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Hilbert-Pachpatte-type inequality due to fractional differential inequalities

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Abstract. The main objective of this paper is a study of some generalizations of Hilbert-Pachpatte-type inequality. We apply our general results to homogeneous functions. Also, this paper presents improvements and weighted versions of Hilbert-Pachpatte type inequalities involving the fractional derivatives.

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1. Introduction

Although classical, Hilbert's inequality is still of interest to numerous mathematicians. Through the years, Hilbert-type inequalities were discussed by numerous authors, who either reproved them using various techniques, or applied and generalized them in many different ways. For more details as regards Hilbert's inequality the reader is referred to [4], [7] and [8]. In particular, in [9], Pachpatte proved some new inequalities similar to Hilbert's inequality. In this paper, we establish some new integral Hilbert-Pachpatte type inequalities.

We start with the following result of Zhongxue Lü from [11]: let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $s > 2 - \min\{p, q\}$, and $f(x)$, $g(y)$ be real-valued continuous functions defined on $[0, \infty)$, respectively, and let $f(0) = g(0) = 0$, and

$$0 < \int_0^\infty \int_0^x x^{1-s}|f'(\tau)|^p d\tau dx < \infty, \qquad 0 < \int_0^\infty \int_0^y y^{1-s}|g'(\delta)|^q d\delta dy < \infty,$$

then

$$\int_0^\infty \int_0^\infty \frac{|f(x)||g(y)|}{(qx^{p-1} + py^{q-1})(x+y)^s} dxdy \tag{1}$$
$$\leq \frac{B\left(\frac{q+s-2}{q}, \frac{p+s-2}{p}\right)}{pq} \left(\int_0^\infty \int_0^x x^{1-s}|f'(\tau)|^p d\tau dx\right)^{\frac{1}{p}} \left(\int_0^\infty \int_0^y y^{1-s}|g'(\delta)|^q d\delta dy\right)^{\frac{1}{q}}.$$

Here, $B(\cdot,\cdot)$ denotes the usual Beta function. In this paper we shall consider more general form of inequality (1). Moreover, the main objective of this paper is to deduce Hilbert-Pachpatte type inequalities using the Taylor series of function and refinement of arithmetic-geometric inequality from [5]. Also, this paper presents improvements and weighted versions of Hilbert-Pachpatte type inequalities involving the fractional derivatives. Our results will be based on the following result of Krnić and Pečarić

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(see [6]) for conjugate exponents p and q. More precisely, they obtained the following two equivalent inequalities:

$$\int_{\Omega\times\Omega} K(x,y)f(x)g(y)d\mu_1(x)d\mu_2(y) \le \left[\int_\Omega \varphi^p(x)F(x)f^p(x)d\mu_1(x)\right]^{\frac{1}{p}} \left[\int_\Omega \psi^q(y)G(y)g^q(y)d\mu_2(y)\right]^{\frac{1}{q}} \tag{2}$$

and

$$\int_\Omega G^{1-p}(y)\psi^{-p}(y)\left[\int_\Omega K(x,y)f(x)d\mu_1(x)\right]^p d\mu_2(y) \le \int_\Omega \varphi^p(x)F(x)f^p(x)d\mu_1(x), \tag{3}$$

where $p>1$, μ_1,μ_2 are positive σ-finite measures, $K:\Omega\times\Omega\to\mathbb{R}$, $f,g,\varphi,\psi:\Omega\to\mathbb{R}$ are measurable, non-negative functions and

$$F(x)=\int_\Omega \frac{K(x,y)}{\psi^p(y)}d\mu_2(y) \quad \text{and} \quad G(y)=\int_\Omega \frac{K(x,y)}{\varphi^q(x)}d\mu_1(x). \tag{4}$$

On the other hand, here we also refer to a paper of Brnetić *et al*, [10], where a general Hilbert-type inequality was obtained for $l\ge 2$ conjugate exponents, that is, real parameters $p_1,\dots,p_l>1$, such that $\sum_{i=1}^l \frac{1}{p_i}=1$. Namely, let $K:\Omega^l\to\mathbb{R}$ and $\phi_{ij}:\Omega\to\mathbb{R}$, $i,j=1,\dots,l$, be non-negative measurable functions. If $\prod_{i,j=1}^l \phi_{ij}(x_j)=1$, then the inequality

$$\int_{\Omega^l} K(x_1,\dots,x_l)\prod_{i=1}^l f_i(x_i)\,dx_1\dots dx_l \le \prod_{i=1}^l \left(\int_\Omega F_i(x_i)(\phi_{ii}f_i)^{p_i}(x_i)\,dx_i\right)^{\frac{1}{p_i}}, \tag{5}$$

holds for all non-negative measurable functions $f_1,\dots,f_l:\Omega\to\mathbb{R}$, where

$$F_i(x_i)=\int_{\Omega^{l-1}} K(x_1,\dots,x_l)\prod_{j=1,j\ne i}^l \phi_{ij}^{p_i}(x_j)dx_1\dots dx_{i-1}dx_{i+1}\dots dx_l, \tag{6}$$

for $i=1,\dots,l$.

2. Main results

In this section we shall state our main results. We suppose that all integrals converge and shall omit these types of conditions. To obtain the main result we need some lemmas.

Lemma 2.1. *For $f\in C^n[a,b]$, $n\in\mathbb{N}$, the Taylor series of function f is given by*

$$f(x)=\frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}f^{(n)}(t)dt+\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k. \tag{7}$$

Define the subspace $C_a^n[a,b]$ of $C^n[a,b]$ as

$$C_a^n[a,b]=\{f\in C^n[a,b]: f^{(k)}(a)=0, k=0,1,\dots,n-1\}.$$

Obviously, if $f\in C_a^n[a,b]$, then the right-hand side of (7) can be written as

$$f(x)=\frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}f^{(n)}(t)dt. \tag{8}$$

M. Krnić et *al.* in [5] proved the following refinements and converses of Young's inequality in quotient and difference form. For that sake, if $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, we denote $P_n = \sum_{i=1}^n p_i$,

$$A_n(\mathbf{x}) = \frac{\sum_{i=1}^n x_i}{n}, \quad G_n(\mathbf{x}) = \left(\prod_{i=1}^n x_i\right)^{\frac{1}{n}},$$

and

$$M_r(\mathbf{x}, \mathbf{p}) = \begin{cases} \left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i^r\right)^{\frac{1}{r}}, & r \neq 0 \\ \left(\prod_{i=1}^n x_i^{p_i}\right)^{\frac{1}{P_n}}, & r = 0 \end{cases}.$$

Lemma 2.2. ([5]) *Let* $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ *and* $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$ *be positive* $n-$*tuples such that* $\sum_{i=1}^n \frac{1}{p_i} = 1$, *and*

$$\mathbf{x}^{\mathbf{p}} = (x_1^{p_1}, x_2^{p_2}, \ldots, x_n^{p_n}), \quad \boldsymbol{p}^{-1} = \left(\frac{1}{p_1}, \frac{1}{p_2}, \ldots, \frac{1}{p_n}\right).$$

Then
(i)

$$\left[\frac{A_n(\mathbf{x}^{\mathbf{p}})}{G_n(\mathbf{x}^{\mathbf{p}})}\right]^{n\min_{1\leq i\leq n}\{\frac{1}{p_i}\}} \leq \frac{M_1(\mathbf{x}^{\mathbf{p}}, \boldsymbol{p}^{-1})}{M_0(\mathbf{x}^{\mathbf{p}}, \boldsymbol{p}^{-1})} \leq \left[\frac{A_n(\mathbf{x}^{\mathbf{p}})}{G_n(\mathbf{x}^{\mathbf{p}})}\right]^{n\max_{1\leq i\leq n}\{\frac{1}{p_i}\}},$$

and
(ii)

$$n\min_{1\leq i\leq n}\left\{\frac{1}{p_i}\right\}[A_n(\mathbf{x}^{\mathbf{p}}) - G_n(\mathbf{x}^{\mathbf{p}})] \leq M_1(\mathbf{x}^{\mathbf{p}}, \boldsymbol{p}^{-1}) - M_0(\mathbf{x}^{\mathbf{p}}, \boldsymbol{p}^{-1})$$
$$\leq n\max_{1\leq i\leq n}\left\{\frac{1}{p_i}\right\}[A_n(\mathbf{x}^{\mathbf{p}}) - G_n(\mathbf{x}^{\mathbf{p}})].$$

We start with the refinement of Hilbert-Pachpatte type inequalities with the general kernel.

Theorem 2.3. *Let* $\frac{1}{p} + \frac{1}{q} = 1$ *with* $p, q > 1$, *and* $0 \leq a < b \leq \infty$. *If* $K : [a, b] \times [a, b] \to \mathbb{R}$ *is non-negative function,* $\varphi(x)$, $\psi(y)$ *are non-negative functions on* $[a, b]$ *and* $f, g \in C_a^n[a, b]$, *then the following inequalities hold*

$$\begin{aligned} &\int_a^b\int_a^b \frac{K(x,y)|f(x)|\,|g(y)|}{\left((x-a)^{\frac{1}{q(M-m)}} + (y-a)^{\frac{1}{p(M-m)}}\right)^{2(M-m)}}dxdy \\ &\leq \frac{1}{4^{M-m}}\int_a^b\int_a^b \frac{K(x,y)|f(x)|\,|g(y)|}{(x-a)^{\frac{1}{q}}(y-a)^{\frac{1}{p}}}dxdy \\ &\leq \frac{1}{4^{M-m}[(n-1)!]^2}\left(\int_a^b\int_a^x (x-t)^{p(n-1)}\varphi^p(x)F(x)|f^{(n)}(t)|^p dtdx\right)^{\frac{1}{p}} \\ &\quad\times\left(\int_a^b\int_a^y (y-t)^{q(n-1)}\psi^q(y)G(y)|g^{(n)}(t)|^q dtdy\right)^{\frac{1}{q}}, \end{aligned} \tag{9}$$

and

$$\int_a^b G^{1-p}(y)\psi^{-p}(y)\left(\int_a^b K(x,y)\left(\int_a^x (x-t)^{p(n-1)}|f^{(n)}(t)|^p dt\right)^{\frac{1}{p}} dx\right)^p dy$$
$$\leq \int_a^b \int_a^x (x-t)^{p(n-1)}\varphi^p(x)F(x)|f^{(n)}(t)|^p dtdx, \tag{10}$$

where $m = \min\{\frac{1}{p}, \frac{1}{q}\}$, $M = \max\{\frac{1}{p}, \frac{1}{q}\}$, *and* $F(x)$ *and* $G(y)$ *are defined as in (4).*

Proof. By using (8) and Hölder's inequality, we have

$$\begin{aligned} |f(x)| &= \frac{1}{(n-1)!}\left|\int_a^x (x-t)^{n-1} f^{(n)}(t)dt\right| \\ &\leq \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}|f^{(n)}(t)|\cdot 1dt \\ &\leq \frac{1}{(n-1)!}\left(\int_a^x (x-t)^{p(n-1)}|f^{(n)}(t)|^p dt\right)^{\frac{1}{p}}\left(\int_a^x 1^q dt\right)^{\frac{1}{q}} \\ &= \frac{(x-a)^{\frac{1}{q}}}{(n-1)!}\left(\int_a^x (x-t)^{p(n-1)}|f^{(n)}(t)|^p dt\right)^{\frac{1}{p}}, \end{aligned} \tag{11}$$

and similarly

$$|g(x)| \leq \frac{(y-a)^{\frac{1}{p}}}{(n-1)!}\left(\int_a^y (y-t)^{q(n-1)}|g^{(n)}(t)|^q dt\right)^{\frac{1}{q}}. \tag{12}$$

Now, from (11) and (12) we get

$$\begin{aligned} |f(x)|\,|g(y)| \leq\ & \frac{1}{[(n-1)!]^2}(x-a)^{\frac{1}{q}}(y-a)^{\frac{1}{p}} \times \left(\int_a^x (x-t)^{p(n-1)}|f^{(n)}(t)|^p dt\right)^{\frac{1}{p}} \\ & \times \left(\int_a^y (y-t)^{q(n-1)}|g^{(n)}(t)|^q dt\right)^{\frac{1}{q}}. \end{aligned} \tag{13}$$

Applying Lemma 2.2(i) (see also [5]), we have

$$4^{M-m}(x^p y^q)^{M-m} \leq (x^p + y^q)^{2(M-m)}, \quad x \geq 0,\ y \geq 0, \tag{14}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$, and $m = \min\{\frac{1}{p}, \frac{1}{q}\}$, $M = \max\{\frac{1}{p}, \frac{1}{q}\}$. From (13) and (14) we observe that

$$\frac{4^{M-m}|f(x)|\,|g(y)|}{\left((x-a)^{\frac{1}{q(M-m)}} + (y-a)^{\frac{1}{p(M-m)}}\right)^{2(M-m)}} \leq \frac{|f(x)|\,|g(y)|}{(x-a)^{\frac{1}{q}}(y-a)^{\frac{1}{p}}}$$
$$\leq \frac{1}{[(n-1)!]^2}\left(\int_a^x (x-t)^{p(n-1)}|f^{(n)}(t)|^p dt\right)^{\frac{1}{p}}\left(\int_a^y (y-t)^{q(n-1)}|g^{(n)}(t)|^q dt\right)^{\frac{1}{q}},$$

and therefore

$$
\begin{aligned}
&4^{M-m}\int_a^b\int_a^b \frac{K(x,y)|f(x)|\,|g(y)|}{\left((x-a)^{\frac{1}{q(M-m)}}+(y-a)^{\frac{1}{p(M-m)}}\right)^{2(M-m)}}dxdy \\
&\le \int_a^b\int_a^b \frac{K(x,y)|f(x)|\,|g(y)|}{(x-a)^{\frac{1}{q}}(y-a)^{\frac{1}{p}}}dxdy \qquad (15)\\
&\le \frac{1}{[(n-1)!]^2}\int_a^b\int_a^b K(x,y)\left(\int_a^x (x-t)^{p(n-1)}|f^{(n)}(t)|^p dt\right)^{\frac{1}{p}} \\
&\times\left(\int_a^y (y-t)^{q(n-1)}|g^{(n)}(t)|^q dt\right)^{\frac{1}{q}} dxdy.
\end{aligned}
$$

Applying the substitutions

$$
f_1(x)=\left(\int_a^x (x-t)^{p(n-1)}|f^{(n)}(t)|^p dt\right)^{\frac{1}{p}},\quad g_1(y)=\left(\int_a^y (y-t)^{q(n-1)}|g^{(n)}(t)|^q dt\right)^{\frac{1}{q}}
$$

and (2), we have

$$
\begin{aligned}
\int_a^b\int_a^b K(x,y)f_1(x)g_1(y)dxdy &\le \left(\int_a^b \varphi^p(x)F(x)f_1^p(x)dx\right)^{\frac{1}{p}}\left(\int_a^b \psi^q(y)G(y)g_1^q(y)dy\right)^{\frac{1}{q}} \\
&= \left(\int_a^b\int_a^x (x-t)^{p(n-1)}\varphi^p(x)F(x)|f^{(n)}(t)|^p dtdx\right)^{\frac{1}{p}} \\
&\times\left(\int_a^b\int_a^y (y-t)^{q(n-1)}\psi^q(y)G(y)|g^{(n)}(t)|^q dtdy\right)^{\frac{1}{q}}. \qquad (16)
\end{aligned}
$$

By using (15) and (16) we obtain (9). The second inequality (10) can be proved by applying (3). □

Now we can apply our main result on non-negative homogeneous functions. Recall that for homogeneous function of degree $-s$, $s>0$, the equality $K(tx,ty)=t^{-s}K(x,y)$ is satisfied. Further, we define

$$
k(\alpha):=\int_0^\infty K(1,u)u^{-\alpha}du
$$

and suppose that $k(\alpha)<\infty$ for $1-s<\alpha<1$. To prove first application of our main results we need the following lemma.

Lemma 2.4. *If $\lambda>0$, $1-\lambda<\alpha<1$ and $K:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ is a non-negative homogeneous function of degree $-\lambda$, then*

$$
\int_0^\infty K(x,y)\left(\frac{x}{y}\right)^\alpha dy = x^{1-\lambda}k(\alpha), \qquad (17)
$$

and

$$
\int_0^\infty K(x,y)\left(\frac{y}{x}\right)^\alpha dx = y^{1-\lambda}k(2-\lambda-\alpha). \qquad (18)
$$

Proof. We use the substitution $y=ux$. The proof follows easily from homogeneity of the function $K(x,y)$. □

Corollary 2.5. *Let* $\frac{1}{p}+\frac{1}{q}=1$, *with* $p,q>1$. *If* $K:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ *is a non-negative and homogeneous function of degree* $-\lambda$, $\lambda>0$, *and* $f,g\in C_0^n[0,\infty]$, *then the following inequalities hold*

$$\int_0^\infty\int_0^\infty \frac{K(x,y)|f(x)|\,|g(y)|}{\left(x^{\frac{1}{q(M-m)}}+y^{\frac{1}{p(M-m)}}\right)^{2(M-m)}}dxdy \tag{19}$$
$$\leq \frac{pq}{4^{M-m}}\int_0^\infty\int_0^\infty K(x,y)|f(x)|\,|g(y)|d(x^{\frac{1}{p}})d(y^{\frac{1}{q}})$$
$$\leq \frac{L}{4^{M-m}[(n-1)!]^2}\left(\int_0^\infty\int_0^x x^{p(A_1-A_2+n-1)+1-\lambda}|f^{(n)}(t)|^p dtdx\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^\infty\int_0^y y^{q(A_2-A_1+n-1)+1-\lambda}|g^{(n)}(t)|^q dtdy\right)^{\frac{1}{q}},$$

and

$$\int_0^\infty y^{(p-1)(\lambda-1)+p(A_1-A_2)}\left(\int_0^\infty K(x,y)\left(\int_0^x (x-t)^{p(n-1)}|f^{(n)}(t)|^p dt\right)^{\frac{1}{p}}dx\right)^p dy$$
$$\leq L^p\int_0^\infty\int_0^x x^{p(A_1-A_2+n-1)+1-\lambda}|f^{(n)}(t)|^p dtdx, \tag{20}$$

where $A_1\in(\frac{1-\lambda}{q},\frac{1}{q})$, $A_2\in(\frac{1-\lambda}{p},\frac{1}{p})$, $L=k(pA_2)^{\frac{1}{p}}k(2-\lambda-qA_1)^{\frac{1}{q}}$, *and* M,m *are defined as in Theorem 2.3.*

Proof. Let $F(x)$, $G(y)$ be the functions defined by (4). Setting $\varphi(x)=x^{A_1}$ and $\psi(y)=y^{A_2}$ in (9), using the fact $(x-t)^{p(n-1)}\leq x^{p(n-1)}$, for $x\geq 0$ and $t\in[0,x]$, and Lemma 2.4, we get

$$\int_0^\infty\int_0^x (x-t)^{p(n-1)}\varphi^p(x)F(x)|f^{(n)}(t)|^p dtdx$$
$$\leq \int_0^\infty\int_0^x x^{p(A_1-A_2+n-1)}\left(\int_0^\infty K(x,y)\left(\frac{x}{y}\right)^{pA_2}dy\right)|f^{(n)}(t)|^p dtdx$$
$$= k(pA_2)\int_0^\infty\int_0^x x^{p(A_1-A_2+n-1)+1-\lambda}|f^{(n)}(t)|^p dtdx, \tag{21}$$

and similarly

$$\int_0^\infty\int_0^y (y-t)^{q(n-1)}\psi^q(y)G(y)|g^{(n)}(t)|^q dtdy \tag{22}$$
$$\leq k(2-\lambda-qA_1)\int_0^\infty\int_0^y y^{p(A_2-A_1+n-1)+1-\lambda}|g^{(n)}(t)|^q dtdy.$$

From (9), (21) and (22), we get (19). □

We proceed with some special homogeneous functions. First, by putting $K(x,y)=\frac{\ln\frac{y}{x}}{y-x}$ in Corollary 2.5, we get the following result.

Corollary 2.6. *Let* $\frac{1}{p}+\frac{1}{q}=1$, *with* $p,q>1$. *Let* M,m,f,g *be defined as in Corollary 2.5. Then the following inequalities hold*

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \frac{\ln\frac{y}{x}|f(x)|\,|g(y)|}{(y-x)\left(x^{\frac{1}{q(M-m)}}+y^{\frac{1}{p(M-m)}}\right)^{2(M-m)}}dxdy\\
&\leq \frac{pq}{4^{M-m}}\int_0^\infty\int_0^\infty \frac{\ln\frac{y}{x}|f(x)|\,|g(y)|}{y-x}d(x^{\frac{1}{p}})d(y^{\frac{1}{q}})\\
&\leq \frac{L_1}{4^{M-m}[(n-1)!]^2}\left(\int_0^\infty\int_0^x x^{p(A_1-A_2+n-1)}|f^{(n)}(t)|^p dtdx\right)^{\frac{1}{p}}\\
&\quad\times\left(\int_0^\infty\int_0^y y^{q(A_2-A_1+n-1)}|g^{(n)}(t)|^q dtdy\right)^{\frac{1}{q}},
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^\infty y^{p(A_1-A_2)}\left(\int_0^\infty \frac{\ln\frac{y}{x}}{y-x}\left(\int_0^x (x-t)^{p(n-1)}|f^{(n)}(t)|^p dt\right)^{\frac{1}{p}}dx\right)^p dy\\
&\leq L_1^p\int_0^\infty\int_0^x x^{p(A_1-A_2+n-1)}|f^{(n)}(t)|^p dtdx,
\end{aligned}$$

where $A_1\in(0,\frac{1}{q})$, $A_2\in(0,\frac{1}{p})$, *and*

$$L_1=\pi^2(\sin pA_2\pi)^{-\frac{2}{p}}(\sin qA_1\pi)^{-\frac{2}{q}}.$$

Similarly, for the homogeneous function of degree $-\lambda$, $\lambda>0$, $K(x,y)=(\max\{x,y\})^{-\lambda}$, $A_1=A_2=\frac{2-\lambda}{pq}$, with the condition $\lambda>2-\min\{p,q\}$, we have:

Corollary 2.7. *Let* $\frac{1}{p}+\frac{1}{q}=1$, *with* $p,q>1$. *Let* M,m,f,g *be defined as in Corollary 2.5. Then the following inequalities hold*

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \frac{(\max\{x,y\})^{-\lambda}|f(x)|\,|g(y)|}{\left(x^{\frac{1}{q(M-m)}}+y^{\frac{1}{p(M-m)}}\right)^{2(M-m)}}dxdy \leq \frac{pq}{4^{M-m}}\int_0^\infty\int_0^\infty \frac{|f(x)|\,|g(y)|}{(\max\{x,y\})^{\lambda}}d(x^{\frac{1}{p}})d(y^{\frac{1}{q}})\\
&\leq \frac{L_2}{4^{M-m}[(n-1)!]^2}\left(\int_0^\infty\int_0^x x^{p(n-1)+1-\lambda}|f^{(n)}(t)|^p dtdx\right)^{\frac{1}{p}}\\
&\quad\times\left(\int_0^\infty\int_0^y y^{q(n-1)+1-\lambda}|g^{(n)}(t)|^q dtdy\right)^{\frac{1}{q}},
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^\infty y^{(p-1)(\lambda-1)}\left(\int_0^\infty (\max\{x,y\})^{-\lambda}\left(\int_0^x (x-t)^{p(n-1)}|f^{(n)}(t)|^p dt\right)^{\frac{1}{p}}dx\right)^p dy\\
&\leq L_2^p\int_0^\infty\int_0^x x^{p(n-1)+1-\lambda}|f^{(n)}(t)|^p dtdx,
\end{aligned}$$

where $L_2=k(\frac{2-\lambda}{q})$ *and* $k(\alpha)=\frac{\lambda}{(1-\alpha)(\lambda+\alpha-1)}$.

In the proof of the following result we used a general Hilbert-type inequality (5) of Brnetić et al, [10].

Theorem 2.8. *Let* $n,l\in\mathbb{N}$, $l\geq 2$, $\sum_{i=1}^{l}\frac{1}{p_i}=1$ *with* $p_i>1$, $i=1,\ldots,l$. *Let* α_i, $i=1,\ldots,l$, *is defined by* $\alpha_i=\prod_{j=1,j\neq i}^{l}p_j$. *If* $K:[a,b]^l\to\mathbb{R}$ *is non-negative function,*

$\phi_{ij}(x_j)$, $i,j = 1,\dots,l$, are non-negative functions on $[a,b]$, such that $\prod_{i,j=1}^{l}\phi_{ij}(x_j) = 1$, and $f_i \in C_a^n[a,b]$, $i = 1,\dots,l$, then the following inequality holds

$$\int_{(a,b)^l} \frac{K(x_1,\dots,x_l)\prod_{i=1}^{l}|f_i(x_i)|}{\left(\sum_{i=1}^{l}(x_i-a)^{\frac{1}{\alpha_i(M-m)}}\right)^{l(M-m)}}dx_1\dots dx_l$$
$$\le \frac{1}{l^{(M-m)l}}\int_{(a,b)^l}\frac{K(x_1,\dots,x_l)\prod_{i=1}^{l}|f_i(x_i)|}{\prod_{i=1}^{l}(x_i-a)^{\frac{1}{\alpha_i}}}dx_1\dots dx_l$$
$$\le \frac{1}{l^{(M-m)l}[(n-1)!]^l}\prod_{i=1}^{l}\left(\int_a^b\int_a^{x_i}(x_i-t)^{p_i(n-1)}\phi_{ii}^{p_i}(x_i)F_i(x_i)|f_i^{(n)}(t)|^{p_i}dtdx_i\right)^{\frac{1}{p_i}},$$

where $m = \min_{1\le i\le l}\{\frac{1}{p_i}\}$, and $M = \max_{1\le i\le l}\{\frac{1}{p_i}\}$, and $F_i(x_i)$, $i = 1,\dots,l$ is defined by (6).

Obviously, Theorem 2.8 is a generalization of Theorem 2.3.

Remark 2.1. Applying the second refinement of arithmetic-geometric inequality (see Lemma 2.2(ii)) we obtain

$$x^p y^q \le \left(\frac{x^p+y^q}{2} - \frac{1}{M-m}\right)^2, \quad x \ge 0,\ y \ge 0, \tag{23}$$

where $\frac{1}{p}+\frac{1}{q} = 1$ with $p > 1$, and $m = \min\{\frac{1}{p},\frac{1}{q}\}$, $M = \max\{\frac{1}{p},\frac{1}{q}\}$. If we take (23) and proceed as in the proof of Theorem 2.3, then

$$\int_a^b\int_a^b\frac{K(x,y)|f(x)|\,|g(y)|}{\left(\frac{1}{2}[(x-a)^{\frac{1}{q}}+(y-a)^{\frac{1}{p}}]-\frac{1}{M-m}\right)^2}dxdy \le \int_a^b\int_a^b\frac{K(x,y)|f(x)|\,|g(y)|}{(x-a)^{\frac{1}{q}}(y-a)^{\frac{1}{p}}}dxdy$$
$$\le \frac{1}{[(n-1)!]^2}\left(\int_a^b\int_a^x(x-t)^{p(n-1)}\varphi^p(x)F(x)|f^{(n)}(t)|^p dtdx\right)^{\frac{1}{p}}$$
$$\times\left(\int_a^b\int_a^y(y-t)^{q(n-1)}\psi^q(y)G(y)|g^{(n)}(t)|^q dtdy\right)^{\frac{1}{q}},$$

where $F(x)$ and $G(y)$ are defined by (4).

3. The fractional derivatives and applications to Hilbert-Pachpatte type inequalities

First, we introduce some facts about fractional derivatives (see [3]). Let $[a,b]$, $-\infty < a < b < \infty$, be a finite interval on real axis $\mathbb{R}$. By $L_p[a,b]$, $1 \le p < \infty$, we denote the space of all Lebesgue measurable functions f for which $|f^p|$ is Lebesgue integrable on $[a,b]$. For $f \in L_1[a,b]$ the left-sided and right-sided *the Riemann-Liouville integral* of f of order α are defined by

$$J_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}f(t)dt, \quad x > a,$$
$$J_{b-}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}f(t)dt, \quad x < b.$$

For $f : [a,b] \to \mathbb{R}$ the left-sided *the Riemann-Liouville derivative* of f of order α is defined by

$$D^\alpha_{a+}f(x)=\frac{d^n}{dx^n}J^{n-\alpha}_{a+}f(x)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x(x-t)^{n-\alpha-1}f(t)dt.$$

Our result with the Riemann-Liouville fractional derivative is based on the following lemma (see [1]). By $AC^m[a,b]$ we denote the space of all functions $g\in C^{m-1}[a,b]$ with $g^{(m-1)}\in AC[a,b]$, where $AC[a,b]$ is the space of all absolutely continuous functions fnctions on $[a,b]$. For $\alpha>0$, $[\alpha]$ denotes the integral part of α.

Lemma 3.1. ([1]) *Let $\beta>\alpha\geq 0$, $m=[\beta]+1$, $n=[\alpha]+1$. The composition identity*

$$D^\alpha_{a+}f(x)=\frac{1}{\Gamma(\beta-\alpha)}\int_0^x(x-t)^{\beta-\alpha-1}D^\beta_{a+}f(t)dt,\quad x\in[a,b],$$

is valid if one of the following conditions holds:

(i) $f\in J^\beta_{a+}(L_1[a,b])=\{f:f=J^\beta_{a+}\varphi,\varphi\in L_1[a,b]\}$.
(ii) $J^{m-\beta}_{a+}f\in AC^m[a,b]$ *and* $D^{\beta-k}_{a+}f(a)=0$ *for* $k=1,\ldots,m$.
(iii) $D^{\beta-1}_{a+}f\in AC[a,b]$, $D^{\beta-k}_{a+}f\in C[a,b]$ *and* $D^{\beta-k}_{a+}f(a)=0$ *for* $k=1,\ldots,m$.
(iv) $f\in AC^m[a,b]$, $D^\beta_{a+}f$, $D^\alpha_{a+}f\in L_1[a,b]$, $\beta-\alpha\notin\mathbb{N}$, $D^{\beta-k}_{a+}f(a)=0$ *for* $k=1,\ldots,m$ *and* $D^{\alpha-k}_{a+}f(a)=0$ *for* $k=1,\ldots,n$.
(v) $f\in AC^m[a,b]$, $D^\beta_{a+}f$, $D^\alpha_{a+}f\in L_1[a,b]$, $\beta-\alpha=l\in\mathbb{N}$, $D^{\beta-k}_{a+}f(a)=0$ *for* $k=1,\ldots,l$.
(vi) $f\in AC^m[a,b]$, $D^\beta_{a+}f$, $D^\alpha_{a+}f\in L_1[a,b]$, *and* $f^{(k)}(a)=0$ *for* $k=0,\ldots,m-2$.
(vii) $f\in AC^m[a,b]$, $D^\beta_{a+}f$, $D^\alpha_{a+}f\in L_1[a,b]$, $\beta\notin\mathbb{N}$ *and* $D^{\beta-1}_{a+}f$ *is bounded in a neighborhood of* $m=a$.

By using Lemma 2.2 (see also Remark 2.1) and Lemma 3.1 we obtain our first result with the fractional derivative.

Theorem 3.2. *Let α, β, f, g be defined as in Theorem 3.1. If $K:[a,b]^2\to\mathbb{R}$ is non-negative function, $\varphi(x)$, $\psi(y)$ are non-negative functions on $[a,b]$, then the following inequality holds*

$$\begin{aligned}
&\int_a^b\int_a^b\frac{K(x,y)|D^\alpha_{a+}f(x)|\,|D^\alpha_{a+}g(y)|}{\left(\frac12[(x-a)^{\frac1q}+(y-a)^{\frac1p}]-\frac{1}{M-m}\right)^2}dxdy\\
&\leq\int_a^b\int_a^b\frac{K(x,y)|D^\alpha_{a+}f(x)|\,|D^\alpha_{a+}g(y)|}{(x-a)^{\frac1q}(y-a)^{\frac1p}}dxdy\\
&\leq\frac{1}{[\Gamma(\beta-\alpha)]^2}\left(\int_a^b\int_a^x(x-t)^{p(\beta-\alpha-1)}\varphi^p(x)F(x)|D^\alpha_{a+}f(t)|^pdtdx\right)^{\frac1p}\\
&\quad\times\left(\int_a^b\int_a^y(y-t)^{q(\beta-\alpha-1)}\psi^q(y)G(y)|D^\alpha_{a+}g(t)|^qdtdy\right)^{\frac1q},
\end{aligned}$$

where $m,M,F(x),G(y)$ are defined as in Theorem 2.3.

Proof. The proof is similar to the proof of Theorem 2.3. □

Let $\nu>0$, $n=[\nu]$, and $\overline{\nu}=\nu-n$, $0\leq\overline{\nu}<1$. Let $[a,b]\subseteq\mathbb{R}$ and $x_0,x\in[a,b]$ such that $x\geq x_0$ where x_0 is fixed. For $f\in C[a,b]$ *the generalized Riemann-Liouville fractional integral* of f of order ν is given by

$$(J^{x_0}_\nu f)(x)=\frac{1}{\Gamma(\nu)}\int_{x_0}^x(x-t)^{\nu-1}f(t)dt,\quad x\in[x_0,b].$$

Further, define the subspace $C^\nu_{x_0}[a,b]$ of $C^n[a,b]$ as

$$C^\nu_{x_0}[a,b] = \{f \in C^n[a,b] : J^{x_0}_{1-\overline{\nu}} f^{(n)} \in C^1[x_0,b]\}.$$

For $f \in C^\nu_{x_0}[a,b]$ *the generalized Canavati $\nu-$fractional derivative* of f over $[x_0,b]$ is given by

$$D^\nu_{x_0} f = D\, J^{x_0}_{1-\overline{\nu}} f^{(n)},$$

where $D = d/dx$. Notice that

$$(J^{x_0}_{1-\overline{\nu}} f^{(n)})(x) = \frac{1}{\Gamma(1-\overline{\nu})} \int_{x_0}^x (x-t)^{-\overline{\nu}} f^{(n)}(t)dt$$

exists for $f \in C^\nu_{x_0}[a,b]$.
To obtain the result with generalized Canavati $\nu-$fractional derivative of f we need the following lemma.

Lemma 3.3. ([3]) *Let $f \in C^\nu_{x_0}[a,b]$, $\nu > 0$ and $f^{(i)}(x_0) = 0$, $i = 0,1,\dots,n-1$, $n = [\nu]$. Then*

$$f(x) = \frac{1}{\Gamma(\nu)} \int_{x_0}^x (x-t)^{\nu-1} (D^\nu_{x_0} f)(t)dt,$$

for all $x \in [a,b]$ with $x \geq x_0$.

Theorem 3.4. *Let $\nu > 0$ and $x_0, y_0 \in [a,b]$. Let $K : [a,b]^2 \to \mathbb{R}$ is non-negative function, $\varphi(x)$, $\psi(y)$ are non-negative functions on $[a,b]$. If $f \in C^\nu_{x_0}[a,b]$ and $g \in C^\nu_{y_0}[a,b]$ such that $f^{(i)}(x_0) = g^{(i)}(y_0) = 0$, $i = 0,1,\dots,n-1$, $n = [\nu]$, then the following inequalities hold*

$$\begin{aligned}
&\int_a^b \int_a^b \frac{K(x,y)|f(x)|\,|g(y)|}{\left((x-x_0)^{\frac{1}{q(M-m)}} + (y-y_0)^{\frac{1}{p(M-m)}}\right)^{2(M-m)}} dxdy \\
&\leq \frac{1}{4^{M-m}} \int_a^b \int_a^b \frac{K(x,y)|f(x)|\,|g(y)|}{(x-x_0)^{\frac{1}{q}} (y-y_0)^{\frac{1}{p}}} dxdy \qquad (24)\\
&\leq \frac{1}{4^{M-m}[\Gamma(\nu)]^2} \left(\int_a^b \int_{x_0}^x (x-t)^{p(\nu-1)} \varphi^p(x) F(x) |(D^\nu_{x_0} f)(t)|^p dtdx \right)^{\frac{1}{p}} \\
&\quad \times \left(\int_a^b \int_{y_0}^y (y-t)^{q(\nu-1)} \psi^q(y) G(y) |(D^\nu_{y_0} g)(t)|^q dtdy \right)^{\frac{1}{q}},
\end{aligned}$$

and

$$\begin{aligned}
&\int_a^b G^{1-p}(y) \psi^{-p}(y) \left(\int_a^b K(x,y) \left(\int_{x_0}^x (x-t)^{p(\nu-1)} |(D^\nu_{x_0} f)(t)|^p dt \right)^{\frac{1}{p}} dx \right)^p dy \\
&\leq \int_a^b \int_{x_0}^x (x-t)^{p(\nu-1)} \varphi^p(x) F(x) |(D^\nu_{x_0} f)(t)|^p dtdx, \qquad (25)
\end{aligned}$$

where $m = \min\{\frac{1}{p}, \frac{1}{q}\}$, $M = \max\{\frac{1}{p}, \frac{1}{q}\}$, and $F(x)$ and $G(y)$ are defined by (4).

Proof. To prove the inequalities (24) and (25) we follow the same procedure as in the proof of Theorem 2.3, except we use Lemma 3.3 instead Lemma 2.1. □

In a similar manner as in the previous section, using the inequality (5) we obtain a generalization of Theorem 3.4.

Theorem 3.5. *Let $\nu > 0$ and α_i, $i = 1, \dots, l$, is defined by $\alpha_i = \prod_{j=1, j\neq i}^{l} p_j$, where $\sum_{i=1}^{l} \frac{1}{p_i} = 1$ with $p_i > 1$, $i = 1, \dots, l$. Let $K(x_1, \dots, x_l)$, ϕ_{ij}, $i, j = 1, \dots, l$, are defined as in Theorem 2.8. If $f_i \in C^{\nu}_{x_0^{(i)}}[a, b]$ $(x_0^{(i)} \in [a, b])$, $i = 1, \dots, l$, such that $f_i^{(j)}(x_0^{(i)}) = 0$, $j = 0, 1, \dots, n-1$, $n = [\nu]$, then the following inequality holds*

$$\int_{(a,b)^l} \frac{K(x_1, \dots, x_l) \prod_{i=1}^{l} |f_i(x_i)|}{\left(\sum_{i=1}^{l} (x_i - x_0^{(i)})^{\frac{1}{\alpha_i (M-m)}}\right)^{l(M-m)}} dx_1 \dots dx_l$$
$$\leq \frac{1}{l^{(M-m)l}} \int_{(a,b)^l} \frac{K(x_1, \dots, x_l) \prod_{i=1}^{l} |f_i(x_i)|}{\prod_{i=1}^{l} (x_i - x_0^{(i)})^{\frac{1}{\alpha_i}}} dx_1 \dots dx_l$$
$$\leq \frac{1}{l^{(M-m)l} [\Gamma(\nu)]^l} \prod_{i=1}^{l} \left(\int_a^b \int_{x_0^{(i)}}^{x_i} (x_i - t)^{p_i(\nu-1)} \phi_{ii}^{p_i}(x_i) F_i(x_i) |(D^{\nu}_{x_0^{(i)}} f_i)(t)|^{p_i} dt dx_i \right)^{\frac{1}{p_i}}$$

where $m = \min_{1\leq i\leq l}\{\frac{1}{p_i}\}$, and $M = \max_{1\leq i\leq l}\{\frac{1}{p_i}\}$, and $F_i(x_i)$, $i = 1, \dots, l$ is defined by (6).

For $\alpha > 0$, $f \in AC^n[a, b]$, where $n = [\alpha] + 1$ if $\alpha \notin \mathbb{N}_0$ and $n = \alpha$ if $\alpha \in \mathbb{N}_0$, *the Caputo fractional derivative* of f of order α ${}^cD^{\alpha}_{a+} f$ (left-sided) and ${}^cD^{\alpha}_{b-} f$ (right-sided) are defined by

$$ {}^cD^{\alpha}_{a+} f(x) = D^{\alpha}_{a+} \left[f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k+1)} (x-a)^k \right],$$
$$ {}^cD^{\alpha}_{b-} f(x) = D^{\alpha}_{b-} \left[f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{\Gamma(k+1)} (b-x)^k \right],$$

where D^{α}_{a+}, D^{α}_{b-} denote the left-hand sided and the right-hand sided Riemann-Liouville derivatives.

Very recently, Andrić et al [2] proved the following result.

Theorem 3.6. *Let $\nu > \gamma \geq 0$, $n = [\nu] + 1$, $m = [\gamma] + 1$ and $f \in AC^k[a, b]$, $k = n$ if $\nu \notin \mathbb{N}_0$ and $k = n - 1$ if $\nu \in \mathbb{N}_0$. Let ${}^cD^{\nu}_{a+} f$, ${}^cD^{\gamma}_{a+} f \in L^1[a, b]$. Suppose that one of the following conditions holds:*
(a) *$\nu, \gamma \notin \mathbb{N}_0$ and $f^{(i)}(a) = 0$ for $i = m, \dots, n-1$.*
(b) *$\nu \in \mathbb{N}$, $\gamma \notin \mathbb{N}_0$ and $f^{(i)}(a) = 0$ for $i = m, \dots, n-2$.*
(c) *$\nu \notin \mathbb{N}$, $\gamma \in \mathbb{N}_0$ and $f^{(i)}(a) = 0$ for $i = m-1, \dots, n-1$.*
(d) *$\nu \in \mathbb{N}$, $\gamma \in \mathbb{N}_0$ and $f^{(i)}(a) = 0$ for $i = m-1, \dots, n-2$.*
Then

$$ {}^cD^{\gamma}_{a+} f(x) = \frac{1}{\Gamma(\nu - \gamma)} \int_a^x (x-t)^{\nu-\gamma-1} D^{\nu}_{a+} f(t) dt.$$

Applying Lemma 2.2(i) and Theorem 3.6 (see also [2]) we obtain the following result.

Theorem 3.7. *Let ν, γ, f, g be defined as in Theorem 3.6. If $K : [a, b]^2 \to \mathbb{R}$ is non-negative function, $\varphi(x)$, $\psi(y)$ are non-negative functions on $[a, b]$, then the following inequality holds*

$$\int_a^b \int_a^b \frac{K(x,y)|^c D^{\gamma}_{a+} f(x)|\,|^c D^{\gamma}_{a+} g(y)|}{\left((x-a)^{\frac{1}{q(M-m)}} + (y-a)^{\frac{1}{p(M-m)}}\right)^{2(M-m)}} dxdy$$
$$\leq \frac{1}{4^{M-m}} \int_a^b \int_a^b \frac{K(x,y)|^c D^{\gamma}_{a+} f(x)|\,|^c D^{\gamma}_{a+} g(y)|}{(x-x_0)^{\frac{1}{q}}(y-y_0)^{\frac{1}{p}}} dxdy$$
$$\leq \frac{1}{4^{M-m}[\Gamma(\nu-\gamma)]^2} \left(\int_a^b \int_a^x (x-t)^{p(\nu-\gamma-1)} \varphi^p(x) F(x) |^c D^{\nu}_{a+} f(t)|^p dtdx\right)^{\frac{1}{p}}$$
$$\times \left(\int_a^b \int_a^y (y-t)^{q(\nu-\gamma-1)} \psi^q(y) G(y) |^c D^{\nu}_{a+} g(t)|^q dtdy\right)^{\frac{1}{q}},$$

where $m, M, F(x), G(y)$ are defined as in Theorem 2.3.

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