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Existence of renormalized solution for a class of doubly nonlinear parabolic equations with nonstandard growth

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Abstract. We prove the existence of a renormalized solution to a class of doubly nonlinear parabolic equation

$$\frac{\partial b(x,u)}{\partial t} - \operatorname{div}(a(x,t,u,\nabla u) + \Phi(u)) = f - \operatorname{div}(F) \quad \text{in } Q,$$

where $-\operatorname{div} a(x,t,u,\nabla u)$ is a Leray-Lions operator which is coercive and which grows like $|\nabla u|^{p(x)-1}$ with respect to ∇u, but which is not restricted by any growth condition with respect to u and where $b(x,u)$ is an $C^1(\mathbb{R})$-function strictly increasing with respect u. The data f, F and u_0 respectively belong to $L^1(Q)$, $(L^{p'(\cdot)}(Q))^N$ and $L^1(\Omega)$.

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1. Introduction

We consider the following doubly nonlinear parabolic equation:

$$\begin{cases} \dfrac{\partial b(x,u)}{\partial t} - \operatorname{div}(a(x,t,u,\nabla u) + \Phi(u)) = f - \operatorname{div}(F) \quad \text{in} \quad Q \\ u(x,t) = 0 \quad \text{on} \quad \partial\Omega \times (0,T) \\ b(x,u(x,0)) = b(x,u_0(x)) \quad \text{in} \quad \Omega. \end{cases} \tag{1}$$

In the problem (1) Ω is a bounded domain of $\mathbb{R}^N$ $(N \geq 2)$, with a Lipschitz boundary, T is a positive real number, $Q = \Omega \times (0,T)$ and $-\operatorname{div}(a(x,t,u,\nabla u))$ is a Leray-Lions operator acting from the space X defined as in [7] and [26] by

$$X_T = X := \left\{ u \in L^{p^-}(0,T; W_0^{1,p(x)}(\Omega)); \quad \nabla u \in (L^{p(x)}(Q))^N \right\}, \tag{2}$$

to its dual X' for some variable exponent $p(\cdot) : \overline{\Omega} \to [1,+\infty[$ which is assumed *Log-Hölder* continuous function only dependent on the space variable x (see definitions below), b is a Carathéodory function such that for every $x \in \Omega$, $b(x,\cdot)$ is a strictly increasing C^1-function, Φ is a function which just assumed to be continuous on $\mathbb{R}$. The initial data u_0 is in $L^1(\Omega)$ such that $b(\cdot,u_0)$ belongs to $L^1(\Omega)$. The source terms f belongs to $L^1(Q)$ and F is in $(L^{p'(\cdot)}(Q))^N$.

Under our assumptions, problem (1) does not admit a weak solution since the field $a(x,t,u,\nabla u)$ do not belong to $(L^1_{loc}(Q))^N$ in general. In order to overcome this difficulty, we work with the framework of renormalized solutions. The notion of renormalized solutions was introduced by R.-J. DiPerna and P.-L. Lions in [14] for the study of the Boltzmann equation. It was adapted by many authors in the framework of a constant exponent $p(\cdot) = p$ to the study of some nonlinear elliptic or parabolic problems (see e.g.

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[11],[8],[23]). In the framework of a variable exponent $p(\cdot)$ satisfying the so-called *Log-Hölder* condition, the existence and uniqueness of renormalized solutions for parabolic equation involving $p(x)$-Laplacien for $b(x,u) = u$, $\Phi = 0$ and $F = 0$ has been established by Bendahmane et al. in [7] and by Zhang and Zhou in [26].

Recently, doubly nonlinear parabolic problem with variable exponents have attracted attention. We refer to the works [4],[2] and [9]. In [5] Azroul et al. obtained the existence of renormalized solution for the problem (1) with $F = 0$ by assuming that $p^- \geq 2$. The aim of this work is to extend the result of [23] for the case of variable nonlinearity and also extend the result [5] by working with b which depend to (x,u) and F an element of $(L^{p'(\cdot)}(Q))^N$ with $F \neq 0$ and by assuming that $H(x,t,u,\nabla u) = -\mathrm{div}(\Phi(u))$ and that $p^- > \dfrac{2N}{N+2}$. Note that $\dfrac{2N}{N+2} \leq 2$ for a dimension $N \geq 2$.

Due to the recent developments in the studies related to differential equations with $p(\cdot)$-grows, a great interest has arisen in the space $L^{p(\cdot)}$ and $W^{m,p(\cdot)}$ with variable exponent. We refer e.g. [13, 16, 17, 19] for fundamental properties of these spaces. One of the motivations behind the study of (1) comes from electro-rheological fluids called also smart fluids (more details can be found in [24],[12],[1]). Another important motivations are related to image processing (see [10]) and elasticity (see [28]) and also the so-called porous media equation which has been studied for instance in [3].

The plan of the paper is as follows: After this section, we recall in section 2 a basic background of Lebesgue and Sobolev spaces with variable exponents. In Section 3, we set a basic assumptions. In Section 4 we give the definition of solution of problem (1), finally in section 5 we prove the existence of such a solution.

2. Preliminaries

Throughout this section, we suppose that the variable exponent $p(x) : \overline{\Omega} \to [1, +\infty[$ is only dependent on the space variable x and is log-Hölder continuous on Ω, that is there is a real constant $C > 0$ such that for every $x, y \in \overline{\Omega}$, $x \neq y$ with $|x - y| \leq \dfrac{1}{2}$ one has

$$|p(x) - p(y)| \leq \frac{C}{-\log|x-y|} \tag{3}$$

and satisfying

$$1 < p^- \leq p(x) \leq p^+ < +\infty. \tag{4}$$

where $p^- := \operatorname*{ess\,inf}_{x\in\overline{\Omega}} p(x)$ and $p^+ := \operatorname*{ess\,sup}_{x\in\overline{\Omega}} p(x) < +\infty$.

We recall some definitions and basic properties of spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$. (see e.g. [13, 19, 28] for more details and results).

The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined as

$$L^{p(\cdot)}(\Omega) = \{u : \Omega \to \mathbb{R};\ u \text{ is measurable with } \rho_{p(x)}(u) := \int_\Omega |u(x)|^{p(x)}\,dx < +\infty\}.$$

Equipped with the so-called *Luxemburg norm*

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf\Big\{\lambda > 0 : \rho_{p(x)}\Big(\frac{u}{\lambda}\Big) \leq 1\Big\},$$

$L^{p(\cdot)}(\Omega)$ is a reflexive Banach space if $p^- > 1$. The dual space of $L^{p(\cdot)}(\Omega)$ can be identified with $L^{p'(\cdot)}(\Omega)$, where $\dfrac{1}{p(\cdot)} + \dfrac{1}{p'(\cdot)} = 1$. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$ the Hölder type inequality

$$\int_\Omega |uv|\,dx \le \left(\frac{1}{p^-}+\frac{1}{p'^-}\right)\|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{p'(\cdot)}(\Omega)} \tag{5}$$

holds true.

Lemma 2.1.
(i) For any $u \in L^{p(x)}(\Omega)$, we have

$$\min\{\|u\|^{p^-}_{L^{p(x)}(\Omega)}, \|u\|^{p^+}_{L^{p(x)}(\Omega)}\} \le \rho_{p(x)}(u) \le \max\{\|u\|^{p^-}_{L^{p(x)}(\Omega)}, \|u\|^{p^+}_{L^{p(x)}(\Omega)}\}.$$

(ii) If $p_1(x) \le p_2(x)$ for any $x \in \overline{\Omega}$, then $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ and the embedding is continuous.
(iii) $\|u\|^{p^-}_{L^{p(x)}(\Omega)} - 1 \le \rho_{p(x)}(u) \le \|u\|^{p^+}_{L^{p(x)}(\Omega)} + 1$.

Lemma 2.2. *Let $p(\cdot)$ satisfying (3) and (4), we have:*
(i) $\|u\|_{L^{p(x)}(\Omega)} < 1\ (=1; >1) \Leftrightarrow \rho_{p(x)}(u) < 1\ (=1; >1)$,
(ii) $\|u\|_{L^{p(x)}(\Omega)} < 1 \Rightarrow \|u\|^{p^+}_{L^{p(x)}(\Omega)} \le \rho_{p(x)}(u) \le \|u\|^{p^-}_{L^{p(x)}(\Omega)}$,
(iii) $\|u\|_{L^{p(x)}(\Omega)} > 1 \Rightarrow \|u\|^{p^-}_{L^{p(x)}(\Omega)} \le \rho_{p(x)}(u) \le \|u\|^{p^+}_{L^{p(x)}(\Omega)}$,
(iv) $\|u\|_{L^{p(x)}(\Omega)} \to 0 \Leftrightarrow \rho_{p(x)}(u) \to 0$ and $\|u\|_{L^{p(x)}(\Omega)} \to \infty \Leftrightarrow \rho_{p(x)}(u) \to \infty$.

The variable Sobolev space $W^{1,p(\cdot)}(\Omega)$ is defined in the following sense

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega);\ |\nabla u| \in L^{p(\cdot)}(\Omega)\}.$$

It is a Banach space under the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)}$$

Under the condition (3), smooth functions are dense in variable exponent Sobolev spaces and there is no confusion to define the Sobolev space with zero boundary values $W_0^{1,p(\cdot)}$ as $W_0^{1,p(\cdot)}(\Omega) := \overline{C_c^\infty(\Omega)}^{W^{1,p(\cdot)}(\Omega)}$ with respect to the norm $\|\cdot\|_{W^{1,p(\cdot)}(\Omega)}$. Assuming $p^- > 1$ the spaces $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ are reflexive and separable Banach spaces. The dual space $(W_0^{1,p(\cdot)}(\Omega))^*$ is denoted by $W^{-1,p'(\cdot)}(\Omega)$ equipped with the norm

$$\|v\|_{W^{-1,p'(\cdot)}(\Omega)} = \inf \sum_{|\alpha|\le 1} \|v_\alpha\|_{L^{p'(\cdot)}(\Omega)}$$

where the infinimum is taken on all possible decompositions

$$v = \sum_{|\alpha|\le 1} (-1)^{|\alpha|} D^\alpha v_\alpha, \quad v_\alpha \in L^{p'(\cdot)}(\Omega).$$

Let us exhibit Poincaré and Sobolev type inequalities (see [16, 17]).

Lemma 2.3. *Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$.*
(i) If (3) holds, then there exists a constant $C > 0$ depending only on Ω and the function p such that

$$\|u\|_{L^{p(\cdot)}(\Omega)} \le C\|\nabla u\|_{L^{p(\cdot)}(\Omega)} \quad \textit{for all } u \in W_0^{1,p(\cdot)}(\Omega).$$

In particular, the space $W_0^{1,p(\cdot)}(\Omega)$ has a norm $\|\cdot\|_{p(\cdot)}$ given by

$$\|u\|_{p(\cdot)} = \|\nabla u\|_{L^{p(\cdot)}(\Omega)} \textit{ for all } u \in W_0^{1,p(\cdot)}(\Omega),$$

which equivalent to $\|\cdot\|_{W^{1,p(\cdot)}(\Omega)}$.

(ii) If $p \in \mathcal{C}(\overline{\Omega})$, $1 < p^- \le p^+ < N$, $q : \Omega \to [1, +\infty)$ is measurable and $\inf_{x\in\Omega}(p^*(x) - q(x)) > 0$ with $p^*(x) := Np(x)/(N-p(x))$, then $W^{1,p(\cdot)}(\Omega)$ is continuously and compactly embedded in $L^{q(\cdot)}(\Omega)$. Moreover, a real constant $C > 0$ exists such that

$$\|u\|_{L^{q(\cdot)}(\Omega)} \le C\|u\|_{W^{1,p(\cdot)}(\Omega)} \quad \text{for all } u \in W^{1,p(\cdot)}(\Omega).$$

In particular, the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is compact.

Extending a variable exponent $p : \overline{\Omega} \to [1, +\infty[$ to $\overline{Q} \to [1, +\infty[$ by setting $p(x, t) = p(x)$ for all $(x, t) \in \overline{Q}$, we may also consider the generalized Lebesgue space which shares the same type of properties as $L^{p(x)}(\Omega)$:

$$L^{p(x)}(Q) = \Big\{u : Q \to \mathbb{R} \text{ measurable with } \int_Q |u(x,t)|^{p(x)} dxdt < \infty\Big\}.$$

We consider the functional space X defined in (2) which is a separable and reflexive Banach space endowed with the norm

$$\| u \|_X := \| u \|_{L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))} + \| \nabla u \|_{L^{p(x)}(Q)},$$

Or, the equivalent norm

$$|u|_X = \| \nabla u \|_{L^{p(x)}(Q)} \,.$$

The equivalence of the two norms is a consequence of Poincaré inequality and the continuous embedding $L^{p(x)}(Q) \hookrightarrow L^{p^-}(0, T; L^{p(x)}(\Omega))$ (see [7]). Since $0 < |\Omega| < +\infty$, we can see, by using the Hölder inequality, that the space X is continuously embedded in $L^1(Q)$.

The elements of the dual space of X denoted by X' can represent as follows: if $\Psi \in X'$, then there exists $\Phi = (\phi_1, \cdots, \phi_N) \in (L^{p'(x)}(Q))^N$ such that $\Psi = \operatorname{div}\Phi$ and

$$\langle \Psi, \Theta\rangle_{X,X'} = \int_0^T \int_\Omega \Phi . D\Theta \, dx\, dt \quad \text{for any } \ \Theta \in X.$$

By lemma 3.1. in [7], the following continuous and dense embedding

$$X \hookrightarrow_d L^{p^-}(0, T; W_0^{1,p(x)}(\Omega)), \qquad X' \hookrightarrow L^{(p^+)'}(0, T; W^{-1,p'(x)}(\Omega)),$$

and also the following continuous imbedding holds

$$\{u \in X; u_t \in X' + L^1(Q)\} \hookrightarrow C([0, T]; L^1(\Omega)). \tag{6}$$

We recall some basic results that will be used later

Lemma 2.4. (see [18],[22]) *Suppose that $1 \le p(x) < \infty$. Let $\{v_n\}_n$ be bounded in $L^{p(x)}(\Omega)$. If $v_n \to v$ a.e. in Ω, then $v_n \rightharpoonup v$ weakly in $L^{p(x)}(\Omega)$.*

Lemma 2.5. (see [15]) *Let $(u_n)_n$ be a sequence in $L^1(\Omega)$ and $u \in L^1(\Omega)$ such that $u_n \to u$ a.e. in Ω, u_n, $u \ge 0$ a.e. and $\int_\Omega u_n\, dx \to \int_\Omega u\, dx$. Then $u_n \to u$ in $L^1(\Omega)$.*

Lemma 2.6. (see [27]) *Let $\Omega \subset \mathbb{R}^N$ be measurable with finite Lebesgue measure. Suppose that $\{c_n(x)\} \subset L^\infty(\Omega)$ and $\{b_n(x)\} \subset L^1(\Omega)$ are two sequences such that*

$$c_n \to c \text{ a.e. in } \Omega \text{ and } c_n \rightharpoonup c \text{ weak-} * \text{ in } L^\infty(\Omega),$$

and

$$b_n \rightharpoonup b \quad \text{weakly in } L^1(\Omega).$$

Then

$$c_n b_n \rightharpoonup cb \quad \text{weakly in } L^1(\Omega).$$

3. Basic assumptions

Throughout this paper, we assume that the following assumptions hold true: The variable exponent $p(\cdot) : \overline{\Omega} \to [1, +\infty[$ such that

$$p(\cdot) \text{ is Log-Hölder continuous satisfying } \frac{2N}{N+2} < p^- \leq p(x) \leq p^+ < +\infty. \tag{7}$$

$$b : \Omega \times \mathbb{R} \mapsto \mathbb{R} \text{ is a Carathéodory function such that for every } x \in \Omega, \tag{8}$$

$b(x,.)$ is a strictly increasing $\mathcal{C}^1(\mathbb{R})$-function with $b(x,0) = 0$ and for any $k > 0$, there exists a constant $\lambda > 0$ and functions $A_k \in L^\infty(\Omega)$ and $B_k \in L^{p(\cdot)}(\Omega)$ such that for almost every x in Ω

$$\lambda \leq \frac{\partial b(x,s)}{\partial s} \leq A_k(x) \quad \text{and} \quad \left|\nabla_x\Big(\frac{\partial b(x,s)}{\partial s}\Big)\right| \leq B_k(x) \quad \forall s,\ |s| \leq k. \tag{9}$$

Here, $\nabla_x\Big(\frac{\partial b(x,s)}{\partial s}\Big)$ denote the gradient of $\frac{\partial b(x,s)}{\partial s}$ defined in the sense of distributions. $a:\ Q \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function such that there is $\alpha > 0$ and for any $k > 0$, there exists $\nu_k > 0$ and a function $h_k \in L^{p'(\cdot)}(Q)$ such that $\forall s \in \mathbb{R}$, $|s| \leq k$, $\forall \xi \in \mathbb{R}^N$,

$$|a(x,t,s,\xi)| \leq \nu_k\Big(h_k(x,t) + |\xi|^{p(x)-1}\Big), \tag{10}$$

$$a(x,t,s,\xi)\xi \geq \alpha|\xi|^{p(x)} \tag{11}$$

$$(a(x,t,s,\xi) - a(x,t,s,\eta))(\xi - \eta) > 0 \text{ for all } \xi \neq \eta. \tag{12}$$

$$\Phi : \mathbb{R} \to \mathbb{R}^N \quad \text{ is a continuous function,} \tag{13}$$

For the source term and initial data we assume that:

$$f \in L^1(Q), \quad F \in (L^{p'(\cdot)}(Q))^N, \tag{14}$$

$$u_0 \in L^1(\Omega) \text{ such that } b(x,u_0) \in L^1(\Omega). \tag{15}$$

4. Main results

Definition 4.1. A measurable function u defined on Q_T is called a *renormalized* solution of (1) if

$$b(x,u) \in L^\infty(0,T;L^1(\Omega)), \tag{16}$$

$$T_k(u) \in X \text{ for any } k > 0, \tag{17}$$

$$\lim_{h\to+\infty} \int_{\{(x,t)\in Q:\ h\leq|u(x,t)|\leq h+1\}} a(x,t,u,\nabla u)\cdot\nabla u\,dx\,dt = 0, \tag{18}$$

and if for every function S in $W^{2,\infty}(\mathbb{R})$ which is piecewise C^1 and such that S' has a compact support, we have in the sense of distributions:

$$\begin{aligned}\frac{\partial B_S(x,u)}{\partial t} &- \text{ div } \Big(S'(u)a(x,t,u,\nabla u)\Big) + S''(u)a(x,t,u,\nabla u)\cdot\nabla u + \text{ div } (\Phi(u)S'(u)) \\ &- S''(u)\Phi(u)\cdot\nabla u = fS'(u) + \text{ div } (S'(u)F) - S''(u)F\cdot\nabla u\end{aligned} \tag{19}$$

and

$$B_S(x,u)(t=0) = B_S(x,u_0) \quad in \quad \Omega, \tag{20}$$

where $B_S(x,z) = \displaystyle\int_0^z \frac{\partial b(x,s)}{\partial s} S'(s)ds.$

Remark 4.1. Equation (19) is formally obtained through pointwise multiplication of equation (1) by $S'(u)$. Recall that for a renormalized solution, due to (17), each term in (19) has a meaning in $L^1(Q)+X'$. Indeed, since $|T_k(u)|\le k$, we can choose k such that $\mathrm{supp}(S')\subset[-k,k]$. Then by properties of S, the functions S' and S'' are bounded in $\mathbb{R}$. Moreover, by using (9) we can see that $B_s(x,u)\in L^\infty(Q)$, $S(u)a(x,t,u,\nabla u)\in (L^{p'(\cdot)}(Q))^N$ and $S'(u)a(x,t,u,\nabla u).\nabla u\in L^1(Q)$. For the term $S'(u)a(u,\nabla u)$ for example, it is identified with $S'(u)a(T_k(u),\nabla T_k(u))$ as a consequence of (10), (16) and the fact that $S'(u)$ belongs to $L^\infty(Q)$. Thus

$$S'(u)a(T_k(u),\nabla T_k(u))\in (L^{p'(\cdot)}(Q))^N,$$

by consequent, div $(S'(u)a(u,\nabla u))\in X'$. We can prove similarly to [5] and to [23] that each term in (19) belongs either in X' or in $L^1(Q)$. (for the other terms see step 4. in section 5. Thus,

$$\frac{\partial B_S(x,u)}{\partial t} \text{ belongs to } X'+L^1(Q). \tag{21}$$

$$B_S(x,u) \text{ belongs to } X. \tag{22}$$

which implies by using (6) that $B_S(x,u)$ belongs to $C^0([0,T];L^1(\Omega))$, so the initial condition (20) makes sense.

Theorem 4.1. *Assume that* (7)–(15) *hold true, then there exists at least renormalized solution of problem* (1).

5. Existence of renormalized solution

5.1. Proof of the Theorem 4.1.

Step 1: The approximate problem. Let us define the following approximations ($n\in\mathbb{N}^*$)

$$b_n(x,s)=b(x,T_n(s))+\frac{1}{n}s\quad \text{a.e. } x\in\Omega, s\in\mathbb{R},$$

$$a_n(x,t,s,\xi)=a(x,t,T_n(s),\xi)\quad \text{a.e. } x\in Q, s\in\mathbb{R},\xi\in\mathbb{R}^N,$$

$$f_n\in C_0^\infty(Q)\cap X',\ f_n\to f\quad \text{in } L^1(Q) \text{ with } \|f_n\|_{L^1(Q)}\le\|f\|_{L^1(Q)},$$

$$u_{0_n}\in C_0^\infty(\Omega), b_n(x,u_{0_n})\to b(x,u_0) \text{ in } L^1(\Omega) \text{ with } \|b_n(x,u_{0_n})\|_{L^1(\Omega)}\le\|b(x,u_0)\|_{L^1(\Omega)}.$$

Φ_n is a a Lipschitz continuous function which converges uniformly to Φ on any compact subset of $\mathbb{R}$.

Let us consider the following approximate regularized problem

$$\begin{cases} \dfrac{\partial b_n(x,u)}{\partial t}-\mathrm{div}(a_n(x,t,u,\nabla u)+\Phi_n(u))=f_n-\mathrm{div}(F) \text{ in } Q\\ u_n(x,t)=0\quad \text{on}\quad \partial\Omega\times(0,T)\\ b(x,u_n(x,0))=b(x,u_{0_n}(x))\quad \text{in}\quad \Omega. \end{cases} \tag{23}$$

Note that b_n verifies (9) and that a_n is a Carathéodory function verifying (10),(11) and (12), which implies that there exists $\beta_n>0$ and a function $C_n\in(L^{p'(\cdot)}(Q))$ such that

$$|a_n(x,t,s,\xi)|\le C_n(x,t)+\beta_n|\xi|^{\gamma-1}\quad \text{a.e. in } Q,\ s\in\mathbb{R},\ \xi\in\mathbb{R}^N,$$

with $\gamma=p^-$ if $|\xi|\le1$ and $\gamma=p^+$ if $|\xi|>1$. By using classical results (see e.g. [22]), we can see that the problem (23) admits a least weak solution $u_n\in X$.

Step 2: Some estimations. Throughout this work, C denote various positive constant not depending on n and k which may vary from line to line.
For $k > 0$ and $\tau \in [0, T]$, if we take $T_k(u_n)\chi_{(0,\tau)}$ as a test function in (23), one obtain

$$\int_\Omega \overline{B}_k^n(x, u_n)(\tau)dx + \int_{Q_\tau} a_n(x, t, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla T_k(u_n)dxdt + \int_{Q_\tau} F \cdot \nabla T_k(u_n)dxdt = \int_{Q_\tau} \Phi_n(T_k(u_n)\nabla T_k(u_n)dxdt + \int_{Q_\tau} f_n T_k(u_n)dxdt + \int_\Omega \overline{B}_k^n(x, u_{0_n})dx, \tag{24}$$

where $\overline{B}_k^n(x, r) = \int_0^r T_k(s)\frac{\partial b_n(x, s)}{\partial s}\, ds$, note that $0 \le \overline{B}_k^n(x, u_n)$ and

$$0 \le \overline{B}_k^n(x, u_{0_n}) \le k \int_\Omega |b_n(x, u_{0_n})|\, dx \le k\|b(x, u_0)\|_{L^1(\Omega)},$$

The Lipschitz character of Φ_n and Stokes' formula together with the boundary condition in (23) make it possible to obtain

$$\int_0^\tau \int_\Omega \Phi_n(u_n).\nabla T_k(u_n)\, dx\, dt = 0, \quad \text{a.e. } \tau \in]0, T[,$$

by using Young's inequality and (11) we obtain

$$\begin{aligned} \alpha \int_{Q_\tau} |\nabla T_k(u_n)|^{p(x)}dxdt &\le k(\|f_n\|_{L^1(Q)} + \|b_n(x, u_{0_n})\|_{L^1(\Omega)}) + \\ &\quad \frac{\alpha}{2} \int_{Q_\tau} |\nabla T_k(u_n)|^{p(x)}dxdt + C_\alpha \int_{Q_\tau} |F|^{p'(x)}dxdt \\ &\le Ck + \frac{\alpha}{2} \int_{Q_\tau} |\nabla T_k(u_n)|^{p(x)}\, dx\, dt + C_\alpha \int_{Q_\tau} |F|^{p'(x)}dxdt \\ &\le Ck + \frac{\alpha}{2} \int_{Q_\tau} |\nabla T_k(u_n)|^{p(x)}\, dx\, dt, \end{aligned} \tag{25}$$

where C_α denote a positive constant which depends to p^+ and p^- but not depending on n and k. (25) implies that $\int_{Q_\tau} |\nabla T_k(u_n)|^{p(x)}\, dx\, dt \le Ck$ and by virtue of lemma (2.1),

$$\|\nabla T_k(u_n)\|_{L^{p(\cdot)}(Q_\tau)} \le Ck^{\frac{1}{p^-}}, \tag{26}$$

by consequent

$$\|T_k(u_n)\|_X \le Ck^{\frac{1}{p^-}}. \tag{27}$$

Taking in mind (24), we deduce that

$$\int_\Omega \overline{B}_k^n(x, u_n)(\tau)\, dx \le C. \tag{28}$$

We have by Hölder inequality for $k > 1$,

$$\begin{aligned} k \text{ meas}\{(x, t) :\ |u_n| > k\} &= \int_{\{|u_n|>k\}} |T_k(u_n)|\, dx\, dt \le \int_{Q_T} |T_k(u_n)|\, dx\, dt \\ &\le 2\|1\|_{L^{p'(x)}(Q_T)}.\|T_k(u_n)\|_{L^{p(x)}(Q_T)} \\ &\le 2(|Q_T| + 1)^{\frac{1}{p'^-}} \|T_k(u_n)\|_X \\ &\le Ck^{\frac{1}{p^-}}, \end{aligned}$$

by consequent

$$\lim_{k\to\infty} \text{meas}\big\{|u_n| > k\big\} = 0 \quad \text{uniformly with respect to } n. \tag{29}$$

If we multiply in (23) by $\gamma_k'(b_n(x,u_n))$ where γ is a $C^2(\mathbb{R})$ nondecreasing function such that $\gamma(s) = s$ for $|s| \le \frac{k}{2}$ and $\gamma(s) = k$ for $|s| > k$, remark that γ_k' and γ_k'' has compact support. we can deduce with a similar manner to that of [5] that $\gamma_k(b_n(x,u_n))$ is bounded in X and $\dfrac{\partial \gamma_k(b_n(x,u_n))}{\partial t}$ is bounded in $X' + L^1(Q)$ independently of n. Note that the condition (7) ensures that $\big(W_0^{1,p(x)}(\Omega), L^2(\Omega), W^{-1,p'(x)}(\Omega)\big)$ is a Gefland triple. An Aubin's type lemma (see corollary 4. in [25]) implies that for any $k > 1$ and any $n \ge k$, $\gamma_k(b_n(x,u_n))$ lies in a compact set in $L^1(Q)$. Proceeding as in [5], we deduce that there is a measurable function u defined on Q such that $b_n(x,u_n) \to b(x,u)$ a.e. in Q and $u_n \to u$ a.e. in Q for a subsequence. consequently $T_k(u_n) \to T_k(u)$ a.e.in Q.
On the other hand, by (27) $(T_k(u_n))_n$ is bounded in X, then for a subsequence

$$T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } X.$$

Also, we can deduce by (10) that the sequence $(a(x,t,T_k(u_n),\nabla T_k(u_n))_n$ is bounded in $(L^{p'(\cdot)}(Q))^N$.
Summing up, there exists $\overline{a}_k \in (L^{p'(\cdot)}(Q))^N$

$$\begin{aligned}
&u_n \to u \text{ a.e. in } Q,\\
&b_n(x,u_n) \to b(x,u) \text{ a.e. in } Q,\\
&T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } X,\\
&a(x,t,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup \overline{a}_k \text{ weakly in } (L^{p'(\cdot)}(Q))^N.
\end{aligned} \tag{30}$$

Moreover, due to the almost everywhere convergence of u_n and $b(x,u_n)$ to u and $b(x,u)$ in Q, we can pass to the lim inf in (28) as n tends to ∞, to obtain

$$\frac{1}{k}\int_\Omega B_k(x,u)\,dx \le C.$$

The definition of B_k and the fact that $\dfrac{1}{k}B_k(x,u)$ converges pointwise to $b(x,u)$ as k tends to ∞, implies that

$$b(x,u) \in L^\infty(0,T;L^1(\Omega)).$$

Let $h > 0$, taking $T_{h+1}(u_n) - T_h(u_n)$ as a test function in (23) we obtain

$$\begin{aligned}
&\int_\Omega B_h^n(x,u_n)(T)\,dx + \int_Q a(x,t,u_n,\nabla u_n)\cdot(\nabla T_{h+1}(u_n) - \nabla T_h(u_n))\,dx\,dt\\
&\quad + \int_Q \Phi_n(u_n).\nabla(T_{h+1}(u_n) - T_h(u_n))\,dx\,dt + \int_Q F.\nabla(T_{h+1}(u_n) - T_h(u_n))\,dx\,dt\\
&= \int_Q f_n(T_{h+1}(u_n) - T_h(u_n))\,dx\,dt + \int_\Omega B_h^n(x,u_{0_n})\,dx,
\end{aligned}$$

where $B_h^n(x,u_n)(r) = \int_0^r \dfrac{\partial b_n(x,s)}{\partial s}(T_{h+1}(s) - T_h(s))\,ds$.
Note that $T_{h+1} - T_h$ is Lipschitz continuous function verifying $\nabla(T_{h+1}(u_n) - T_h(u_n)) = \chi_{\{h\le|u_n|\le h+1\}}\nabla u_n$, $\|T_{h+1} - T_h\|_{L^\infty(\mathbb{R})} \le 1$ and $T_{h+1}(s) - T_h(s) \underset{h\to\infty}{\to} 0$ for any s.
The Lipschitz character of Φ_n and Stokes' formula together with the boundary condition

in (23) make it possible to obtain again

$$\int_0^\tau \int_\Omega \Phi_n(u_n)\cdot\nabla(T_{h+1}(u_n)-T_h(u_n))\,dx\,dt=0,\ \text{a.e. } \tau\in]0,T[.$$

Since $B_h^n(x,r)\geq 0$ and

$$\begin{aligned} a_n(x,t,u_n,\nabla u_n).\nabla(T_{h+1}(u_n)-T_h(u_n)) &= a(x,t,u_n,\nabla u_n).\nabla(T_{h+1}(u_n)-T_h(u_n)) \\ &= a(x,t,u_n,\nabla u_n).\nabla(u_n)\chi_{\{h\leq|u_n|\leq h+1\}}, \end{aligned}$$

hence for a.e. $(x,t)\in Q$ and $h+1\leq n$, we can write

$$\begin{aligned} &\int_{\{h\leq|u_n|\leq h+1\}} a(x,t,u_n,\nabla u_n)\cdot\nabla(u_n)dxdt \\ &\leq \int_Q f_n(T_{h+1}(u_n)-T_h(u_n))dxdt+\int_Q F.\nabla(T_{h+1}(u_n)-T_h(u_n))dxdt+\int_\Omega B_h^n(x,u_{0_n})dx. \end{aligned} \tag{31}$$

With the help of (27) we can write

$$T_{h+1}(u_n)-T_h(u_n)\underset{h\to\infty}{\rightharpoonup} T_{h+1}(u)-T_h(u) \text{ weakly in } X.$$

Thus

$$\int_Q F.\nabla(T_{h+1}(u_n)-T_h(u_n))\,dx\,dt \underset{h\to\infty}{\to} 0.$$

which implies by using (30) and Lebesgue's convergence theorem that

$$\begin{aligned} \limsup_{n\to\infty}\int_{\{h\leq|u_n|\leq h+1\}} & a(x,t,u_n,\nabla u_n)\cdot\nabla(u_n)\,dx\,dt\leq \\ & \int_Q f(T_{h+1}(u)-T_h(u))dxdt+\int_\Omega B_h(x,u_0)dx, \end{aligned}$$

the convergence everywhere of $T_{h+1}-T_h$ to 0 as h tends to ∞, (9) and the fact that $b(x,u_0)\in L^1(\Omega)$ and allows us by using Lebesgue's convergence theorem,to conclude that

$$\lim_{h\to\infty}\limsup_{n\to\infty}\int_{\{h\leq|u_n|\leq h+1\}} a(x,t,u_n,\nabla u_n)\cdot\nabla(u_n)\,dx\,dt=0. \tag{32}$$

and by (11)

$$\lim_{h\to\infty}\limsup_{n\to\infty}\int_{\{h\leq|u_n|\leq h+1\}} |\nabla u_n|^{p(x)}\,dx\,dt=0. \tag{33}$$

Step 3: Convergence of the gradient of truncations. Our aim is to prove that

$$\overline{a}_k=a(x,t,T_k(u_n),\nabla T_k(u_n))\quad \text{a.e. in } Q, \tag{34}$$

and that as n tends to infinity

$$a(x,t,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u_n)\rightharpoonup a(x,t,T_k(u),\nabla T_k(u))\cdot\nabla T_k(u) \text{ weakly in } L^1(Q). \tag{35}$$

For this end, we need to apply the following lemma, which is more general than e.g. the version in [21].

Lemma 5.1. *Assuming that* (10)- (12) *holds, and let* $(v_n)_n$ *be a sequence of* X *such that*

$$(v_n) \text{ is bounded in } X \text{ such that } v_n\to v \text{ weakly in } X \text{ and a.e. in } Q. \tag{36}$$

and

$$\int_Q \Big(a(x,t,v_n,\nabla v_n)-a(x,t,v_n,\nabla v)\Big)\cdot(\nabla v_n-\nabla v)dxdt \underset{n\to\infty}{\longrightarrow} 0. \tag{37}$$

Then

$$v_n \to v \text{ strongly in } X.$$

Proof. Let $D_n = \big(a(x,t,v_n,\nabla v_n) - a(x,t,v_n,\nabla v)\big)\cdot(\nabla v_n - \nabla v)$, thanks to (12) we have D_n is a positive function. In view of (37) and lemma (2.5), we get $D_n \to 0$ in $L^1(Q)$ as $n\to\infty$. Extracting a subsequence still denoted by v_n such that: $v_n \rightharpoonup v$ in X, $v_n \to v$ a.e in Q, and $D_n \to 0$ a.e in Q. Then there exists a subset B in Q with measure zero such that

$$|v_n(x,t)| < \infty, |\nabla v_n(x,t)| < \infty,\ v_n(x,t) \to\ v(x,t) \ \text{ and } D_n(x,t)\to 0, \forall (x,t)\in Q\setminus B.$$

It follows that there is a constant $C_{x,t}$ without dependence on n such that

$$\begin{aligned} D_n &= \big(a(x,t,v_n,\nabla v_n) - a(x,t,v_n,\nabla v)\big)\cdot(\nabla v_n - \nabla v)\\ &\geq \alpha|\nabla v_n|^{p(x)} - C_{x,t}\Big(1 + |\nabla v_n|^{p(x)-1} + |\nabla v_n|\Big), \end{aligned}$$

thus, we obtain

$$D_n \geq |\nabla v_n|^{p(x)}\Big(\alpha - \frac{C_{x,t}}{|\nabla v_n|^{p(x)}} - \frac{C_{x,t}}{|\nabla v_n|} - \frac{C_{x,t}}{|\nabla v_n|^{p(x)-1}}\Big).$$

The sequence $(\nabla v_n)_n$ is bounded almost everywhere in Q. Indeed, if $|\nabla v_n| \to \infty$ in a measurable subset $E \in Q$ then

$$\lim_{n\to\infty}\int_Q D_n\,dx\,dt \geq \lim_{n\to\infty}\int_E |\nabla v_n|^{p(x)}\big(\alpha - \frac{C_{x,t}}{|\nabla v_n|^{p(x)}} - \frac{C_{x,t}}{|\nabla v_n|} - \frac{C_{x,t}}{|\nabla v_n|^{p(x)-1}}\big)\,dx\,dt = \infty,$$

which is absurd since $D_n \to 0$ in $L^1(Q)$. Let ξ^* a cluster point of $(\nabla v_n)_n$, we have $|\xi^*| < \infty$ and by the continuity of the Carathéodory function $a(x,t,\cdot,\cdot)$, we obtain

$$\Big(a(x,t,v_n,\xi^*) - a(x,t,v,\nabla v)\Big)\cdot(\xi^* - \nabla v) = 0,$$

thanks to (12), we have $\xi^* = \nabla v$, the uniqueness of the cluster point means that for the whole sequence

$$\nabla v_n \to \nabla v \quad \text{a.e in } Q. \tag{38}$$

Since v_n is bounded in X, we can easily prove that $(a(x,t,v_n,\nabla v_n))_n$ is bounded in $(L^{p'(\cdot)}(Q))^N$ and that $a(x,t,v_n,\nabla v_n) \to a(x,t,v,\nabla v)$ a.e in Q, in view of the Lemma (2.4) we can establish that

$$a(x,t,v_n,\nabla v_n) \rightharpoonup a(x,t,v,\nabla v) \text{ in } (L^{p'(\cdot)}(Q))^N. \tag{39}$$

Remark that $a(x,t,v_n,\nabla v_n)\cdot\nabla v_n \geq 0$ and $a(x,t,v_n,\nabla v_n)\cdot\nabla v_n \to a(x,t,v,\nabla v)\cdot\nabla v$ a.e. in Q. By using (36), (37), Vitali's theorem and (39), we deduce that

$$\int_Q a(x,t,v_n,\nabla v_n)\cdot\nabla v_n dxdt \longrightarrow \int_Q a(x,t,v,\nabla v)\cdot\nabla v dxdt. \tag{40}$$

We set $y_n = a(x,t,v_n,\nabla v_n)\cdot\nabla v_n$ and $y = a(x,t,v,\nabla v)\cdot\nabla v$. By using (38) and (40) we deduce that

$$y_n \geq 0, \quad y_n \to y \text{ a.e. in } Q, y\in L^1(Q), \text{ and } \int_Q y_n \to \int_Q y.$$

Using lemma(2.5), we obtain $y_n \to y$ in $L^1(Q)$ i.e.

$$a(x,t,v_n,\nabla v_n)\cdot\nabla v_n \to a(x,t,v,\nabla v)\cdot\nabla v \text{ in } L^1(Q). \tag{41}$$

According (41) to the condition (11), we obtain, using Vitali's theorem

$$\nabla v_n \to \nabla v \quad \text{in} \quad (L^{p(\cdot)}(Q))^N,$$

thus, $v_n \to v$ in X, which finishes our proof. □

In order to deal with the time derivative of truncations, we introduce for fixed $k > 0$ a time regularization of the function $T_k(u)$. This kind of regularization has been first introduced by Landes [22]. Namely, for a fixed $\mu > 0$ and a given $w \in X$, we set

$$w_\mu(x,t) = \mu \int_{-\infty}^{t} \overline{w}(x,s)\exp(\mu(s-t))ds \qquad \text{where } \overline{w}(x,s) = w(x,s)\chi_{(0,T)}(s).$$

Applying this regularization to the truncation $T_k(u)$, following [7, 26] we have

$$\begin{aligned} &(T_k(u))_\mu \in X \cap L^\infty(Q) \quad \text{with } \|T_k(u))_\mu\|_{L^\infty(Q)} \le k,\\ &(T_k(u))_\mu \to T_k(u) \text{ a.e. in } Q,\ \text{weak} - * \text{ in } L^\infty(Q) \text{ and strongly in } X \text{ as } \mu \to \infty. \end{aligned} \tag{42}$$

Moreover,

$$\frac{\partial (T_k(u))_\mu}{\partial t} = \mu(T_k(u) - (T_k(u))_\mu) \text{ and } T_k(u))_\mu(t=0) = v_0^\mu, \tag{43}$$

where the sequence $v_0^\mu \subset L^\infty(\Omega) \cap W_0^{1,p(\cdot)}(\Omega)$ such that $v_0^\mu \to T_k(u_0)$ a.e. in Q. Remark that by (43), we have $\dfrac{\partial (T_k(u))_\mu}{\partial t} \in X$.

We set $w_\mu^n = (T_k(u_n) - (T_k(u))_\mu)$. In view of (30) and (42), we have

$$\begin{aligned} &w_\mu^n \in X \cap L^\infty(Q) \text{ with } \|w_\mu^n\|_{L^\infty(Q)} \le 2k,\\ &w_\mu^n \underset{n\to\infty}{\to} T_k(u) - (T_k(u))_\mu \text{ a.e. in } Q,\ \text{weak-} * \text{ in } L^\infty(Q) \text{ and strongly in } X. \end{aligned} \tag{44}$$

Fix $k > 0$, for $h > k$ we consider the function $S_h \in C^\infty(\mathbb{R})$ such that:
$S_h(0) = 0$, $S'_h(r) = 1$ if $|r| \le h$, $\text{supp}(S'_h) \subset [-h-1, h+1]$ and $\|(S''_h)\|_{L^\infty(\mathbb{R})} \le 1$.
We test in (23) by $S'_h(u_n)w_\mu^n$(we denote $a(x,t,u,\nabla u)$ by $a(u,\nabla u)$ for simplicity) to obtain

$$\begin{aligned} &\int_0^T \int_0^t \langle \frac{\partial b_n(x,u_n)}{\partial t}, S'_h(u_n)w_\mu^n\rangle\, ds\, dt + \int_0^T \int_0^t \int_\Omega S'_h(u_n)a_n(u_n,\nabla u_n)\cdot \nabla w_\mu^n dxdsdt\\ &\quad + \int_0^T \int_0^t \int_\Omega S''_h(u_n)w_\mu^n a_n(u_n,\nabla u_n)\cdot \nabla u_n dxdsdt\\ &\quad + \int_0^T \int_0^t \int_\Omega S'_h(u_n)\Phi_n(u_n)\cdot \nabla w_\mu^n dxdsdt + \int_0^T \int_0^t \int_\Omega S''_h(u_n)w_\mu^n \Phi_n(u_n)\cdot \nabla u_n dxdsdt\\ &= \int_0^T \int_0^t \int_\Omega f_n S'_h(u_n)w_\mu^n dxdsdt + \int_0^T \int_0^t \int_\Omega S'_h(u_n)F\cdot \nabla w_\mu^n dxdsdt\\ &\quad + \int_0^T \int_0^t \int_\Omega S''_h(u_n)w_\mu^n F\cdot \nabla u_n dxdsdt. \end{aligned} \tag{45}$$

Firstly, with the same proof as in [23], lemma 3.2, we can prove the following estimate

$$\liminf_{\mu\to\infty} \lim_{n\to\infty} \int_0^T \int_0^s \langle \frac{\partial b_n(x,u_n)}{\partial t}, H(u_n)(T_k(u_n) - (T_k(u))_\mu)\rangle\, dt\, ds \ge 0. \tag{46}$$

where H is a positive function with compact support belonging to $W^{1,\infty}(\mathbb{R})$ and $\langle , \rangle$ denotes the duality pairing between $X \cap L^\infty(Q)$ and $X' + L^1(Q)$.
In view of (46) and by taking $H = S'_h$ for a fixed $h \ge k$, we have

$$\liminf_{\mu\to\infty} \lim_{n\to\infty} \int_0^T \int_0^s \langle \frac{\partial b_n(x,u_n)}{\partial t}, S'_h(u_n)(T_k(u_n) - (T_k(u))_\mu)\rangle\, dt\, ds \ge 0. \tag{47}$$

Since S'_h is smooth and bounded with $\text{supp}(S'_h) \subset [-h-1, h+1]$, one has by (44) for $n \ge h+1$

$$S'_h(u_n)\Phi_n(u_n)\cdot \nabla w_\mu^n = S'_h(u_n)\Phi_n(T_{h+1}(u_n))\cdot \nabla w_\mu^n, \quad \text{a.e. in } Q,$$

by the convergence a.e. of u_n to u in Q, the character of Φ_n and (44), we deduce that

$$S_h'(u_n)\Phi_n(T_{h+1}(u_n)) \underset{n\to\infty}{\to} S_h'(u)\Phi(T_{h+1}(u)), \quad \text{a.e. in } Q, \ *\text{-weak in } L^\infty(Q). \tag{48}$$

By help to (44), we deduce that

$$\begin{aligned}
&\lim_{\mu\to\infty}\lim_{n\to\infty}\int_0^T\int_0^s\int_\Omega S_h'(u_n)\Phi_n(u_n)\cdot\nabla w_\mu^n\,dx\,dt\,ds \\
&\quad= \lim_{\mu\to\infty}\int_0^T\int_0^s\int_\Omega S_h'(u)\Phi(T_{h+1}(u))\cdot\nabla(T_k(u)-(T_k(u))_\mu)\,dx\,dt\,ds \quad = 0
\end{aligned} \tag{49}$$

Also we have

$$S_h''(u_n)w_\mu^n\Phi_n(u_n)\cdot\nabla u_n = S_h''(u_n)w_\mu^n\Phi_n(T_{h+1}(u_n))\cdot\nabla T_{h+1}(u_n), \quad \text{a.e. in } Q,$$

by similar arguments as for (49), we deduce that

$$\begin{aligned}
&\lim_{\mu\to\infty}\lim_{n\to\infty}\int_0^T\int_0^s\int_\Omega S_h''(u_n)w_\mu^n\Phi_n(u_n)\cdot\nabla u_n\,dx\,dt\,ds \\
&\quad= \lim_{\mu\to\infty}\int_0^T\int_0^s\int_\Omega S_h''(u)(T_k(u)-(T_k(u))_\mu)\Phi(T_{h+1}(u))\cdot\nabla T_{h+1}(u)\,dx\,dt\,ds \\
&\quad= 0
\end{aligned} \tag{50}$$

by definition of f_n and S_h,the fact that $u_n \to u$ a.e. in Q and by help to (44), we have

$$\begin{aligned}
&\lim_{\mu\to\infty}\lim_{n\to\infty}\int_0^T\int_0^s\int_\Omega f_nS_h'(u_n)w_\mu^n\,dx\,dt\,ds \\
&\quad= \lim_{\mu\to\infty}\int_0^T\int_0^s\int_\Omega fS_h'(u)(T_k(u)-(T_k(u))_\mu)\,dx\,dt\,ds \quad = 0
\end{aligned} \tag{51}$$

and

$$\begin{aligned}
&\lim_{\mu\to\infty}\lim_{n\to\infty}\int_0^T\int_0^s\int_\Omega S_h'(u_n)F\cdot\nabla w_\mu^n + S_h''(u_n)w_\mu^nF\cdot\nabla u_n\,dx\,dt\,ds \\
&\quad= \lim_{\mu\to\infty}\int_0^T\int_0^s\int_\Omega S_h'(u)F\cdot\nabla(T_k(u)-(T_k(u))_\mu) \\
&\qquad + S_h''(u)(T_k(u)-(T_k(u))_\mu)F\cdot\nabla T_{h+1}(u)\,dx\,dt\,ds.
\end{aligned}$$

The fact that $S_h'(u)F \in (L^{p'(\cdot)}(Q))^N$ and that $S_h''(u)F\cdot\nabla T_{h+1}(u) \in L^1(Q)$ (since $\nabla T_{h+1}(u) \in L^{p(\cdot)}(Q)$) implies with (42) that

$$\begin{aligned}
&\lim_{\mu\to\infty}\int_0^T\int_0^s\int_\Omega S_h'(u)F\cdot\nabla(T_k(u)-(T_k(u))_\mu) \\
&\qquad + S_h''(u)(T_k(u)-(T_k(u))_\mu)F\cdot\nabla T_{h+1}(u)\,dx\,dt\,ds = 0.
\end{aligned} \tag{52}$$

On other hand, we have $\operatorname{supp}(S_h'') \subset [-h-1,-h]\cup[h,h+1]$ which implies that

$$\begin{aligned}
&\Big|\int_0^T\int_0^t\int_\Omega S_h''(u_n)w_\mu^n a_n(u_n,\nabla u_n)\cdot\nabla u_n\,dx\,ds\,dt\Big| \\
&\qquad \le T\|S_h''\|_{L^\infty(\mathbb{R})}\|w_\mu^n\|_{L^\infty(Q)}\int_{\{h\le|u_n|\le h+1\}} a_n(u_n,\nabla u_n)\cdot\nabla u_n\,dx\,dt \\
&\qquad \le 2kT\int_{\{h\le|u_n|\le h+1\}} a_n(u_n,\nabla u_n)\cdot\nabla u_n\,dx\,dt.
\end{aligned}$$

The above inequality with (32) make it possible to have

$$\lim_{h\to\infty}\limsup_{\mu\to\infty}\limsup_{n\to\infty}\Big|\int_0^T\int_0^t\int_\Omega S_h''(u_n)w_\mu^n a_n(u_n,\nabla u_n)\cdot\nabla u_n\,dx\,ds\,dt\Big|$$
$$\leq C\lim_{h\to\infty}\limsup_{n\to\infty}\int_{\{h\leq|u_n|\leq h+1\}} a_n(u_n,\nabla u_n)\,dx\,dt \quad = 0. \tag{53}$$

Where C is a constant independent of n and h.
Due to (47)–(53), we are in a position to pass to the limit-sup when n tends to infinity, then to the limit-sup when μ tends to infinity and then to the limit as h tends to infinity in (45). We obtain by using (44)

$$\lim_{h\to\infty}\limsup_{\mu\to\infty}\limsup_{n\to\infty}\int_0^T\int_0^t\int_\Omega S_h'(u_n)a_n(u_n,\nabla u_n)\cdot\nabla(T_k(u_n)-(T_k(u))_\mu)dxdsdt\leq 0.$$

By definition of the function S_h, we can write for $k\leq n$ and $k\leq h$ the following identification: $S_h'(u_n)a_n(u_n,\nabla u_n)\nabla T_k(u_n)=a(u_n,\nabla u_n)\nabla T_k(u_n)$. Thus, the above inequality implies that

$$\limsup_{n\to\infty}\int_0^T\int_0^t\int_\Omega a_n(u_n,\nabla u_n)\cdot\nabla T_k(u_n)\,dx\,ds\,dt$$
$$\leq\lim_{h\to\infty}\limsup_{\mu\to\infty}\limsup_{n\to\infty}\int_0^T\int_0^t\int_\Omega S_h'(u_n)a_n(u_n,\nabla u_n)\cdot\nabla(T_k(u))_\mu\,dx\,ds\,dt. \tag{54}$$

For the right side term of (55), we have for $n\geq h+1$:
$S_h'(u_n)a_n(u_n,\nabla u_n)=a(T_{h+1}(u_n),\nabla T_{h+1}(u_n))$ a.e. in Q and due to (30) it follows that

$$S_h'(u_n)a_n(u_n,\nabla u_n)\rightharpoonup S_h'(u)\overline{a}_{h+1},\quad \text{weakly in } (L^{p'(\cdot)}(Q))^N \quad \text{when } n\to\infty.$$

The strong convergence of $(T_k(u))_\mu \underset{\mu\to\infty}{\to} T_k(u)$ in X allows us to write

$$\lim_{\mu\to\infty}\lim_{n\to\infty}\int_0^T\int_0^t\int_\Omega S_h'(u_n)a_n(u_n,\nabla u_n)\cdot\nabla(T_k(u))_\mu\,dx\,ds\,dt$$
$$=\int_0^T\int_0^t\int_\Omega S_h'(u)\overline{a}_{h+1}\cdot\nabla T_k(u)\,dx\,ds\,dt$$
$$=\int_0^T\int_0^t\int_\Omega \overline{a}_{h+1}\cdot\nabla T_k(u)\,dx\,ds\,dt. \tag{55}$$

But for $k\leq h$, we have

$$a(T_{h+1}(u_n),\nabla T_{h+1}(u_n))\chi_{\{|u_n|<k\}}=a(T_k(u_n),\nabla T_k(u_n))\chi_{\{|u_n|<k\}} \text{ a.e. in } Q.$$

Passing to the limit as n tends to ∞ to obtain

$$\overline{a}_{h+1}\chi_{\{|u|<k\}}=\overline{a}_k\chi_{\{|u|<k\}} \text{ a.e. in } Q,$$

and by consequent,

$$\overline{a}_{h+1}\cdot\nabla(T_k(u))=\overline{a}_k\cdot\nabla(T_k(u)) \text{ a.e. in } Q.$$

Thus, we have the following key estimate

$$\limsup_{n\to\infty}\int_0^T\int_0^t\int_\Omega a(u_n,\nabla T_k(u_n))\cdot\nabla T_k(u_n)dxdsdt\leq\int_0^T\int_0^t\int_\Omega \overline{a}_k\cdot\nabla(T_k(u))dxdsdt. \tag{56}$$

On other hand, we deduce with the help of (12) that

$$
\begin{aligned}
0 &\le \int_0^T \int_0^t \int_\Omega \big[a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u))\big] \cdot \big[\nabla T_k(u_n) - \nabla T_k(u)\big] dxdsdt \\
&= \int_0^T \int_0^t \int_\Omega a(T_k(u_n), \nabla T_k(u_n)) \cdot \nabla T_k(u_n)\, dx\, ds\, dt \\
&\quad - \int_0^T \int_0^t \int_\Omega a(T_k(u_n), \nabla T_k(u_n)) \cdot \nabla T_k(u)\, dx\, ds\, dt \\
&\quad - \int_0^T \int_0^t \int_\Omega a(T_k(u_n), \nabla T_k(u)) \cdot (\nabla T_k(u_n) - \nabla T_k(u))\, dx\, ds\, dt \\
&= J_n^1 + J_n^2 + J_n^3.
\end{aligned} \tag{57}
$$

Using (56) we have

$$
\begin{aligned}
\limsup_{n\to\infty} J_n^1 &= \limsup_{n\to\infty} \int_0^T \int_0^t \int_\Omega a(u_n, \nabla T_k(u_n)) \cdot \nabla T_k(u_n) dxdsdt \\
&\le \int_0^T \int_0^t \int_\Omega \overline{a}_k \cdot \nabla(T_k(u)) dxdsdt.
\end{aligned} \tag{58}
$$

In view of (30), we deduce that

$$
\lim_{n\to\infty} J_n^2 = -\int_0^T \int_0^t \int_\Omega \overline{a}_k \cdot \nabla(T_k(u)) dxdsdt. \tag{59}
$$

Since $u_n \to u$ a.e. in Q and $a(x,t,\cdot,\cdot)$ is continuous, we have $a(T_k(u_n), \nabla T_k(u) \to a(T_k(u), \nabla T_k(u))$ a.e. in Q as $n \to \infty$, and by (10), $|a(T_k(u_n), \nabla T_k(u)| \le \nu_k\Big(h_k(x,t) + |\nabla T_k(u)|^{p(x)-1}\Big)$ a.e. in Q uniformly with respect to n. It follows by using lemma (2.5) and Lebesgue's theorem that

$$
a(T_k(u_n), \nabla T_k(u) \to a(T_k(u), \nabla T_k(u)) \quad \text{strongly in } (L^{p'(\cdot)}(Q))^N. \tag{60}
$$

Thanks to this strong convergence and the weak convergence of $\nabla T_k(u_n)$ to $\nabla T_k(u)$ in $(L^{p(\cdot)}(Q))^N$ it is possible to pass to the limit-sup as n tends to ∞ in the term J_n^3 to obtain

$$
\lim_{n\to\infty} J_n^3 = 0. \tag{61}
$$

By combining (58), (59) and (61), we conclude that

$$
\begin{aligned}
\lim_{n\to\infty} \int_0^T \int_0^t \int_\Omega &\big[a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u))\big] \cdot \\
&\big[\nabla T_k(u_n) - \nabla T_k(u)\big] dxdsdt = 0.
\end{aligned} \tag{62}
$$

We apply now the lemma (5.1) to conclude that for all $k > 0$,

$$
\begin{aligned}
&T_k(u_n) \to T_k(u) \text{ strongly in } X \\
&\nabla T_k(u_n) \to \nabla T_k(u) \text{ strongly in } (L^{p(\cdot)}(Q))^N, \\
&a(x,t,T_k(u_n), \nabla T_k(u_n)) \rightharpoonup a(x,t,T_k(u), \nabla T_k(u)) \text{ weakly in } (L^{p'(\cdot)}(Q))^N.
\end{aligned} \tag{63}
$$

Thus (34) and (35) holds true.

Step 4: u is a renormalized solution. For any fixed $h>0$, for all $n\geq h+1$, we have by using (35) and the estimate (32)

$$\begin{aligned}
\lim_{n\to\infty}&\int_{\{h\leq|u_n|\leq h+1\}} a_n(u_n,\nabla u_n)\cdot\nabla u_n dxdt\\
&=\lim_{n\to\infty}\int_Q a_n(T_{h+1}(u_n),\nabla T_{h+1}(u_n))\cdot\nabla T_{h+1}(u_n)dxdt\\
&\quad-\int_Q a_n(T_h(u_n),\nabla T_h(u_n))\cdot\nabla T_h(u_n)dxdt\\
&=\int_Q a(T_{h+1}(u),\nabla T_{h+1}(u))\cdot\nabla T_{h+1}(u)dxdt-\int_Q a(T_h(u),\nabla T_h(u))\cdot\nabla T_h(u)dxdt\\
&=\int_{\{h\leq|u|\leq h+1\}} a(u,\nabla u)\cdot\nabla u dxdt. \qquad (64)
\end{aligned}$$

which implies that u satisfies (18).
Let S be a function in $W^{2,\infty}(\mathbb{R})$ with compact support $\subset[-k,k]$ for some $k>0$. The multiplication of (23) by $S'(u_n)$ leads, in the sense of distributions, to

$$\begin{aligned}
&\frac{\partial B_S^n(x,u_n)}{\partial t}-\operatorname{div}\Big(S'(u_n)a(x,t,u_n,\nabla u_n)\Big)+S''(u_n)a(x,t,u_n,\nabla u_n).\nabla u_n\\
&+\operatorname{div}(\Phi(u_n)S'(u_n))-S''(u_n)\Phi(u_n).\nabla u_n=fS'(u_n)+\operatorname{div}(S'(u_n)F)-S''(u_n)F.\nabla u_n, \qquad (65)
\end{aligned}$$

where $B_S^n(x,z)=\int_0^z\frac{\partial b_n(x,r)}{\partial r}S'(r)dr$.
Since S' is bounded and $b_n(x,\cdot)$ is $\mathcal{C}^1(\mathbb{R})$-function satisfying (9), $B_S^n(x,\cdot)$ is continuous and bounded. Then the convergence $u_n\to u$ a.e. in Q implies that $B_S^n(x,u_n)\to B_S^n(x,u)$ a.e. in Q and in $L^\infty(Q)$ weak-$*$. Therefore

$$\frac{\partial B_S^n(x,u_n)}{\partial t}\to\frac{\partial B_S^n(x,u)}{\partial t}\quad\text{in } \mathcal{D}'(Q)\quad\text{as } n\to\infty. \qquad (66)$$

The fact that $\text{supp}(S)\subset[-k,k]$ implies that for $n\geq k$,

$$S'(u_n)a_n(u_n,\nabla u_n)=S'(u_n)a_n(T_k(u_n),\nabla T_k(u_n))\quad\text{a.e. in } Q.$$

and

$$S''(u_n)a_n(u_n,\nabla u_n)\cdot\nabla u_n=S''(u_n)a(T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u_n)\quad\text{a.e. in } Q.$$

The convergence of u_n to u a.e. in Q, the bounded character of S' and (63) implies that

$$S'(u_n)a_n(u_n,\nabla u_n)\rightharpoonup S'(u)a(T_k(u),\nabla T_k(u))\ \text{ weakly in } (L^{p'(\cdot)}(Q))^N\ \text{ as } n\to\infty. \qquad (67)$$

Remark that $S'(u)=0$ for $|u|\geq k$, hence

$$S'(u)a(T_k(u),\nabla T_k(u))=S'(u)a(u,\nabla u)\text{ a.e. in } Q.$$

As mentioned below Definition (4.1), the term $S'(u)a(T_k(u),\nabla T_k(u))$ can be identified by $S'(u)a(u,\nabla u)$ in equation (19). On other hand, the almost everywhere convergence of $S''(u_n)$ to $S''(u)$ in Q, the bounded character of S'' and (35) imply, by lemma (2.6), that

$$S''(u_n)a_n(u_n,\nabla u_n)\cdot\nabla u_n\rightharpoonup S''(u)a(T_k(u),\nabla T_k(u))\cdot\nabla T_k(u)\ \text{ weakly in } L^1(Q)\ \text{ as } n\to\infty. \qquad (68)$$

The term $S''(u)a(T_k(u),\nabla T_k(u))\cdot\nabla T_k(u)$ can be identified with $S''(u)a(u,\nabla u)\cdot\nabla u$.
Since $\text{supp}(S')\subset[-k,k]$, we can write

$$S'(u_n)\Phi_n(u_n)=S'(u_n)\Phi_n(T_k(u_n))\quad\text{a.e. in } Q.$$

The Lipschitz character of Φ_n, the continuity of $T_k(\cdot)$ and the pointwise convergence of u_n to u in Q imply that $S'(u_n)$ is bounded and converges almost everywhere in Q and that $\Phi_n(T_k(u_n))$ is uniformly bounded with respect n and converges a.e. to $\Phi(T_k(u))$. Thus

$$S'(u_n)\Phi_n(u_n) \to S'(u)\Phi(T_k(u)) \quad \text{strongly in } \ L^1(Q) \ \text{ as } n \to \infty. \tag{69}$$

The term $S'(u)\Phi(T_k(u))$ is identified with $S'(u)\Phi(u)$ in (19).
Moreover, we have

$$S''(u_n)\Phi_n(u_n) \cdot \nabla u_n = \Phi_n(T_k(u_n)) \cdot \nabla S'(u_n) \quad \text{a.e. in } \ Q.$$

taking in mind that $S' \in W^{1,\infty}(\mathbb{R})$ and in view of (63) that $\nabla S'(u_n)$ converges weakly in $(L^{p(\cdot)}(Q))^N$ to $\nabla S'(u)$ and since $\Phi_n(T_k(u_n))$ is uniformly bounded with respect n and converges a.e. to $\Phi(T_k(u))$, we can write

$$S''(u_n)\Phi_n(u_n) \cdot \nabla u_n \rightharpoonup \Phi(T_k(u)) \cdot \nabla S'(u) \quad \text{weakly in } L^{p(\cdot)}(Q) \ \text{ as } n \to \infty. \tag{70}$$

We can identifies the term $\Phi(T_k(u)) \cdot \nabla S'(u)$ with $\Phi(u) \cdot \nabla S'(u)$ in (19).
the pointwise convergence of $S'(u_n)$ to $S'(u)$ and the $L^1(Q)$– strong convergence of f_n to f yield

$$f_n S'(u_n) \to f S'(u) \quad \text{stronly in } \ L^1(Q) \ \text{ as } n \to \infty. \tag{71}$$

Recalling that $F \in L^{p'(\cdot)}(Q)$ and that $S'(u_n)$ is bounded and converges almost everywhere in Q makes it possible to obtain

$$\text{div } (F S'(u_n)) \to \text{ div } (F S'(u)) \quad \text{stronly in } \ X' \ \text{ as } n \to \infty. \tag{72}$$

Finally, we recall that $\nabla S'(u_n) \to \nabla S'(u)$ weakly in $(L^{p(\cdot)}(Q))^N$. Then the term $S''(u_n)F \cdot \nabla u_n$ which is equal to $F \cdot \nabla S'(u_n)$ verifies the following convergence result

$$S''(u_n)F \cdot \nabla u_n \rightharpoonup F \cdot \nabla S'(u) \quad \text{weakly in } L^1(Q) \ \text{ as } n \to \infty. \tag{73}$$

We can identifies the term $F \cdot \nabla S'(u)$ with $S''(u)F \cdot \nabla u$ in (19).
As a consequence of the convergence results (66)–(73), we are in a position to pass to the limit as n tends to ∞ in (65) and to conclude that u satisfies (19).
It remains to show that $B_S(x,u)$ satisfies the initial condition (20). To this end, we take in mind the convergence results (66)–(73) of the terms of equation (65), which imply that

$$\frac{\partial B_S^n(x,u_n)}{\partial t} \quad \text{is bounded in } \ X' + L^1(Q).$$

Due to (9), we deduce that

$$\frac{\partial S(u_n)}{\partial t} \quad \text{is bounded in } \ X' + L^1(Q).$$

while $S(u_n)$ being bounded in $L^\infty(Q)$. Then by (6), we conclude that $S(u_n)$ lies in $C([0,T];L^1(\Omega))$. It follows that

$$S(u_n)(t=0) = S(u_{0_n}) \to S(u_0) \quad \text{strongly in } L^1(\Omega). \tag{74}$$

If We turn back to condition (9) which is verified by b_n and properties of S (S' is bounded), one has

$$|B_S^n(x,r) - B_S^n(x,r')| \le A_k|S(r) - S(r')| \quad \forall r, r'. \tag{75}$$

Which implies, with the fact that $b_n(x,u_{0_n}) \to b(x,u_0)$ strongly in $L^1(\Omega)$, that

$$B_S^n(x,u_n)(t=0) = B_S^n(x,u_{0_n}) \to B_S(x,u_0) \quad \text{strongly in } L^1(\Omega).$$

Hence (20) is fulfilled. Thus, the proof of theorem 4.1 is complete.

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