_k|^{p(x)-2}u_k u\,dx\,dt \\ &\qquad + \int_{Q_T} a(x,t,\nabla u)\cdot(\nabla u_k - \nabla u)\,dx\,dt + \delta\int_{Q_T}|u|^{p(x)-2}u(u_k-u)\,dx\,dt,\end{aligned}$$

in view of (92) and (93), we get

$$\liminf_{k\to\infty}\Big\{\int_{Q_T} a(x,t,\nabla u_k)\cdot\nabla u_k dxdt + \delta\int_{Q_T}|u_k|^{p(x)}dxdt\Big\} \ge \int_{Q_T}\vartheta\cdot\nabla u dxdt + \delta\int_{Q_T}|u|^{p(x)}dxdt,$$

this implies, thanks to (99), that

$$\lim_{k\to\infty}\Big\{\int_{Q_T} a(x,t,\nabla u_k)\cdot\nabla u_k dxdt + \delta\int_{Q_T}|u_k|^{p(x)}dxdt\Big\} = \int_{Q_T}\vartheta\cdot\nabla u dxdt + \delta\int_{Q_T}|u|^{p(x)}dxdt, \tag{101}$$

thanks to (98), we obtain $\langle B_n u_k, u_k\rangle \to \langle \chi_n, u\rangle$ as $k \to +\infty$.
Now, by (101) we can obtain

$$\lim_{k\to+\infty}\Big\{\int_{Q_T}(a(x,t,\nabla u_k)-a(x,t,\nabla u))\cdot(\nabla u_k-\nabla u)\,dx\,dt \\ +\delta\int_{Q_T}(|u_k|^{p(x)-2}u_k-|u|^{p(x)-2}u)(u_k-u)dx\,dt\Big\}=0,$$

in view of the Lemma 5.4, we get

$$u_k \longrightarrow u \quad \text{in} \quad V \quad \text{and} \quad \nabla u_k \longrightarrow \nabla u \quad \text{a.e in } Q_T,$$

it follows that

$$a(x,t,\nabla u_k) \rightharpoonup a(x,t,\nabla u) \quad \text{in} \quad (L^{p'(x)}(Q_T))^N,$$

and

$$g_n(x,t,u_k,\nabla u_k) \rightharpoonup g_n(x,t,u,\nabla u) \quad \text{in} \quad L^{p'(x)}(Q_T),$$

we deduce that $\chi_n = B_n u$, which completes the proof of Lemma 7.1. □

Proof of the convergence (77). Let $h>0$ and n, m large enough. We have

$$\begin{aligned}\Big|\int_0^s\int_\Omega&\big(\phi_n(u_n)-\phi_m(u_m)\big)\cdot\nabla T_1(u_n-u_m)\,dx\,dt\Big|\\ &\le\int_{\{|u_n|\le h\}\cap\{|u_m|\le h\}}|\phi(T_h(u_n))-\phi(T_h(u_m))|\;|\nabla T_h(u_n)-\nabla T_h(u_m)|\,dx\,dt\\ &\quad+\int_{\{|u_n|>h\}\cup\{|u_m|>h\}}|\phi_n(u_n)-\phi_m(u_m)|\;|\nabla u_n-\nabla u_m|.\chi_{\{|u_n-u_m|\le1\}}\,dx\,dt\end{aligned}\tag{102}$$

Concerning the first term on the right-hand side of (102), since $\nabla T_h(u_n)$ and $\nabla T_h(u_m)$ converge strongly to $\nabla T_h(u)$ in $(L^{p(x)}(Q_T))^N$, and since $|\phi_n(T_h(u_n))-\phi_n(T_h(u_m))|$ is bounded in $L^{p'(x)}(Q_T)$, then

$$\int_{\{|u_n|\le h\}\cap\{|u_m|\le h\}}|\phi(T_h(u_n))-\phi(T_h(u_m))|\;|\nabla T_h(u_n)-\nabla T_h(u_m)|\,dx\,dt\to0,\tag{103}$$

as m and n tend to infinity. Concerning the second term, we have $\phi_n(\cdot)$ is a continuous function, then there exists $M_1>1$ such that $\sup_{|r-s|\le1}|\phi(s)-\phi(r)|\le M_1$. Also, it's clear that

$$\forall s\in I\!R \quad \text{and} \quad \forall M_2>1 \quad \text{we have} \quad |\phi_n(s)-\phi_m(s)|\le M_2 \quad \text{for} \quad m,n\ge n_0(s,M_2).$$

Taking m and n large enough, By using (31), (43) and Young's inequality, we obtain

$$\begin{aligned}\int_{\{|u_n|>h\}\cup\{|u_m|>h\}}&|\phi_n(u_n)-\phi_m(u_m)|\;|\nabla u_n-\nabla u_m|.\chi_{\{|u_n-u_m|\le1\}}\,dx\,dt\\ &\le\int_{\{|u_n|>h\}\cup\{|u_m|>h\}}|\phi_n(u_n)-\phi_n(u_m)|^{p'(x)}\;.\chi_{\{|u_n-u_m|\le1\}}\,dx\,dt\\ &\quad+\int_{\{|u_n|>h\}\cup\{|u_m|>h\}}|\phi_n(u_m)-\phi_m(u_m)|^{p'(x)}\;.\chi_{\{|u_n-u_m|\le1\}}\,dx\,dt\\ &\quad+2\int_{\{|u_n|>h\}\cup\{|u_m|>h\}}|\nabla u_n-\nabla u_m|^{p(x)}.\chi_{\{|u_n-u_m|\le1\}}\,dx\,dt\\ &\le(M_1^{p'_+}+M_2^{p'_+})\text{meas }(\{|u_n|>h\}\cup\{|u_m|>h\})\\ &\quad+2^{p+}\int_{\{|u_n|>h-1\}\cap\{|u_n|-1\le|u_m|\le|u_n|+1\}}|\nabla u_m|^{p(x)}\,dx\,dt\end{aligned}$$

$$+2^{p+}\int_{\{|u_m|>h-1\}\cap\{|u_m|-1\leq|u_n|\leq|u_m|+1\}}|\nabla u_n|^{p(x)}\,dx\,dt\longrightarrow 0 \qquad \text{as} \quad h\to\infty. \qquad (104)$$

By combining (102) − (104), we get

$$\int_0^s\int_\Omega\big(\phi_n(u_n)-\phi_m(u_m)\big)\cdot\nabla T_1(u_n-u_m)\,dx\,dt\longrightarrow 0 \quad \text{as} \quad n,m\to\infty. \qquad (105)$$

□

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