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Generalized $\Phi-$dichotomous linear part for a class of generalized differential equations

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Abstract. A dichotomy: ordinary or exponential, is a category of conditional stability. In this disquisition, we deal with nonlinear fractional differential equations (NFDE) involving generalized Φ-exponential and Φ-ordinary dichotomous (in the sense of fractional calculus) linear part in a Banach space. By employing of the Banach fixed point principle, the satisfactory conditions are located for the existence of Φ-bounded outcomes of these equations in the real case.

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1. Introduction and preliminaries

The class of fractional differential equations are considered in preference layouts to NDE. Heterogeneities of them performance important tools and roles not only in mathematics, but also in dynamical systems, control systems, physical sciences and engineering to construct the mathematical modeling. Fractional differential equations concerning the Caputo derivative or the Riemann-Liouville fractional operators have been organized in different classes of fractional differential equations. In general, problems with fractional differential equations are great emphasis, because fractional differential equations accrue the whole information of the function in a full form [1]-[3].

There are different types of stability of fractional differential equations. Li et al.[4], considered the Mittag- Leffler stability and the Lyapunov's methods for various classes of fractional differential equations. While Deng [5] imposed acceptable statuses for the local asymptotical stability of NFDE.

Here, in this work, we generalize the dichotomy: ordinary and exponential, by utilizing the Mittag- Leffler function. Nonlinear fractional differential equations with $\Phi-$dichotomous linear part are studied in an arbitrary Banach space. We will illustrate that some equities of these equations will be effected by the analogous $\Phi-$dichotomous homogeneous linear equation. Adequate actions for the existence of $\Phi-$bounded solutions of this equations on $\mathbb{R}_+$ in situation of $\Phi-$exponential or $\Phi-$ordinary dichotomy are constructed.

During this work, let Ξ be an arbitrary Banach space with norm $| \, . \, |$ and identity $\imath$. Let $LB(\Xi)$ be the space of all linear bounded operators performing in Ξ with the norm $\| \, . \, \|$. By $\jmath$ we shall indicate $\mathbb{R}_+ = [0, \infty)$.

We introduce the following NFDE:

$$\mathcal{D}^{\alpha} w(t) = \Lambda(t) w + \phi(t, w), \tag{1}$$

where

$$\mathcal{D}^{\alpha}w(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t\frac{w(\delta)}{(t-\delta)^{\alpha}}d\delta.$$

The appropriate linear homogeneous equation can be described in the form

$$\mathcal{D}^{\alpha}w(t)=\Lambda(t)w, \tag{2}$$

and the related inhomogeneous equation can be viewed as

$$\mathcal{D}^{\alpha}w(t)=\Lambda(t)w+\varphi(t), \tag{3}$$

where $\Lambda(.):\jmath\to LB(\Xi)$, $\varphi(.):\jmath\to\Xi$ are strongly measurable and Bochner integrable on the finite subintervals of $\jmath$ and $\phi(.,.):\jmath\times\Xi\to\Xi$ is a continuous function w. r. t. t.

The outcomes of the above FDE can be defined as continuous functions $v(t)$ that are differentiable (in the sense that it is considerable. We impose the following concepts:

Definition 1.1. A function $v(.):\jmath\to\Xi$ is said to be $\Phi-$bounded on $\jmath$ if $\Phi(t)v(t)$ is bounded on $\jmath$.

Let $\mathfrak{B}_{\Phi}(\Xi)$ indicate the Banach space of all $\Phi-$bounded and continuous functions acting on Ξ with the norm

$$\parallel f\parallel_{\mathfrak{B}_{\Phi}}=\sup_{t\in\jmath}\mid\Phi(t)f(t)\mid.$$

Recall that a linear transformation P from a vector space to itself with $P^2=P$ is called a projection.

Definition 1.2. The equation (2) is said to have a $\Phi-$exponential dichotomy on $\jmath$ if there exist a pair of mutually complementary projections P_1, $P_2=\imath-P_1$ and positive constants C_1,C_2,δ_1 and δ_2 such that

$$\parallel\Phi(t)W(t)P_1W^{-1}(s)\Phi^{-1}(s)\parallel\leq C_1E_{\alpha}(-\delta_1(t-s)^{\alpha})\quad(s\leq t;\ s,t\in\jmath) \tag{4}$$

$$\parallel\Phi(t)W(t)P_2W^{-1}(s)\Phi^{-1}(s)\parallel\leq C_2E_{\alpha}(-\delta_2(s-t)^{\alpha})\quad(t\leq s;\ s,t\in\jmath) \tag{5}$$

where E_{α} is the Mittag-Leffler function, which defined as follows:

$$E_{\alpha}(\tau)=\sum_{n=0}^{\infty}\frac{\tau^n}{\Gamma(\alpha n+1)},$$

and W is the fractional Cauchy operator (2). Equation(1) is said to has a Φ-ordinary dichotomy on $\jmath$ if $\delta_1=\delta_2=0$.

The above concepts are generalizations of the usual cases, which can be found in [6, 7]. It is well known that the solution of (2), for initial condition $w(0)=1$, can be expressed as

$$w(t)=E_{\alpha}(\Lambda t^{\alpha}),\quad t\in\jmath.$$

It is well known that the Green's function is a very robust practical and theoretical tool for studying solutions of differential equations. In [8], has been introduced the concept of fractional Green's function. It has been shown that the inverse transform of the Cauchy problem is also a solution for the homogeneous equation. Therefore, we consider the principal Green function of (3) with the projections P_1 and P_2 from the definition for Φ−exponential dichotomy as follows:

$$G_\alpha(t,s) = \begin{cases} W(t)P_1W^{-1}(s) & (s < t;\ t,s \in \jmath) \\ \\ -W(t)P_2W^{-1}(s) & (t < s;\ t,s \in \jmath). \end{cases} \tag{6}$$

Clearly G_α is continuous except at $t = s$, where it has a jump discontinuity. The main connections are the Mittag- Leffler function and the fractional Green's function. These functions allow us to define the concept of the Φ−dichotomous and to find a mild solution, by utilizing some properties in the Cauchy problem.

Next, we illustrate the following assumptions:

(**A1**) For arbitrary number $\rho > 0$ and $\forall\, w \in \Xi$, we put

$$\mid \Phi(t)w \mid \leq \rho.$$

(A2) There exists a positive functions $\mu(t)$ such that $\mid \Phi(t)\phi(t,w) \mid \leq \mu(t), \quad (t \in \jmath)$.

(A3) There exists a positive function $\kappa(t)$ such that

$$\mid \Phi(t)\big(\varphi(t,w_1) - \phi(t,w_2)\big) \mid \leq \kappa(t) \mid \Phi(t)(w_1 - w_2) \mid \quad (t \in \jmath).$$

(A4) The nonnegative function $\mu(t)$ satisfies the following inequality :

$$B(\mu(t)) = \sup_{t\in\jmath} \int_t^{t+1} \mu(\tau)d\tau < \infty.$$

(A5) There are two positive constants c_1 and c_2 such that

$$B(\mu(t)) \leq c_1, \quad B(\kappa(t)) \leq c_2.$$

2. Outcomes

In this section, we establish the presence of solutions of the fractional system (1). We shall apply the following result:

Lemma 2.1. *If the linear homogeneous equation* (2) *has* Φ*-exponential dichotomy on* $\jmath$*, then the inhomogeneous equation* (3) *has, for every* Φ*-bounded function* $\varphi \in \mathfrak{B}_\Phi(\Xi)$*, at least one* Φ*-bounded solution* $w \in \mathfrak{B}_\Phi(\Xi)$.

Proof. Let (2) have a Φ-exponential dichotomy on $\jmath$. Then by (4), we have

$$\| \Phi(t)W(t)P_1W^{-1}(s)\Phi^{-1}(s)w \| \leq C_1E_\alpha(-\delta_1(t-s)^\alpha)|w|, \quad (s \leq t;\ s,t \in \jmath), \quad w \in \Xi.$$

Putting $w(s) := \Phi(s)W(s)P_1$, therefore we obtain that $\Phi(t)w(t)$ is bounded and consequently, we attain that w is Φ−bounded for Eq. (2). Since $\varphi \in \mathfrak{B}_\Phi(\Xi)$ this yields that w is Φ-bounded solution for Eq. (3). This completes the proof. □

Note that, Lemma 2.1 can be extended to include Eq. (1).

Theorem 2.2. *Let the assumptions **(A1)-(A5)** be achieved. If the linear part of* (1) *has* Φ−*exponential dichotomy on* $\mathbb{R}$ *with projections* P_1 *and* P_2 *then the equation* (1) *has a unique solution* $w(t)$*, which is defined for* $t \in \jmath = \mathbb{R}_+$ *and for which* $\mid \Phi(t)w(t) \mid \leq r, \quad (t \in \jmath,\ r > 0)$.

Proof. Define the operator $\Theta : \mathfrak{B}_\Phi(\Xi) \to \mathfrak{B}_\Phi(\Xi)$ as follows:

$$\Theta w(t) = \int_\jmath G_\alpha(t,\tau)\phi(\tau, w(\tau))d\tau \tag{7}$$

where G_α is defined by (6). Let $w(t)$ be a solution of equation(1) that appears for $t \in \jmath$ in the ball

$$\flat_{\Phi,r} = \{w :\| w \|_{\mathfrak{B}_\Phi} \leq r,\ r > 0\}.$$

In view of Lemma 2.1, the function $\phi(t, w(t))$ is $\Phi-$ bound on $\jmath$ and it follows that the solution w verifies the integral equation

$$w(t) = \Theta w(t). \tag{8}$$

Step 1. Θ maps the ball $\flat_{\Phi,r}$ into itself.
By employing **(A2), (A4)-(A5)**, a computation implies that

$$\begin{aligned}
\mid \Phi(t)\Theta w(t) \mid &\leq \left|\Phi(t) \int_\jmath G_\alpha(t,\tau)\phi(\tau, w(\tau))d\tau\right| \\
&\leq \int_\jmath \| \Phi(t)G_\alpha(t,\tau)\Phi^{-1}(\tau) \| \mid \Phi(\tau)\phi(\tau, w(\tau)) \mid d\tau \\
&= \int_{t\geq\tau} \| \Phi(t)G_\alpha(t,\tau)\Phi^{-1}(\tau) \| \mid \Phi(\tau)\phi(\tau, w(\tau)) \mid d\tau \\
&\quad\quad + \int_{t\leq\tau} \| \Phi(t)G_\alpha(t,\tau)\Phi^{-1}(\tau) \| \mid \Phi(\tau)\phi(\tau, w(\tau)) \mid d\tau \\
&\leq C_1 \int_{\tau\leq t} E_\alpha(-\delta_1(t-\tau)^\alpha)\mu(\tau)d\tau + C_2 \int_{\tau\geq t} E_\alpha(-\delta_2(\tau-t)^\alpha)\mu(\tau)d\tau.
\end{aligned}$$

By assuming $t - \tau = \varsigma$, we conclude that

$$\begin{aligned}
\mid \Phi(t)\Theta w(t) \mid &\leq C_1 \int_{\varsigma\geq 0} E_\alpha(-\delta_1(\varsigma)^\alpha)\mu(t-\varsigma)d\varsigma + C_2 \int_{\varsigma\leq 0} E_\alpha(\delta_2(\varsigma)^\alpha)\mu(t-\varsigma)d\varsigma \\
&\leq B(\mu(t-\varsigma))\big(C_1E_\alpha(-\delta_1(\varsigma)^\alpha) + C_2E_\alpha(-\delta_2(\varsigma)^\alpha)\big) \\
&\leq c_1\Big(\frac{C_1}{1-E_\alpha(-\delta_1)} + \frac{C_2}{1-E_\alpha(-\delta_2)}\Big).
\end{aligned}$$

Hence, by letting

$$c_1 \leq r\Big(\frac{C_1}{1-E_\alpha(-\delta_1)} + \frac{C_2}{1-E_\alpha(-\delta_2)}\Big)^{-1},$$

we receive

$$\mid \Phi(t)\Theta w(t) \mid\leq r.$$

Thus, the operator Θ maps the ball $\flat_{\Phi,r}$ into itself.
Step 2. The operator Θ is a contraction mapping in $\flat_{\Phi,r}$.

By employing **(A1), (A3), (A5)**, a calculation yields for $w_1, w_2 \in \flat_{\Phi,r}$. We get

$$\begin{aligned}
\mid \Phi(t)\Theta w_1(t) - \Phi(t)\Theta w_2(t) \mid &\leq \left| \Phi(t) \int_{\jmath} [G_\alpha(t,\tau)\phi(\tau, w_1(\tau)) - \phi(\tau, w_1(\tau))] d\tau \right| \\
&\leq \int_J \| \Phi(t) G_\alpha(t,\tau)\Phi^{-1}(\tau) \| \| \Phi(\tau)(\phi(\tau, w_1(\tau)) - \phi(\tau, w_2(\tau)) \mid d\tau \\
&\leq \int_{\jmath} \Big(\sup_{\tau\in\jmath} \mid \Phi(\tau)(w_1(\tau)) - w_2(\tau)) \mid \Big) \| \Phi(t) G_\alpha(t,\tau)\Phi^{-1}(\tau) \| \kappa(\tau) d\tau \\
&\leq \int_{\jmath} \Big(\sup_{\tau\in\jmath} \mid \Phi(\tau)(w_1(\tau)) - w_2(\tau)) \mid \Big) \Big(C_1 E_\alpha(-\delta_1(t-\tau)^\alpha) \\
&\qquad + C_2 E_\alpha(-\delta_2(\tau - t)^\alpha) \Big) \kappa(\tau) d\tau \\
&\leq \Big(\sup_{\tau\in\jmath} \mid \Phi(\tau)(w_1(\tau)) - w_2(\tau)) \mid \Big) \Big(\frac{C_1}{1 - E_\alpha(-\delta_1)} + \frac{C_2}{1 - E_\alpha(-\delta_2)} \Big) c_2
\end{aligned}$$

Hence, we have

$$\| \Theta w_1 - \Theta w_2 \|_{\mathfrak{B}_\Phi} \leq \Big(\frac{C_1}{1 - E_\alpha(-\delta_1)} + \frac{C_2}{1 - E_\alpha(-\delta_2)} \Big) c_2 \| w_1 - w_2 \|_{\mathfrak{B}_\Phi} .$$

Thus, by considering

$$c_2 < \Big(\frac{C_1}{1 - E_\alpha(-\delta_1)} + \frac{C_2}{1 - E_\alpha(-\delta_2)} \Big)^{-1},$$

the operator Θ is a contraction in the ball $\flat_{\Phi,r}$. In virtue of Banach's fixed point Theorem, the existence of a unique fixed point of operator Θ follows. □

Corollary 2.3. *If the conditions of Theorem 2.2 are achieved and if, furthermore, $\phi(t, 0) = 0$ $(t \in \jmath)$ then w= 0 is a unique solution of* (1) *in* $\mathfrak{B}_\Phi(\Xi)$.

Proof. Let $\phi(t, 0) = 0,\ t \in \jmath$. Then from **(A1), (A3), (A5)**, it leads

$$\begin{aligned}
\mid \Phi(t)(\phi(t, w(t)) - \phi(t, 0)) \mid &\leq \kappa(t) \mid \Phi(t) w(t) \mid \quad (t \in \jmath) \\
&\leq c_1 \rho.
\end{aligned}$$

Thus for large $\rho > 0$, every solution $w(t)$ except $w(t) \equiv 0$ $(t \in \jmath)$ will drop any ball $\flat_{\Phi,r}$.□ □

Theorem 2.4. *Let the following conditions be achieved:*
1. The linear part of (1) *has* $\Phi-$*ordinary dichotomy on* $\mathbb{R}$ *with projections* P_1 *and* P_2.
2. The function $\phi(t, w)$ *satisfies conditions* ***(A2)-(A5)***.
Then for all $r > 0$ *and sufficient small values of* c_1, c_2, *the equation* (1) *has a unique solution* $w(t)$, *which is defined for* $t \in \jmath$ *and for which* $\mid \Phi(t) w(t) \mid \leq r$ $(t \in \jmath)$.

Proof. Let $\jmath = \mathbb{R}_+$. In view of Lemma 2.1, every solution $w(t)$ of equation (1) remains in the ball $\flat_{\Phi,r}$ and verifies the integral equation

$$w(t) = \int_{\jmath} G_\alpha(t,\tau)\phi(\tau, w(\tau)) d\tau$$

and vice versa. Consider the operator $\Theta : \mathfrak{B}_\Phi(\Xi) \to \mathfrak{B}_\Phi(\Xi)$ introduced in (7).
Step 1. Θ maps the ball $\flat_{\Phi,r}$ into itself.

For $\mid \Phi(t)\Theta w(t) \mid$, we get the following estimate:

$$\mid \Phi(t)\Theta w(t) \mid \leq \left| \Phi(t) \int_{\jmath} G_\alpha(t,\tau)\phi(\tau, w(\tau))d\tau \right|.$$

By letting $c_1 \leq \frac{r}{(C_1+C_2)}$, we receive

$$\begin{aligned}
\mid \Phi(t)\Theta w(t) \mid &\leq \left| \Phi(t) \int_{\jmath} G_\alpha(t,\tau)\phi(\tau, w(\tau))d\tau \right| \\
&\leq \left(\int_{\jmath} \| \Phi(t)G_\alpha(t,\tau)\Phi^{-1}(\tau) \| \mid \Phi(\tau)\phi(\tau, w(\tau)) \mid d\tau \right) \\
&= \int_{t\leq\tau} \| \Phi(t)G_\alpha(t,\tau)\Phi^{-1}(\tau) \| \mid \Phi(\tau)\phi(\tau, w(\tau)) \mid d\tau \\
&\qquad + \int_{t\geq\tau} \| \Phi(t)G_\alpha(t,\tau)\Phi^{-1}(\tau) \| \mid \Phi(\tau)\phi(\tau, w(\tau)) \mid d\tau \\
&\leq C_2 \int_{t\leq\tau} \mu(\tau)d\tau + C_1 \int_{t\geq\tau} \mu(\tau)d\tau \\
&\leq C_2 B(\mu(t+\tau)) + C_1 B(\mu(t+\tau)) \\
&\leq (C_1+C_2)B(\mu(t+\tau)) \\
&\leq (C_1+C_2)c_1 \leq r.
\end{aligned}$$

Thus the operator Θ maps the ball $\flat_{\Phi,r}$ into itself.
Step 2. The operator Θ is a contraction mapping in $\flat_{\Phi,r}$.
For all $w_1, w_2 \in \flat_{\Phi,r}$, we may conclude that

$$\begin{aligned}
\mid \Phi(t)\Theta w_1(t) - \Phi(t)\Theta w_2(t) \mid &\leq \left| \int_{\jmath} G_\alpha(t,\tau)\Big(\phi(\tau, w_1(\tau)) - \phi(\tau, w_2(\tau))\Big)d\tau \right| \\
&\leq \int_{\jmath} \| \Phi(t)G_\alpha(t,\tau)\Phi^{-1}(\tau) \| \mid \Phi(\tau)(\phi(\tau, w_1(\tau)) - \phi(\tau, w_2(\tau)) \mid d\tau \\
&\leq \frac{1}{\Gamma(\alpha)} \int_{\jmath} \| \Phi(t)G_\alpha(t,\tau)\Phi^{-1}(\tau) \| \kappa(\tau) \mid \Phi(\tau)\big(w_1(\tau)) - w_2(\tau)\big) \mid d\tau \\
&\leq \int_{\jmath} \| \Phi(t)G_\alpha(t,\tau)\Phi^{-1}(\tau) \| \kappa(\tau)d\tau \sup_{\tau\in\jmath} \mid \Phi(\tau)(w_1(\tau)) - w_2(\tau)) \mid \\
&\leq (max\{C_1, C_2\}c_2) \sup_{\tau\in\jmath} \mid \Phi(\tau)(w_1(\tau)) - w_2(\tau)).
\end{aligned}$$

Hence

$$\| \Theta w_1 - \Theta w_2 \|_{C_\Phi} \leq (c_2 max\{C_1, C_2\}) \| w_1 - w_2 \|_{C_\Phi}.$$

Hence by $c_2 < (max\{C_1, C_2\})^{-1}$, the operator Θ is a contraction in the ball $\flat_{\Phi,r}$. By employing Banach's fixed point theorem, the existence of a unique fixed point of the operator Θ yields. □

Theorem 2.5. *Let the following conditions be achieved:*
1. The linear part of (1) has $\Phi-$exponential dichotomy on $\jmath$ with projections P_1 and P_2.
*2. The function $\phi(t, w)$ fulfills the assumptions **(A2)-(A5)**.*

Then for all $r > 0$ *and acceptable small* c_1 *and* c_2*, there exists* $\rho < r$ *such that the equation* (1) *has a unique solution* $w(t)$ *on* $\jmath$ *for which* $\zeta \in \Xi_1 = P_1\Xi$ *along* $\mid \Phi(0)\zeta \mid\leq \rho$ *with* $P_1 w(0) = \zeta$ *and* $\mid \Phi(t)w(t) \mid\leq r$ $(t \in \jmath)$.

Proof. In virtue of Lemma 2.1, implies that $w(t)$ verifies the integral equation

$$w(t) = W(t)\zeta + \int_\jmath G_\alpha(t,\tau)\phi(\tau, w(\tau))d\tau \tag{9}$$

where $\zeta = P_1 w(0)$. The converse is also true: a solution of the integral equation (9) verifies the differential equation (1) for $t \in \jmath$. Let $\zeta \in \Xi_1$ and $\mid \Phi(0)\zeta \mid\leq \rho < r$. We assume the space $\mathfrak{B}_\Phi(\Xi)$ and the operator $\Theta : \mathfrak{B}_\Phi(\Xi) \to \mathfrak{B}_\Phi(\Xi)$ defined by the formula

$$\Theta w(t) = W(t)\zeta + \int_\jmath G_\alpha(t,\tau)\phi(\tau, w(\tau))d\tau \tag{10}$$

Step 1. Θ maps the ball $\flat_{\Phi,r}$ into itself.
A computation leads to

$$\mid \Phi(t)\Theta w(t) \mid\leq\mid \Phi(t)W(t)\zeta \mid + \left|\Phi(t)\int_\jmath G_\alpha(t,\tau)\phi(\tau, w(\tau))d\tau\right|.$$

Assume that $\rho \leq \dfrac{r}{2C_1}$, then we have

$$\mid \Phi(t)W(t)\zeta \mid\leq C_1 E_\alpha(-\delta_1(t-\tau)^\alpha) \mid \Phi(0)\zeta \mid\leq C_1\rho \leq \frac{r}{2}.$$

Employing the same method in the proof of Theorem 2.2, we have

$$\left|\Phi(t)\int_\jmath G_\alpha(t,\tau)\phi(\tau, w(\tau))d\tau\right| \leq \frac{r}{2}.$$

Thus the operator Θ maps the ball $\flat_{\Phi,r}$ into itself.
Step 2. The operator Θ is a contraction mapping in $\flat_{\Phi,r}$.
Assume $w_1, w_2 \in \flat_{\Phi,r}$. We get as in the proof of Theorem2.2, the bound

$$\||\ \Theta w_1 - \Theta w_2\ \||_{\mathfrak{B}_\Phi}\leq \Big(\Big(\frac{C_1}{1-E_\alpha(-\delta_1)} + \frac{C_2}{1-E_\alpha(-\delta_2)}\Big)c_2\Big) \parallel w_1 - w_2 \parallel_{\mathfrak{B}_\Phi}.$$

By

$$c_2 < \Big(\frac{C_1}{1-E_\alpha(-\delta_1)} + \frac{C_2}{1-E_\alpha(-\delta_2)}\Big)^{-1},$$

the operator Θ is a contraction in the ball $\flat_{\Phi,r}$. Finally, by applying Banach's fixed point theorem, the existence of a unique fixed point of the operator Θ implies. □

Theorem 2.6. *Let the following conditions be satisfied:*
1. The linear part of (1) *has* $\Phi-$*ordinary dichotomy on* $\mathbb{R}_+$ *with projections* P_1 *and* P_2.
2. The function $\phi(t, w)$ *archives the conditions* ***(A2)-(A5)***.
Then for all $r > 0$ *and small values of* c_1, c_2*, there exists* $\sigma < r$ *such that the equation* (1) *has for each* $\eta \in \Xi_1 = P_1\Xi$ *with* $\mid \Phi(0) \mid\leq \sigma$ *a unique solution* $w(t)$ *on* $\jmath$ *for which* $P_1 w(0) = \eta$ *and* $\mid \Phi(t)w(t) \mid\leq r(t \in \jmath)$.

Proof. Let $\eta \in X_1$ and $\mid \Phi(0)\eta \mid \leq \sigma < r$. We concern about the space $\mathfrak{B}_\Phi(\Xi)$ and the operator $\Theta : \mathfrak{B}_\Phi(\Xi) \to \mathfrak{B}_\Phi(\Xi)$ defined by the formula

$$\Theta w(t) = W(t)\eta + \int_\jmath G_\alpha(t,\tau)\phi(\tau, w(\tau))d\tau. \tag{11}$$

First we have to establish, that the operator Θ maps the all $S_{\Phi,r}$ into itself. We obtain

$$\mid \Phi(t)\Theta w(t) \mid \leq \mid \Phi(t)W(t)\eta \mid + \left|\Phi(t)\int_\jmath G_\alpha(t,\tau)\phi(\tau, w(\tau))d\tau\right|.$$

By letting $\sigma \leq \dfrac{r}{2C_1}$, we have

$$\mid \Phi(t)W(t)\zeta \mid \leq C_1 \mid \Phi(0)\eta \mid \leq C_1\sigma \leq \frac{r}{2}.$$

Now by assuming $c_1 \leq \dfrac{r}{2max\{C_1, C_2\}}$ as in the proof of Theorem 2.4, we get

$$\left|\Phi(t)\int_\jmath G_\alpha(t,\tau)\phi(\tau, w(\tau))d\tau\right| \leq max\{C_1, C_2\}c_1 \leq \frac{r}{2}.$$

Thus the operator Q maps the ball $\flat_{\Phi,r}$ into itself.

Let $w_1, w_2 \in \flat_{\Phi,r}$. As in the proof of Theorem 2.2, we have the computation

$$\| \Theta w_1 - \Theta w_2 \|_{\mathfrak{B}_\Phi} \leq (c_2 max\{C_1, C_2\}) \| w_1 - w_2 \|_{\mathfrak{B}_\Phi}.$$

Hence by $c_2 < (max\{C_1, C_2\})^{-1}$ the operator Θ is a contraction in the ball $\flat_{\Phi,r}$. The fixed point principle of Banach it implies the existence of a unique fixed point of the operator Θ. □

Corollary 2.7. *Let the conditions of Theorem 2.6 hold and let $w_1(t)$ and $w_2(t)$ be two solutions whose initial values achieve $P_1 w_1(0) = \eta$ and $P_1 w_2(0) = \theta$. Let $C_\alpha = max\{C_1, C_2\}$.*

Then for $C_\alpha c_2 < 1$, the following condition holds:

$$\mid \Phi(t)(w_1(t) - w_2(t)) \mid \leq \frac{C_\alpha}{1 - C_\alpha c_2} \mid \Phi(0)(\eta - \theta) \mid \quad (t \in \jmath).$$

Proof. Applying the presentation (11) for the solutions w_1 and w_2 we have

$$w_1(t) - w_2(t) = W(t)(\eta - \theta) + \int_\jmath G_\alpha(t,\tau)(\phi(\tau, w_1(\tau)) - \phi(\tau, w_2(\tau)))d\tau$$

In view of Theorem 2.6, for $v(t) = \Phi(t)(w_1(t) - w_2(t))$, we realize that

$$\mid v(t) \mid \leq C_\alpha \mid \Phi(0)(\eta - \theta) \mid + C_\alpha \int_\jmath \kappa(\tau)v(\tau)d\tau,$$

where C_α is a positive constant depending on α. Let us study the equation

$$v(t) := \beta + C_\alpha \int_\jmath \kappa(\tau)v(\tau)d\tau,$$

where $\beta = C_\alpha \mid \Phi(0)(\eta - \theta) \mid$. Define the functional

$$\Omega : \mathcal{B} \to \jmath,$$

where $\mathcal{B}$ denotes the space of all bounded functions on $\jmath$ by the formula

$$(\Omega v)(t) = C_\alpha \int_\jmath \kappa(\tau) v(\tau) d\tau.$$

We proceed to determine the bound of Ω.

$$\| \Omega \| \leq C_\alpha \int_\jmath \kappa(\tau) d\tau \leq C_\alpha c_2.$$

For sufficiently small c_2, we compute that $\| \Omega \| \leq 1$. Let $\imath_c$ be the identity of the space $\mathcal{B}$. Then the equation $(\imath_c - \Omega)_v = \beta$ has a bounded solution v(t), i.e. there exists a constant $\gamma = sup_{t \in \jmath} \mid v(t) \mid < \infty$. We ought to estimate the constant γ from equation(11),

$$\gamma \leq \beta + C_\alpha \gamma \int_\jmath k(\tau) d\tau \leq \beta + C_\alpha c_2,$$

i.e.

$$\gamma \leq \frac{\beta}{1 - C_\alpha c_2}.$$

Finally, we receive

$$\mid \Phi(t)(w_1(t) - w_2(t)) \mid \leq \frac{C_\alpha \mid \Phi(0)(\eta - \theta) \mid}{1 - C_\alpha c_2}.$$

Hence the proof. □

3. Conclusion

It was shown that the Φ-exponential dichotomy of the homogenous equation (1) is a sufficient condition for the existence of Φ-bounded solutions of the inhomogeneous equation (3) and consequently for the NFDE (1) with Φ-bounded. By applying the principal of fractional Green function of (3) with the projections P_1 and P_2 from the definition for Φ-exponential dichotomy, the solution of (1) can be formulated as

$$w(t) = \int_\jmath G_\alpha(t, s) \phi(s) ds.$$

The solution of (1) remains Φ-bounded when the condition for Φ- boundedness of the function ϕ is replaced by the more general condition (**A4**).

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