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Existence of solution for Liouville-Weyl Fractional Hamiltonian systems

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Abstract. In this paper, we investigate the existence of solution for the following fractional Hamiltonian systems:

$$\begin{aligned} {}_tD^{\alpha}_{\infty}({}_{-\infty}D^{\alpha}_{t}u(t)) + L(t)u(t) = & \quad \nabla W(t,u(t)) \\ u \in H^{\alpha}(\mathbb{R},\mathbb{R}^N). & \end{aligned} \tag{1}$$

where $\alpha \in (1/2, 1)$, $t \in \mathbb{R}$, $u \in \mathbb{R}^n$, $L \in C(\mathbb{R}, \mathbb{R}^{n^2})$ is a symmetric and positive definite matrix for all $t \in \mathbb{R}$, $W \in C^1(\mathbb{R}\times\mathbb{R}^n, \mathbb{R})$ and ∇W is the gradient of W at u. The novelty of this paper is that, assuming there exists $l \in C(\mathbb{R},\mathbb{R})$ such that $(L(t)u, u) \geq l(t)|u|^2$ for all $t \in \mathbb{R}$, $u \in \mathbb{R}^n$ and the following conditions on l: $\inf_{t\in\mathbb{R}} l(t) > 0$ and there exists $r_0 > 0$ such that, for any $M > 0$

$$m(\{t \in (y - r_0, y + r_0)/ \ \ l(t) \leq M\}) \to 0 \ \text{ as } \ |y| \to \infty.$$

are satisfied and W is superquadratic growth as $|u| \to +\infty$, we show that (1) possesses at least one nontrivial solution via mountain pass theorem. Recent results in [21] are significantly improved. We do not assume that $l(t)$ have a limit for $|t| \to \infty$.

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1. Introduction

Fractional differential equations appear naturally in a number of fields such as physics, chemistry, biology, economics, control theory, signal and image processing, and blood flow phenomena. During last decades, the theory of fractional differential equations is an area intensively developed, due mainly to the fact that fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes, see for example [1, 6, 7, 12, 13, 15, 18, 26]. Therein, the composition of fractional differential operators has got much attention from many scientists, mainly due to its wide applications in modeling physical phenomena exhibiting anomalous diffusion. Specifically, the models involving a fractional differential oscillator equation, which contains a composition of left and right fractional derivatives, are proposed for the description of the processes of emptying the silo [10] and the heat flow through a bulkhead filled with granular material [20], respectively. Their studies show that the proposed models based on fractional calculus are efficient and describe well the processes.

In the aspect of theory, the study of fractional differential equations including both left and right fractional derivatives has attracted much attention by using fixed point theory and variational methods [2, 3, 8, 21, 22, 23, 24, 25, 27, 28] and their references. We note that, it is not easy to use the critical point theory to study the fractional

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differential equations including both left and right fractional derivatives, since it is often very difficult to establish a suitable space and a variational functional for the fractional boundary value problem.

Very recently in [21] the author considered the following fractional Hamiltonian systems

$$ {}_tD^{\alpha}_{\infty}({}_{-\infty}D^{\alpha}_{t}u(t)) + L(t)u(t) = \quad \nabla W(t,u(t)) \tag{2}$$

where $\alpha \in (1/2,1)$, $t \in \mathbb{R}$, $u \in \mathbb{R}^n$, $L \in C(\mathbb{R},\mathbb{R}^{n^2})$ is a symmetric matrix valued function for all $t \in \mathbb{R}$, $W \in C^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$ and $\nabla W(t,u(t))$ is the gradient of W at u. Assuming that L and W satisfy the following hypotheses:

(L) $L(t)$ is positive definite symmetric matrix for all $t \in \mathbb{R}$, and there exists an $l \in C(\mathbb{R},(0,\infty))$ such that $l(t) \to +\infty$ as $t \to \infty$ and

$$(L(t)x,x) \geq l(t)|x|^2, \text{ for all } t \in \mathbb{R} \text{ and } x \in \mathbb{R}^n. \tag{3}$$

(W_1) $W \in C^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$, and there is a constant $\mu > 2$ such that

$$0 < \mu W(t,x) \leq (x, \nabla W(t,x)), \text{ for all } t \in \mathbb{R} \text{ and } x \in \mathbb{R}^n \setminus \{0\}.$$

(W_2) $|\nabla W(t,x)| = o(|x|)$ as $x \to 0$ uniformly with respect to $t \in \mathbb{R}$.

(W_3) There exists $\overline{W} \in C(\mathbb{R}^n,\mathbb{R})$ such that

$$|W(t,x)| + |\nabla W(t,x)| \leq |\overline{W(x)}| \text{ for every } x \in \mathbb{R}^n \text{ and } t \in \mathbb{R}.$$

It showed that (2) has at least one nontrivial solution via Mountain pass Theorem.

In particular, if $\alpha = 1$, (2) reduces to the standard second order differential equation

$$u'' - L(t)u + \nabla W(t,u) = 0, \tag{4}$$

where $W : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ is a given function and $\nabla W(t,u)$ is the gradient of W at u. The existence of homoclinic solution is one of the most important problems in the history of that kind of equations, and has been studied intensively by many mathematicians. Assuming that $L(t)$ and $W(t,u)$ are independent of t, or T-periodic in t, many authors have studied the existence of homoclinic solutions for (4) via critical point theory and variational methods. In this case, the existence of homoclinic solution can be obtained by going to the limit of periodic solutions of approximating problems.

If $L(t)$ and $W(t,u)$ are neither autonomous nor periodic in t, this problem is quite different from the ones just described, because the lack of compacteness of the Sobolev embedding. In [16] the authors considered (4) without periodicity assumptions on L and W and showed that (4) possesses one homoclinic solution by using a variant of the mountain pass theorem without the Palais-Smale contidion. In [14], under the same assumptions of [16], the authors, by employing a new compact embedding theorem, obtained the existence of homoclinic solution of (4).

Motivated by this previous result, in this paper we consider the existence of nontrivial solution to (2) under some weaker condition than (L). More precisely we consider

(L_w) $L(t)$ is positive definite symmetric matrix for all $t \in \mathbb{R}$, and there exists an $l \in C(\mathbb{R},\mathbb{R})$ such that

(L_w^1) $\inf_{t\in\mathbb{R}} l(t) > 0$,

(L_w^2) There exists $r_0 > 0$ such that, for any $M > 0$

$$m(\{t \in (y-r_0, y+r_0)/ \ l(t) \leq M\}) \to 0 \text{ as } |y| \to \infty.$$

and

$$(L(t)x,x) \geq l(t)|x|^2, \text{ for all } t \in \mathbb{R} \text{ and } x \in \mathbb{R}^n. \tag{5}$$

We note that, mountain pass theorem can be used to get existence results for (2). However, the direct application of the mountain pass theorem is not enough since the Palais-Smale sequences might lose compactness in the whole space $\mathbb{R}$. Therefore the main difficulty of this paper is to show that the Palais-Smale condition holds. Before stating our results let us introduce the main ingredients involved in our approach. We define the space

$$X^\alpha = \left\{ u \in H^\alpha(\mathbb{R}, \mathbb{R}^n) | \ \int_{\mathbb{R}} \left[|_{-\infty}D_t^\alpha u(t)|^2 + (L(t)u(t), u(t)) \right] dt < \infty \right\},$$

with the inner product

$$\langle u, v \rangle_{X^\alpha} = \int_{\mathbb{R}} \left[({}_{-\infty}D_t^\alpha u(t), \ {}_{-\infty}D_t^\alpha v(t)) + (L(t)u(t), v(t)) \right] dt$$

and the corresponding norm

$$\|u\|_{X^\alpha}^2 = \langle u, u \rangle_{X^\alpha}.$$

Now we say that $u \in X^\alpha$ is a weak solution of (2) if

$$\int_{-\infty}^{\infty} [({}_{-\infty}D_t^\alpha u(t), {}_{-\infty}D_t^\alpha v(t)) + (L(t)u(t), v(t))] dt = \int_{-\infty}^{\infty} (\nabla W(t, u(t)), v(t)) dt,$$

for all $v \in X^\alpha$. For $u \in X^\alpha$ we may define the functional

$$I(u) = \frac{1}{2} \int_{-\infty}^{\infty} [|_{-\infty}D_t^\alpha u(t)|^2 + (L(t)u(t), u(t))] dt - \int_{-\infty}^{\infty} W(t, u(t)) dt. \tag{6}$$

which is of class C^1. We say that $u \in X^\alpha$ is a weak solution of (2) if u is a critical point of I.

Up until now, we can state our main result.

Theorem 1.1. *Suppose that* (L_w), $(W_1) - (W_3)$ *are satisfied. Then, (2) possesses at least one nontrivial solution.*

Remark 1.1. In [21], assuming (L) holds, the author introduced some compact embedding Lemma, (see its Lemma 2.2.) In the present paper, we weaken (L) to (L_w) and we get a new compact embedding Lemma (see Lemma 2.2 below), using this new compact embedding lemma we can verify the Palais-Smale condition.

The rest of the paper is organized as follows: In section §2, we describe the Liouville-Weyl fractional calculus and we introduce the fractional space that we use in our work and some proposition are proven which will aid in our analysis. In section §3, we prove Theorem 1.1.

2. Preliminary Results

2.1. Liouville-Weyl Fractional Calculus. In this section we introduce some basic definitions of fractional calculus which are used further in this paper. For more details we refer the reader to [6].

The Liouville-Weyl fractional integrals of order $0 < \alpha < 1$ are defined as

$${}_{-\infty}I_x^\alpha u(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} (x - \xi)^{\alpha - 1} u(\xi) d\xi \tag{7}$$

$${}_xI_\infty^\alpha u(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} (\xi - x)^{\alpha - 1} u(\xi) d\xi \tag{8}$$

The Liouville-Weyl fractional derivative of order $0 < \alpha < 1$ are defined as the left-inverse operators of the corresponding Liouville-Weyl fractional integrals

$$_{-\infty}D_x^{\alpha}u(x) = \frac{d}{dx}\,{}_{-\infty}I_x^{1-\alpha}u(x) \tag{9}$$

$$_{x}D_{\infty}^{\alpha}u(x) = -\frac{d}{dx}\,{}_{x}I_{\infty}^{1-\alpha}u(x) \tag{10}$$

The definitions (9) and (10) may be written in an alternative form:

$$_{-\infty}D_x^{\alpha}u(x) = \frac{\alpha}{\Gamma(1-\alpha)}\int_0^{\infty}\frac{u(x)-u(x-\xi)}{\xi^{\alpha+1}}d\xi \tag{11}$$

$$_{x}D_{\infty}^{\alpha}u(x) = \frac{\alpha}{\Gamma(1-\alpha)}\int_0^{\infty}\frac{u(x)-u(x+\xi)}{\xi^{\alpha+1}}d\xi \tag{12}$$

We establish the Fourier transform properties of the fractional integral and fractional differential operators. Recall that the Fourier transform $\widehat{u}(w)$ of $u(x)$ is defined by

$$\widehat{u}(w) = \int_{-\infty}^{\infty} e^{-ix.w}u(x)dx.$$

Let $u(x)$ be defined on $(-\infty, \infty)$. Then the Fourier transform of the Liouville-Weyl integral and differential operator satisfies

$$\widehat{_{-\infty}I_x^{\alpha}u(x)}(w) = (iw)^{-\alpha}\widehat{u}(w), \quad \widehat{_{x}I_{\infty}^{\alpha}u(x)}(w) = (-iw)^{-\alpha}\widehat{u}(w) \tag{13}$$

$$\widehat{_{-\infty}D_x^{\alpha}u(x)}(w) = (iw)^{\alpha}\widehat{u}(w), \quad \widehat{_{x}D_{\infty}^{\alpha}u(x)}(w) = (-iw)^{\alpha}\widehat{u}(w) \tag{14}$$

2.2. Fractional Derivative Spaces. In this section we introduce some fractional spaces for more detail see [4].

Let $\alpha > 0$. Define the semi-norm

$$|u|_{I_{-\infty}^{\alpha}} = \|_{-\infty}D_x^{\alpha}u\|_{L^2}$$

and norm

$$\|u\|_{I_{-\infty}^{\alpha}} = \left(\|u\|_{L^2}^2 + |u|_{I_{-\infty}^{\alpha}}^2\right)^{1/2}, \tag{15}$$

and

$$I_{-\infty}^{\alpha}(\mathbb{R}, \mathbb{R}^n) = \overline{C_0^{\infty}(\mathbb{R})}^{\|\cdot\|_{I_{-\infty}^{\alpha}}}.$$

Now we define the fractional Sobolev space $H^{\alpha}(\mathbb{R}, \mathbb{R}^n)$ in terms of the fourier transform. For $0 < \alpha < 1$, let the semi-norm

$$|u|_{\alpha} = \||w|^{\alpha}\widehat{u}\|_{L^2} \tag{16}$$

and norm

$$\|u\|_{\alpha} = \left(\|u\|_{L^2}^2 + |u|_{\alpha}^2\right)^{1/2},$$

and

$$H^{\alpha}(\mathbb{R}, \mathbb{R}^n) = \overline{C_0^{\infty}(\mathbb{R}, \mathbb{R}^n)}^{\|\cdot\|_{\alpha}}.$$

We note a function $u \in L^2(\mathbb{R}, \mathbb{R}^n)$ belong to $I_{-\infty}^{\alpha}(\mathbb{R}, \mathbb{R}^n)$ if and only if

$$|w|^{\alpha}\widehat{u} \in L^2(\mathbb{R}, \mathbb{R}^n). \tag{17}$$

Especially

$$|u|_{I_{-\infty}^{\alpha}} = \||w|^{\alpha}\widehat{u}\|_{L^2}. \tag{18}$$

Therefore $I_{-\infty}^{\alpha}(\mathbb{R}, \mathbb{R}^n)$ and $H^{\alpha}(\mathbb{R}, \mathbb{R}^n)$ are equivalent with equivalent semi-norm and norm. Analogous to $I_{-\infty}^{\alpha}(\mathbb{R}, \mathbb{R}^n)$ we introduce $I_{\infty}^{\alpha}(\mathbb{R}, \mathbb{R}^n)$. Let the semi-norm

$$|u|_{I_{\infty}^{\alpha}} = \|_{x}D_{\infty}^{\alpha}u\|_{L^2}$$

and norm

$$\|u\|_{I^\alpha_\infty} = \left(\|u\|^2_{L^2} + |u|^2_{I^\alpha_\infty}\right)^{1/2}, \tag{19}$$

and

$$I^\alpha_\infty(\mathbb{R},\mathbb{R}^n) = \overline{C_0^\infty(\mathbb{R},\mathbb{R}^n)}^{\|\cdot\|_{I^\alpha_\infty}}.$$

Moreover $I^\alpha_{-\infty}(\mathbb{R},\mathbb{R}^n)$ and $I^\alpha_\infty(\mathbb{R},\mathbb{R}^n)$ are equivalent, with equivalent semi-norm and norm [4]. We recall the Sobolev Lemma.

Theorem 2.1. [21] *If $\alpha > \frac{1}{2}$, then $H^\alpha(\mathbb{R},\mathbb{R}^n) \subset C(\mathbb{R},\mathbb{R}^n)$ and there is a constant $C = C_\alpha$ such that*

$$\sup_{x\in\mathbb{R}} |u(x)| \leq C\|u\|_\alpha \tag{20}$$

Remark 2.1. From Theorem 2.1, we now that if $u \in H^\alpha(\mathbb{R},\mathbb{R}^n)$ with $1/2 < \alpha < 1$, then $u \in L^q(\mathbb{R})$ for all $q \in [2,\infty)$, because

$$\int_{\mathbb{R}} |u(x)|^q dx \leq \|u\|^{q-2}_\infty \|u\|^2_{L^2}.$$

In what follows, we introduce the fractional space in which we will construct the variational framework of (2). Let

$$X^\alpha = \left\{ u \in H^\alpha(\mathbb{R},\mathbb{R}^n) | \quad \int_{\mathbb{R}} \left[|_{-\infty}D_t^\alpha u(t)|^2 + (L(t)u(t),u(t)) \right] dt < \infty \right\},$$

then X^α is a reflexive and separable Hilbert space with the inner product

$$\langle u,v\rangle_{X^\alpha} = \int_{\mathbb{R}} \left[({}_{-\infty}D_t^\alpha u(t),\ {}_{-\infty}D_t^\alpha v(t)) + (L(t)u(t),v(t)) \right] dt$$

and the corresponding norm

$$\|u\|^2_{X^\alpha} = \langle u,u\rangle_{X^\alpha}.$$

Similar to Lemma 2.1 in [21], we have the following conclusion.

Lemma 2.2. *Suppose L satisfies (L_w). Then X^α is continuously embedded in $H^\alpha(\mathbb{R},\mathbb{R}^n)$.*

Proof. Let $l_{min} = \inf_{t\in\mathbb{R}} l(t) > 0$, so we have

$$(L(t)u(t),u(t)) \geq l(t)|u(t)|^2 \geq l_{min}|u(t)|^2,\ \forall t \in \mathbb{R}.$$

Then

$$\begin{aligned} l_{min}\|u\|^2_\alpha &= l_{min}\left(\int_{\mathbb{R}} |_{-\infty}D_t^\alpha u(t)|^2 + |u(t)|^2 dt\right) \\ &\leq l_{min}\int_{\mathbb{R}} |_{-\infty}D_t^\alpha u(t)|^2 dt + \int_{\mathbb{R}} (L(t)u(t),u(t))dt. \end{aligned}$$

So

$$\|u\|^2_\alpha \leq K\|u\|^2_{X^\alpha}, \tag{21}$$

where $K = \frac{\max\{l_{min},1\}}{l_{min}}$. □

Lemma 2.3. *Suppose L satisfies (L_w). Then the imbedding of X^α in $L^2(\mathbb{R},\mathbb{R}^n)$ is compact.*

Proof. We note first that by Lemma 2.2 and Remark 2.1 we have

$$X^\alpha \hookrightarrow L^2(\mathbb{R},\mathbb{R}^n) \text{ is continuous.}$$

Now, let $(u_k) \in X^\alpha$ be a sequence such that $u_k \rightharpoonup u$ in X^α. We will show that $u_k \to u$ in $L^2(\mathbb{R},\mathbb{R}^n)$. Suppose, without loss of generality, that $u_k \rightharpoonup 0$ in X^α. The Banach-Steinhaus theorem implies that

$$A = \sup_k \|u_k\|_{X^\alpha} < +\infty.$$

For any $y \in \mathbb{R}$, $\forall M > 0$ set

$$\begin{aligned} I_M(y) &= \{t \in (y-r, y+r)/\ l(t) \le M\}, \\ \overline{I}_M(y) &= \{t \in (y-r, y+r)/\ l(t) > M\}. \end{aligned}$$

Choose $\{y_i\} \subset \mathbb{R}$ such that $\mathbb{R} \subset \cup_{i=1}^\infty (y_i - r, y_i + r)$ and each $t \in \mathbb{R}$ is covered by at most 2 such intervals. Then, for any $M > 0$ and $R > 2r$, we have

$$\begin{aligned} \int_{(-R,R)^c} |u_k(t)|^2 dt &\le \sum_{|y_i| \ge R-r}^{\infty} \int_{(y_i-r, y_i+r)} |u_k(t)|^2 dt \\ &\le \sum_{|y_i| \ge R-r}^{\infty} \left[\int_{(y_i-r,y_i+r)\cap I_M(y_i)} |u_k(t)|^2 dt + \int_{(y_i-r,y_i+r)\cap \overline{I}_M(y_i)} |u_k(t)|^2 dt \right] \\ &\le \sum_{|y_i| \ge R-r}^{\infty} \left[\int_{I_M(y_i)} |u_k(t)|^2 dt + \frac{1}{M} \int_{(y_i-r,y_i+r)} l(t)|u_k(t)|^2 dt \right] \\ &\le \sum_{|y_i| \ge R-r}^{\infty} \left[(\sup_{(y_i-r,y_i+r)} |u_k(t)|)^2 m(I_M(y_i)) + \frac{1}{M} \int_{(y_i-r,y_i+r)} l(t)|u_k(t)|^2 dt \right] \\ &\le \sum_{|y_i| \ge R-r}^{\infty} \left[Cm(I_M(y_i)) \|u_k\|^2_{X^\alpha(y_i-r,y_i+r)} + \frac{1}{M} \int_{(y_i-r,y_i+r)} l(t)|u_k(t)|^2 dt \right] \\ &\le 2\|u_k\|^2_{X^\alpha} \left[C \sup_{|y| \ge R-r} (m(I_M(y))) + \frac{1}{M} \right] \\ &\le 2A^2 \left[C \sup_{|y| \ge R-r} (m(I_M(y))) + \frac{1}{M} \right]. \end{aligned}$$

For any $\epsilon > 0$, taking M and R large enough, we can obtain that

$$\int_{(-R,R)^c} |u_k(t)|^2 dt \le \frac{\epsilon}{2}.$$

By Sobolev Theorem, $u_k \to 0$ uniformly on $[-R, R]$. Then, for such $R > 0$, there exists $k_0 > 0$ such that

$$\int_{[-R,R]} |u_k(t)|^2 dt \le \frac{\epsilon}{2}, \text{ for all } k \ge k_0.$$

Hence, by the arbitrary of ϵ we can obtain that $u_k \to 0$ in $L^2(\mathbb{R},\mathbb{R}^n)$. $\square$

Lemma 2.4. [21] *There are constants $c_1 > 0$ and $c_2 > 0$ such that*

$$W(t,u) \ge c_1|u|^\mu, \quad |u| \ge 1 \tag{22}$$

and

$$W(t,u) \le c_2|u|^\mu, \quad |u| \le 1. \tag{23}$$

Remark 2.2. By Lemma 2.4, we have

$$W(t,u) = o(|u|^2) \text{ as } u \to 0 \text{ uniformly in } t \in \mathbb{R}. \tag{24}$$

In addition, by (W_2), we have, for any $u \in \mathbb{R}^n$ such that $|u| \leq M_1$, there exists some constant $d > 0$ (dependent on M_1) such that

$$|\nabla W(t,u(t))| \leq d|u(t)|. \tag{25}$$

Similar to Lemma 2.4 of [21], we can get the following result.

Lemma 2.5. *Suppose that (L_w), (W_1)-(W_2) are satisfied. If $u_k \rightharpoonup u$ in X^α, then $\nabla W(t,u_k) \to \nabla W(t,u)$ in $L^2(\mathbb{R},\mathbb{R}^n)$.*

Proof. Assume that $u_k \rightharpoonup u$ in X^α. Then there exists a constant $d_1 > 0$ such that, by Banach-Steinhaus Theorem and (20),

$$\sup_{k\in\mathbb{N}} \|u_k\|_\infty \leq d_1, \quad \|u\|_\infty \leq d_1.$$

By (W_2), for any $\epsilon > 0$ there is $\delta > 0$ such that

$$|u_k| < \delta \text{ implies } |\nabla W(t,u_k)| \leq \epsilon |u_k|$$

and by (W_3) there is $M > 0$ such that

$$|\nabla W(t,u_k)| \leq M, \quad \text{for all } \delta < u_k \leq d_1.$$

Therefore, there exists a constant $d_2 > 0$ (dependent on d_1) such that

$$|\nabla W(t,u_k(t))| \leq d_2|u_k(t)|, \quad |\nabla W(t,u(t))| \leq d_2|u(t)|$$

for all $k \in \mathbb{N}$ and $t \in \mathbb{R}$. Hence,

$$|\nabla W(t,u_k(t)) - \nabla W(t,u(t))| \leq d_2(|u_k(t)| + |u(t)|) \leq d_2(|u_k(t) - u(t)| + 2|u(t)|),$$

Since, by Lemma 2.3, $u_k \to u$ in $L^2(\mathbb{R},\mathbb{R}^n)$, passing to a subsequence if necessary, it can be assumed that

$$\sum_{k=1}^\infty \|u_k - u\|_{L^2} < \infty.$$

But this implies $u_k(t) \to u(t)$ almost every where $t \in \mathbb{R}$ and

$$\sum_{k=1}^\infty |u_k(t) - u(t)| = v(t) \in L^2(\mathbb{R},\mathbb{R}^n).$$

Therefore

$$|\nabla W(t,u_k(t)) - \nabla W(t,u(t))| \leq d_2(v(t) + 2|u(t)|).$$

Then, using the Lebesgue's convergence theorem, the Lemma is proved. □

Now we introduce more notations and some necessary definitions. Let $\mathfrak{B}$ be a real Banach space, $I \in C^1(\mathfrak{B},\mathbb{R})$, which means that I is a continuously Fréchet differentiable functional defined on $\mathfrak{B}$. Recall that $I \in C^1(\mathfrak{B},\mathbb{R})$ is said to satisfy the (PS) condition if any sequence $\{u_k\}_{k\in\mathbb{N}} \in \mathfrak{B}$, for which $\{I(u_k)\}_{k\in\mathbb{N}}$ is bounded and $I'(u_k) \to 0$ as $k \to +\infty$, possesses a convergent subsequence in $\mathfrak{B}$.

Moreover, let B_r be the open ball in $\mathfrak{B}$ with the radius r and centered at 0 and ∂B_r denote its boundary. We obtain the existence of weak solutions of (2) by use of the following well-known Mountain Pass Theorems, see [17].

Theorem 2.6. *Let $\mathfrak{B}$ be a real Banach space and $I \in C^1(\mathfrak{B},\mathbb{R})$ satisfying (PS) condition. Suppose that $I(0) = 0$ and*

i. *There are constants $\rho, \beta > 0$ such that $I|_{\partial B_\rho} \geq \beta$, and*

ii. *There is and* $e \in \mathfrak{B} \setminus \overline{B_\rho}$ *such that* $I(e) \le 0$.
Then I *possesses a critical value* $c \ge \beta$. *Moreover* c *can be characterized as*

$$c = \inf_{\gamma \in \Gamma} \max_{s \in [0,1]} I(\gamma(s))$$

where

$$\Gamma = \{\gamma \in C([0,1], \mathfrak{B}) : \ \ \gamma(0) = 0, \ \ \gamma(1) = e\}.$$

3. Proof of Theorem 1.1

Now we are in position to proof Theorem 1.1. Although its proof is just the repetition of the process of Theorem 1.1 in [21], for the reader's convenience, we give the details.

Define the functional $I : X^\alpha \to \mathbb{R}$ by

$$\begin{aligned} I(u) &= \int_{\mathbb{R}} \left[\frac{1}{2}|_{-\infty}D_t^\alpha u(t)|^2 + \frac{1}{2}(L(t)u(t), u(t)) - W(t, u(t))\right] dt \\ &= \frac{1}{2}\|u\|^2_{X^\alpha} - \int_{\mathbb{R}} W(t, u(t))dt. \end{aligned} \tag{26}$$

Lemma 3.1. *Under the conditions of Theorem 1.1, we have*

$$I'(u)v = \int_{\mathbb{R}} [({}_{-\infty}D_t^\alpha u(t), {}_{-\infty}D_t^\alpha v(t)) + (L(t)u(t), v(t)) - (\nabla W(t, u(t)), v(t))]\, dt \tag{27}$$

for all $u, v \in X^\alpha$, *which yields that*

$$I'(u)u = \|u\|^2_{X^\alpha} - \int_{\mathbb{R}} (\nabla W(t, u(t)), u(t))dt. \tag{28}$$

Moreover, I *is a continuously Fréchet differentiable functional defined on* X^α, *i.e.,* $I \in C^1(X^\alpha, \mathbb{R})$.

Proof. We firstly show that $I : X^\alpha \to \mathbb{R}$. By (24), there is a $\delta > 0$ such that $|u| \le \delta$ implies that

$$W(t, u) \le \epsilon|u|^2 \ \text{ for all } t \in \mathbb{R}. \tag{29}$$

Let $u \in X^\alpha$, then $u \in C(\mathbb{R}, \mathbb{R}^n)$, the space of continuous function $u \in \mathbb{R}$ such that $u(t) \to 0$ as $|t| \to +\infty$. Therefore there is a constant $R > 0$ such that $|t| \ge R$ implies $|u(t)| \le \delta$. Hence, by (29), we have

$$\int_{\mathbb{R}} W(t, u(t)) \le \int_{-R}^{R} W(t, u(t))dt + \epsilon \int_{|t| \ge R} |u(t)|^2 dt < +\infty. \tag{30}$$

Combining (26) and (30), we show that $I : X^\alpha \to \mathbb{R}$.
Now we prove that $I \in C^1(X^\alpha, \mathbb{R})$. Rewrite I as follows

$$I = I_1 - I_2,$$

where

$$I_1 = \frac{1}{2}\int_{\mathbb{R}} [|_{-\infty}D_t^\alpha u(t)|^2 + (L(t)u(t), u(t))]dt, \ \ I_2 = \int_{\mathbb{R}} W(t, u(t))dt.$$

It is easy to check that $I_1 \in C^1(X^\alpha, \mathbb{R})$ and

$$I_1'(u)v = \int_{\mathbb{R}} [({}_{-\infty}D_t^\alpha u(t), {}_{-\infty}D_t^\alpha v(t)) + (L(t)u(t), v(t))]\, dt. \tag{31}$$

Thus it is sufficient to show this is the case for I_2. In the process we will see that

$$I_2'(u)v = \int_{\mathbb{R}} (\nabla W(t,u(t)), v(t))dt, \tag{32}$$

which is defined for all $u, v \in X^\alpha$. For any given $u \in X^\alpha$, let us define $J(u) : X^\alpha \to \mathbb{R}$ as follows

$$J(u)v = \int_{\mathbb{R}} (\nabla W(t,u(t)), v(t))dt, \quad \forall v \in X^\alpha.$$

It is obvious that $J(u)$ is linear. Now we show that $J(u)$ is bounded. Indeed, for any given $u \in X^\alpha$, by (25), there is a constant $d_3 > 0$ such that

$$|\nabla W(t,u(t))| \leq d_3 |u(t)|,$$

which yields that, by the Hölder inequality and Lemma 2.2

$$\begin{aligned} |J(u)v| &= \left| \int_{\mathbb{R}} (\nabla W(t,u(t)), v(t))dt \right| \leq d_3 \int_{\mathbb{R}} |u(t)||v(t)|dt \\ &\leq \frac{d_3}{l_{min}} \|u\|_{X^\alpha} \|v\|_{X^\alpha}. \end{aligned} \tag{33}$$

Moreover, for u and $v \in X^\alpha$, by Mean Value theorem, we have

$$\int_{\mathbb{R}} W(t, u(t) + v(t))dt - \int_{\mathbb{R}} W(t,u(t))dt = \int_{\mathbb{R}} (\nabla W(t, u(t) + h(t)v(t)))dt,$$

where $h(t) \in (0,1)$. Therefore, by Lemma 2.3 and the Hölder inequality, we have

$$\begin{aligned} &\int_{\mathbb{R}} (\nabla W(t, u(t) + h(t)v(t)), v(t))dt - \int_{\mathbb{R}} (\nabla W(t,u(t)), v(t))dt \\ &\quad = \int_{\mathbb{R}} (\nabla W(t,u(t)) + h(t)v(t) - \nabla W(t,u(t)), v(t))dt \to 0 \end{aligned} \tag{34}$$

as $v \to 0$ in X^α. Combining (33) and (34), we see that (32) holds. It remains to prove that I_2' is continuous. Suppose that $u \to u_0$ in X^α and note that

$$\begin{aligned} \sup_{\|v\|_{X^\alpha}=1} |I_2'(u)v - I_2'(u_0)v| &= \sup_{\|v\|_{X^\alpha}=1} \left| \int_{\mathbb{R}} (\nabla W(t,u(t)) - \nabla W(t,u_0(t)), v(t))dt \right| \\ &\leq \sup_{\|v\|_{X^\alpha}=1} \|\nabla W(.,u(.)) - \nabla W(.,u_0(.))\|_{L^2} \|v\|_{L^2} \\ &\leq \frac{1}{\sqrt{l_{\min}}} \|\nabla W(.,u(.)) - \nabla W(.,u_0(.))\|_{L^2} \end{aligned}$$

By Lemma 2.3, we obtain that $I_2'(u)v - I_2'(u_0)v \to 0$ as $\|u\|_{X^\alpha} \to \|u_0\|_{X^\alpha}$ uniformly with respect to v, which implies the continuity of I_2' and $I \in C^1(X^\alpha, \mathbb{R})$. $\square$

Lemma 3.2. *Suppose that* $(L_w), (W_1) - (W_2)$ *are satisfied. I satisfies the (PS) condition.*

Proof. Assume that $(u_k)_{k\in\mathbb{N}} \in X^\alpha$ is a sequence such that $\{I(u_k)\}_{k\in\mathbb{N}}$ is bounded and $I'(u_k) \to 0$ as $k \to +\infty$. Then there exists a constant $C_1 > 0$ such that

$$|I(u_k)| \leq C_1, \quad \|I'(u_k)\|_{(X^\alpha)^*} \leq C_1 \tag{35}$$

for every $k \in \mathbb{N}$.

We firstly prove that $\{u_k\}_{k\in\mathbb{N}}$ is bounded in X^α. By (26), (28) and (W_1), we have

$$\begin{aligned} C_1 + \|u_k\|_{X^\alpha} &\geq I(u_k) - \frac{1}{\mu} I'(u_k)u_k \\ &= \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_k\|^2_{X^\alpha} - \int_{\mathbb{R}} [W(t,u_k(t)) - \frac{1}{\mu}(\nabla W(t,u_k(t)), u_k(t))]dt \\ &\geq \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_k\|^2_{X^\alpha}. \end{aligned} \tag{36}$$

Since $\mu > 2$, the inequality (36) shows that $\{u_k\}_{k\in\mathbb{N}}$ is bounded in X^α. So passing to a subsequence if necessary, it can be assumed that $u_k \rightharpoonup u$ in X^α and hence, by Lemma 2.3, $u_k \to u$ in $L^2(\mathbb{R},\mathbb{R}^n)$. It follows from the definition of I that

$$\begin{aligned} &(I'(u_k) - I'(u))(u_k - u) \\ &= \|u_k - u\|^2_{X^\alpha} - \int_{\mathbb{R}} [\nabla W(t,u_k) - \nabla W(t,u)](u_k - u)dt. \end{aligned} \tag{37}$$

Since $u_k \to u$ in $L^2(\mathbb{R},\mathbb{R}^n)$, we have (see Lemma 2.5) $\nabla W(t,u_k(t)) \to \nabla W(t,u(t))$ in $L^2(\mathbb{R},\mathbb{R}^n)$. Hence

$$\int_{\mathbb{R}} (\nabla W(t,u_k(t)) - \nabla W(t,u(t)), u_k(t) - u(t))dt \to 0$$

as $k \to +\infty$. So (37) implies

$$\|u_k - u\|_{X^\alpha} \to 0 \text{ as } k \to +\infty.$$

□

Now we are in the position to give the proof of theorem 1.1. We divide the proof into several steps.

Proof. **Step 1.** It is clear that $I(0) = 0$ and $I \in C^1(X^\alpha,\mathbb{R})$ satisfies the (PS) condition by lemma 3.1 and 3.2.

Step 2. We show that there exist constant $\rho > 0$ and $\beta > 0$ such that I satisfies the condition (i) of Theorem 2.6. By Lemma 2.3, there is a $C_0 > 0$ such that

$$\|u\|_{L^2} \leq C_0\|u\|_{X^\alpha}.$$

On the other hand by Theorem 2.1, there is $C_\alpha > 0$ such that

$$\|u\|_\infty \leq C_\alpha\|u\|_{X^\alpha}.$$

By (24), for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$W(t,u(t)) \leq \epsilon|u(t)|^2 \text{ wherever } |u(t)| < \delta.$$

Let $\rho = \frac{\delta}{C_\alpha}$ and $\|u\|_{X^\alpha} \leq \rho$; we have $\|u\|_\infty \leq \frac{\delta}{C_\alpha}.C_\alpha = \delta$. Hence

$$|W(t,u(t))| \leq \epsilon|u(t)|^2 \text{ for all } t \in \mathbb{R}.$$

Integrating on $\mathbb{R}$, we get

$$\int_{\mathbb{R}} W(t,u(t))dt \leq \epsilon\|u\|^2_{L^2} \leq \epsilon C_0^2\|u\|^2_{X^\alpha}$$

So, if $\|u\|_{X^\alpha} = \rho$, then

$$I(u) = \frac{1}{2}\|u\|^2_{X^\alpha} - \int_{\mathbb{R}} W(t,u(t))dt \geq (\frac{1}{2} - \epsilon C_0^2)\|u\|^2_{X^\alpha} = (\frac{1}{2} - \epsilon C_0^2)\rho^2.$$

And it suffices to choose $\epsilon = \frac{1}{4C_0^2}$ to get

$$I(u) \geq \frac{\rho^2}{4C_0^2} = \beta > 0. \tag{38}$$

Step 3. It remains to prove that there exists an $e \in X^\alpha$ such that $\|e\|_{X^\alpha} > \rho$ and $I(e) \leq 0$, where ρ is defined in Step 2. Consider

$$I(\sigma u) = \frac{\sigma^2}{2}\|u\|_{X^\alpha}^2 - \int_{\mathbb{R}} W(t, \sigma u(t))dt$$

for all $\sigma \in \mathbb{R}$. By (22), there is $c_1 > 0$ such that

$$W(t, u(t)) \geq c_1|u(t)|^\mu \text{ for all } |u(t)| \geq 1. \tag{39}$$

Take some $u \in X^\alpha$ such that $\|u\|_{X^\alpha} = 1$. Then there exists a subset Ω of positive measure of $\mathbb{R}$ such that $u(t) \neq 0$ for $t \in \Omega$. Take $\sigma > 0$ such that $\sigma|u(t)| \geq 1$ for $t \in \Omega$. Then by (39), we obtain

$$I(\sigma u) \leq \frac{\sigma^2}{2} - c_1\sigma^\mu \int_\Omega |u(t)|^\mu dt. \tag{40}$$

Since $c_1 > 0$ and $\mu > 2$, (40) implies that $I(\sigma u) < 0$ for some $\sigma > 0$ with $\sigma|u(t)| \geq 1$ for $t \in \Omega$ and $\|\sigma u\|_{X^\alpha} > \rho$, where ρ is defined in Step 2. By Theorem 2.6, I possesses a critical value $c \geq \beta > 0$ given by

$$c = \inf_{\gamma \in \Gamma} \max_{s \in [0,1]} I(\gamma(s))$$

where

$$\Gamma = \{\gamma \in C([0,1], X^\alpha) : \ \gamma(0) = 0, \ \gamma(1) = e\}.$$

Hence there is $u \in X^\alpha$ such that

$$I(u) = c, \ \ I'(u) = 0.$$

□

References

[1] O. Agrawal, J. Tenreiro Machado, and J. Sabatier, *Fractional derivatives and their application: Nonlinear dynamics*, Springer-Verlag, Berlin, 2004.

[2] T. Atanackovic and B. Stankovic, On a class of differential equations with left and right fractional derivatives, *ZAMM* **87** (2007), 537–539.

[3] D. Baleanu and J. Trujillo, On exact solutions of a class of fractional Euler-Lagrange equations, *Nonlinear Dyn.* **52** (2008), 331–335.

[4] V. Ervin and J. Roop, Variational formulation for the stationary fractional advection dispersion equation, *Numer. Meth. Part. Diff. Eqs* **22** (2006), 58–76.

[5] R. Herrmann, *Fractional calculus: An introduction for physicists 2 ed.*, World Scientific Publishing, 2014.

[6] R. Hilfer, *Applications of fractional calculus in physics*, World Scientific, Singapore, 2000.

[7] A. Kilbas, H. Srivastava, and J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, **204**, Amsterdam, 2006.

[8] M. Klimek, Existence and uniqueness result for a certain equation of motion in fractional mechanics, *Bull. Polish Acad. Sci. Tech. Sci.* **58**(4) (2010), 573–581.

[9] F. Jiao and Y. Zhou, Existence results for fractional boundary value problem via critical point theory, Intern. *Journal of Bif. and Chaos* **22**(4) (2012), 1–17.

[10] J. Leszczynski and T. Blaszczyk, Modeling the transition between stable and unstable operation while emptying a silo, *Granular Matter* **13**(4) (2011), 429–438.

[11] J. Mawhin and M. Willem, *Critical point theory and Hamiltonian systems*, Applied Mathematical Sciences **74**, Springer, Berlin, 1989.

[12] R. Metsler and J. Klafter, The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics, *J. Phys. A* **37** (2004) , 161–208.
[13] K. Miller and B. Ross, *An introduction to the fractional calculus and fractional differential equations*, Wiley and Sons, New York, 1993.
[14] M. Omana and M. Willem, Homoclinic orbits for a class of Hamiltonian systems, *Differ. Integr. Equ.* **5**(5) (1992), 1115–1120.
[15] I. Podlubny, *Fractional differential equations*, Academic Press, New York, 1999.
[16] P. Rabinowitz and K. Tanaka, Some result on connecting orbits for a class of Hamiltonian systems, *Math. Z.* **206** (1991), 473–499.
[17] P. Rabinowitz, *Minimax method in critical point theory with applications to differential equations*, CBMS Amer. Math. Soc., **65**, 1986.
[18] J. Sabatier, O. Agrawal, and J. Tenreiro Machado, *Advances in fractional calculus. Theoretical developments and applications in physics and engineering*, Springer-Verlag, Berlin, 2007.
[19] S. Samko, A. Kilbas, and O. Marichev, *Fractional integrals and derivatives: Theory and applications*, Gordon and Breach, New York, 1993.
[20] E. Szymanek, The application of fractional order differential calculus for the description of temperature profiles in a granular layer, In: *Advances in the Theory and Applications of Non-integer Order Systems*, vol. **257** of Lecture Notes in Electrical Engineering (2013), 243–248.
[21] C. Torres, Existence of solution for fractional Hamiltonian systems, *Electronic Jour. Diff. Eq.* **259** (2013), 1–12.
[22] C. Torres, Mountain pass solution for a fractional boundary value problem, *Journal of Fractional Calculus and Applications* **5**(1) (2014), 1–10.
[23] C. Torres, Existence of a solution for fractional forced pendulum, *Journal of Applied Mathematics and Computational Mechanics* **13**(1) (2014), 125–142.
[24] A. Mendez Cruz, C. Torres Ledesma, and W. Zubiaga Vera, Liouville-Weyl Fractional Hamiltonian Systems: Existence result, preprint.
[25] C. Torres, Ground state solution for a class of differential equations with left and right fractional derivatives, Mathematical Methods in the Applied Sciences, DOI: 10.1002/mma.3426.
[26] B. West, M. Bologna, and P. Grigolini, *Physics of fractal operators*, Springer-Verlag, Berlin, 2003.
[27] Z. Zhang and R. Yuang, Variational approach to solutions for a class of fractional Hamiltonian systems, *Math. Methods Appl. Sci.* **37**(13) (2014), 1873–1883.
[28] Z. Zhang and R. Yuang, Solutions for subquadratic fractional Hamiltonian systems without coercive conditions, *Math. Methods Appl. Sci.* **37**(18) (2014), 2934–2945.

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