Multiple positive periodic solutions for a delayed predator-prey system with Beddington-DeAngelis functional response and harvesting terms

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ABSTRACT. In this paper, a delayed predator-prey system with Beddington-DeAngelis functional response and harvesting terms is studied. By using Mawhin’s continuation theorem, the sufficient conditions are established for the existence of at least four positive periodic solutions. Finally, an example is presented to illustrate the effectiveness of the results.

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1. Introduction

In the past decades, various mathematical models with delays have been proposed in the study of population dynamics([1-20]). Among these models, predator-prey systems play an important role in population theory. One of the most popular predator-prey models is the one with a Beddington-DeAngelis functional response which was originally proposed by Beddington [1] and DeAngelis et al. [2], independently. The dynamics of this model is described by differential equations in the form

$$\dot{x} = rx(t) \left( 1 - \frac{x(t)}{K} \right) - \frac{bx(t)y(t)}{1+nx(t)+my(t)},$$

$$\dot{y} = y(t) \left( -d + \frac{fh(t)}{1+nx(t)+my(t)} \right).$$

Recently, predator-prey systems with a Beddington-DeAngelis functional response were widely investigated ([3,4,10,12,14,15,17]). For example, in [3], the authors studied the following nonautonomous delayed predator-prey model with the Beddington-DeAngelis functional response

$$\dot{x} = x(t)[a(t) - b(t)x(t - \tau(t, x(t), y(t)))] - \frac{c(t)x(t)y(t)}{1+nx(t)+my(t)},$$

$$\dot{y} = y(t) \left[ \frac{f(t)x(t-\sigma(t,x(t),y(t)))}{1+nx(t-\sigma(t,x(t),y(t)))+my(t-\sigma(t,x(t),y(t)))} - d(t) \right].$$

In addition, since the exploitation of biological resources and the harvest of population species are commonly practiced in fishery, forestry, and wildlife management, the study of population dynamics with harvesting is an important subject in mathematical bioeconomics (see [5-7,11,13,16,18-20], for example). This motivates us to consider the
MUL{TIPLE PERIODIC SOLUTIONS FOR A DELAYED PREDATOR-PREY SYSTEM 331

following nonautonomous delayed predator-prey system with Beddington-DeAngelis functional response and harvesting terms:

\[ \begin{align*}
\dot{x}(t) &= a(t) x(t) \left(1 - \frac{x(t)}{K}\right) - \frac{b(t) x(t) y(t)}{f(t)+m(t)x(t)+n(t)y(t)} - h_1(t), \\
\dot{y}(t) &= y(t) (c(t) - d(t) y(t)) - \frac{r(t) x(t-\tau(t)) y(t)}{f(t)+m(t)x(t-\tau(t))+n(t)y(t-\tau(t))} - h_2(t),
\end{align*} \]

where \( x(t) \) and \( y(t) \) denote prey and predator population, respectively, \( a(t), b(t), c(t), d(t), f(t), m(t), n(t), h_i(t) \) \((i = 1, 2)\), \( \tau(t) \) are all positive continuous \( \omega \)-periodic functions, \( K \) is a positive constant. Here \( a(t), c(t) \) represent the intrinsic growth rate and \( K \) the carrying capacity of the prey, \( d(t) \) is the death rate of the predator, \( m(t), n(t) \) is the conversion factor denoting the number of newly born predators for each captured prey. \( h_i(t), i = 1, 2 \) is the \( i \)-th species harvesting terms standing for the harvests, \( \tau(t) \) is the state dependent delay.

The rest of the paper is arranged as follows. In Section 2, we present Mawhin’s continuation theorem and establish existence of four positive periodic solutions. In section 3, we give an example to illustrate the effectiveness of the results.

2. Existence of four positive periodic solutions

In this section, by using Mawhin’s continuation theorem, we shall show the existence of positive periodic solutions of (1). To do so, we need to make some preparations.

Let \( X \) and \( Z \) be real normed vector spaces. Let \( L : \text{Dom}L \subset X \to Z \) be a linear mapping and \( N : X \times [0,1] \to Z \) be a continuous mapping.

The mapping \( L \) will be called a Fredholm mapping of index zero if \( \dim \text{Ker}L = \text{codim Im}L < \infty \) and \( \text{Im}L \) is closed in \( Z \).

If \( L \) is a Fredholm mapping of index zero, then there exist two continuous projectors \( P : X \to X \) and \( Q : Z \to Z \) such that \( \text{Im}P = \text{Ker}L \) and \( \text{Ker}Q = \text{Im}L = \text{Im}(I - Q) \), and \( X = \text{Ker}L \oplus \text{Ker}P \) and \( Z = \text{Im}L \oplus \text{Im}Q \). It follows that \( L|_{\text{Dom}L \cap \text{Ker}P} : (I - P)X \to \text{Im}L \) is invertible and its inverse is denoted by \( K_P \).

Set \( \Omega \) is a bounded open subset of \( X \), if \( QN\Omega \times [0,1] \) is bounded and \( K_P(I - Q)N : \bar{\Omega} \times [0,1] \to X \) is compact, then the mapping \( N \) is called \( L \)-compact on \( \bar{\Omega} \times [0,1] \).

Because \( \text{Im}Q \) is isomorphic to \( \text{Ker}L \), there exists an isomorphism \( J : \text{Im}Q \to \text{Ker}L \).

The Mawhin’s continuous theorem [8, p. 40] is given as follows.

**Lemma 2.1.** ([8]) Let \( L \) be a Fredholm mapping of index zero and let \( N \) be \( L \)-compact on \( \bar{\Omega} \times [0,1] \). Assume

(a) for each \( \lambda \in (0,1) \), every solution \( x \) of \( Lx = \lambda N(x, \lambda) \) is such that \( x \notin \partial\Omega \cap \text{Dom}L \);

(b) \( QN(x, 0) \neq 0 \) for each \( x \in \partial\Omega \cap \text{Ker}L \);

(c) \( \text{deg}(JQN(x, 0), \Omega \cap \text{Ker}L, 0) \neq 0 \).

Then, \( Lx = N(x, 1) \) has at least one solution in \( \Omega \cap \text{Dom}L \).

**Lemma 2.2.** Let \( x > 0, y > 0, z > 0 \) and \( x > 2\sqrt{\frac{y}{2}} \). For the functions \( f(x, y, z) = \frac{x^3 - 4yz}{2y} \) and \( g(x, y, z) = \frac{x + \sqrt{x^2 - 4yz}}{2y} \), the following assertions hold.

(i) \( f(x, y, z) \) is monotonically decreasing on the variable \( x \in (0, \infty) \), monotonically increasing on the variable \( y \in (0, \infty) \), monotonically increasing on the variable \( z \in (0, \infty) \), respectively.

(ii) \( g(x, y, z) \) is monotonically increasing on the variable \( x \in (0, \infty) \), monotonically
decreasing on the variable $y \in (0, \infty)$, monotonically decreasing on the variable $z \in (0, \infty)$, respectively.

**Proof.** By the relationship of the derivative and the monotonicity, the above assertions are easily proved, and we omit them. \hfill \square

For the sake of convenience, we denote

$$f^l = \min_{t \in [0, \omega]} f(t), \quad f^M = \max_{t \in [0, \omega]} f(t), \quad \bar{f} = \frac{1}{\omega} \int_0^\omega f(t)dt,$$

here $f(t)$ is a continuous $\omega$-periodic function.

Throughout this paper, we need the following assumptions.

(A1) $a^l > 2 \sqrt{\frac{a^M}{K} \left( h_1^M + \frac{b^M H_1}{m^l} \right)}$, and $a^M > 2 \sqrt{\frac{a^M}{K}}$,

(A2) $c^l > \frac{r^M l_1^+}{f_1 + m^l l_1} + 2 \sqrt{d^M h_2^M}$.

Our main result is stated as follows.

**Theorem 2.3.** Assume that (A1) and (A2) hold. Then, system (1) has at least four positive $\omega$-periodic solutions.

**Proof.** By making the substitution $x(t) = \exp\{u_1(t)\}, \ y(t) = \exp\{u_2(t)\}$, system (1) can be reformulated as

$$\begin{align*}
\dot{u}_1(t) &= a(t) \left( 1 - \frac{e^{u_1(t)}}{K} \right) - \frac{b(t)e^{u_2(t)}}{f(t)+m(t)e^{u_1(t)}+n(t)e^{u_2(t)}} - h_1(t)e^{-u_1(t)}, \\
\dot{u}_2(t) &= c(t) - d(t)e^{u_2(t)} - \frac{r(t)e^{u_1(t)-\tau(t)}}{f(t)+m(t)e^{u_1(t)}+n(t)e^{u_2(t)-\tau(t)}} - h_2(t)e^{-u_2(t)}. \\
\end{align*}
$$

(2)

Let

$$X = Z = \{ u = (u_1, u_2)^T \in C(\mathbb{R}, \mathbb{R}^2) : u(t + \omega) = u(t) \},$$

and define

$$\| u \| = \sum_{i=1}^2 \max_{t \in [0, \omega]} |u_i(t)|, \quad u \in X \text{ or } Z.$$

Equipped with the above norm $\| \cdot \|$, $X$ and $Z$ are Banach spaces. For $u \in X$, let

$$N(u, \lambda) = \left( \begin{array}{c}
a(t) \left( 1 - \frac{e^{u_1(t)}}{K} \right) - \frac{b(t)e^{u_2(t)}}{f(t)+m(t)e^{u_1(t)}+n(t)e^{u_2(t)}} - h_1(t)e^{-u_1(t)} \\
c(t) - d(t)e^{u_2(t)} - \frac{r(t)e^{u_1(t)-\tau(t)}}{f(t)+m(t)e^{u_1(t)}+n(t)e^{u_2(t)-\tau(t)}} - h_2(t)e^{-u_2(t)}
\end{array} \right).$$

We put

$$Pu = \frac{1}{\omega} \int_0^\omega u(t)dt, \quad u \in X; \quad Qz = \frac{1}{\omega} \int_0^\omega z(t)dt, \quad z \in Z.$$

Thus, it follows that $\text{Ker} L = \mathbb{R}^2$, $\text{Im} L = \{ z \in Z : \int_0^\omega z(t)dt = 0 \}$ is closed in $Z$, $\text{dim Ker} L = 2 = \text{codim Im} L$, and $P, Q$ are continuous projectors such that

$$\text{Im} P = \text{Ker} L, \quad \text{Ker} Q = \text{Im} L = \text{Im} (I - Q).$$
Hence, $L$ is a Fredholm mapping of index zero. Furthermore, the generalized inverse (to $L$) \(K_p : \text{Im} L \to \text{Ker} P \cap \text{Dom} L\) is given by
\[
K_p(z) = \int_0^t z(s)\,ds - \frac{1}{\omega} \int_0^\omega \int_0^t z(s)\,ds\,dt.
\]
Then
\[
QN(u, \lambda) = \left( \frac{1}{\omega} \int_0^\omega F_1(s, \lambda)\,ds \right)
\]
and
\[
K_p(I-Q)N(u, \lambda) = \left( \int_0^t F_1(s, \lambda)\,ds - \frac{1}{\omega} \int_0^\omega \int_0^t F_1(s, \lambda)\,ds\,dt + \left( \frac{1}{2} - \frac{r}{\omega} \right) \int_0^\omega F_1(s, \lambda)\,ds \right)
\]
where
\[
F_1(s, \lambda) = a(s)(1 - e^{u_1(s)} - \lambda \frac{b(s)e^{u_2(s)}}{f(s) + m(s)e^{u_1(s)} + n(s)e^{u_2(s)}} - h_1(s)e^{-u_1(s)}),
\]
\[
F_2(s, \lambda) = c(s) - d(s)e^{u_2(s)} - \lambda \frac{r(s)e^{u_1(s) - u_2(s)}}{f(s) + m(s)e^{u_1(s) - u_2(s)} + n(s)e^{u_2(s) - u_1(s)}} - h_2(s)e^{-u_2(s)}.
\]
Obviously, $QN$ and $K_p(I-Q)N$ are continuous and, moreover, $QN(\overline{\Omega} \times [0, 1])$, $K_p(I-Q)N(\overline{\Omega} \times [0, 1])$ are relatively compact for any open bounded set $\Omega \subset X$ by using the Arzela-Ascoli theorem. Hence, $N$ is $L$-compact on $\overline{\Omega} \times [0, 1]$, with any open bounded set $\Omega \subset X$.

In order to use Lemma 2.1, we have to find at least four appropriate open bounded subsets in $X$. Corresponding to the operator equation $Lu = \lambda N(u, \lambda)$, $\lambda \in (0, 1)$, we have
\[
\dot{u}_1(t) = \lambda \left( a(t)(1 - e^{u_1(t)} - \lambda \frac{b(t)e^{u_2(t)}}{f(t) + m(t)e^{u_1(t)} + n(t)e^{u_2(t)}} - h_1(t)e^{-u_1(t)}) \right),
\]
\[
\dot{u}_2(t) = \lambda \left( c(t) - d(t)e^{u_2(t)} - \lambda \frac{r(t)e^{u_1(t) - u_2(t)}}{f(t) + m(t)e^{u_1(t) - u_2(t)} + n(t)e^{u_2(t) - u_1(t)}} - h_2(t)e^{-u_2(t)} \right).
\]
(3)

Assume that $u \in X$ is an $\omega$-periodic solution of system (3) for some $\lambda \in (0, 1)$. Then, there exist $\xi_i, \eta_i \in [0, \omega]$ such that
\[
u_i(\xi_i) = \max_{t \in [0, \omega]} u_i(t), \quad u_i(\eta_i) = \min_{t \in [0, \omega]} u_i(t), \quad i = 1, 2.
\]

It is clear that $\dot{u}_i(\xi_i) = 0$, $\dot{u}_i(\eta_i) = 0$, $i = 1, 2$. From this and (3), we have
\[
a(\xi_1) \left( 1 - e^{u_1(\xi_1)} \right) - \lambda \frac{b(\xi_1)e^{u_2(\xi_1)}}{f(\xi_1) + m(\xi_1)e^{u_1(\xi_1)} + n(\xi_1)e^{u_2(\xi_1)}} - h_1(\xi_1)e^{-u_1(\xi_1)} = 0,
\]
\[
c(\xi_2) - d(\xi_2)e^{u_2(\xi_2)} - \lambda \frac{r(\xi_2)e^{u_1(\xi_2) - u_2(\xi_2)}}{f(\xi_2) + m(\xi_2)e^{u_1(\xi_2) - u_2(\xi_2)} + n(\xi_2)e^{u_2(\xi_2) - u_1(\xi_2)}} - h_2(\xi_2)e^{-u_2(\xi_2)} = 0,
\]
and
\[
a(\eta_1) \left( 1 - e^{u_1(\eta_1)} \right) - \lambda \frac{b(\eta_1)e^{u_2(\eta_1)}}{f(\eta_1) + m(\eta_1)e^{u_1(\eta_1)} + n(\eta_1)e^{u_2(\eta_1)}} - h_1(\eta_1)e^{-u_1(\eta_1)} = 0,
\]
\[
c(\eta_2) - d(\eta_2)e^{u_2(\eta_2)} - \lambda \frac{r(\eta_2)e^{u_1(\eta_2) - u_2(\eta_2)}}{f(\eta_2) + m(\eta_2)e^{u_1(\eta_2) - u_2(\eta_2)} + n(\eta_2)e^{u_2(\eta_2) - u_1(\eta_2)}} - h_2(\eta_2)e^{-u_2(\eta_2)} = 0.
\]
According to the first equation of (4), we have
\[ \frac{a_1}{K} e^{a_1(\xi_1)} + h_1 e^{-u_1(\xi_1)} \leq \frac{a(\xi_1)e^{a_1(\xi_1)}}{K} + \lambda \frac{h(\xi_1)e^{a_1(\xi_1)}}{T(\xi_1) + m(\xi_1)e^{2u_1(\xi_1)} + mi_1e^{2u_2(\xi_1)} + h_1(\xi_1)e^{-u_1(\xi_1)}} \]
amely,
\[ \frac{a_1}{K} e^{2u_1(\xi_1)} - a^M e^{u_1(\xi_1)} + h_1^2 < 0, \]
which implies that
\[ \ln l_1^- < u_1(\xi_1) < \ln l_1^+, \] where
\[ l_1^\pm = a^M \pm \sqrt{a^{2M} - \frac{4a^2h_1}{K}}. \]
Similarly, by the first equation of (5), we obtain
\[ \ln l_1^- < u_1(\eta_1) < \ln l_1^+. \] (7)
Denote \( c_1^+ = c - \frac{e^{M l_1^+}}{l_1^+ - m l_1^+} \). The second equation of (4) gives
\[ d^M e^{2u_2(\xi_2)} + h_2^M > d(\xi_2)e^{2u_2(\xi_2)} + h_2(\xi_2) > c_1^+ e^{u_2(\xi_2)}, \]
that is,
\[ d^M e^{2u_2(\xi_2)} - c_1^+ e^{u_2(\xi_2)} + h_2^M > 0, \]
which implies that
\[ u_2(\xi_2) > \ln l_2^+ \quad \text{or} \quad u_2(\xi_2) < \ln l_2^-, \] where
\[ l_2^\pm = c_1^+ \pm \sqrt{(c_1^+)^2 - 4d^M h_2^M}. \]
Similarly, by the second equation of (5), we get
\[ u_2(\eta_2) > \ln l_2^+ \quad \text{or} \quad u_2(\eta_2) < \ln l_2^- \] (9)
Denote \( H_1 = \frac{e^M}{c_1^+} \). Then from the second equation of (4), we have
\[ d^M e^{2u_2(\xi_2)} \leq d(\xi_2)e^{2u_2(\xi_2)} + h_2(\xi_2)e^{-u_2(\xi_2)} < e^M, \]
that is,
\[ u_2(\xi_2) < \ln \frac{e^M}{c_1^+} = \ln H_1. \] (10)
Similarly, denote \( H_2 = \frac{h_1}{c_1^+} \), then from the second equation of (5), we obtain
\[ h_2^M e^{-u_2(\eta_2)} \leq h_2(\eta_2)e^{-u_2(\eta_2)} < h_2(\eta_2)e^{-u_2(\eta_2)} + d(\eta_2)e^{u_2(\eta_2)} < e^M, \]
which implies that
\[ u_2(\eta_2) > \ln H_2. \] (11)
We claim that \( \ln l_1^+ < \ln H_1 \), \( \ln H_2 < \ln l_2^- \). In fact,
\[ l_2^- = \frac{c_1^+ - \sqrt{(c_1^+)^2 - 4d^M h_2^M}}{2d^M} < \frac{c_1^+}{d^M} < \frac{c_1^+}{d^M} = H_1, \]
\[ l_2^+ = \frac{c_1^+ + \sqrt{(c_1^+)^2 - 4d^M h_2^M}}{2d^M} = \frac{2h_2^M}{c_1^+ + \sqrt{(c_1^+)^2 - 4d^M h_2^M}} > \frac{h_2^M}{e^M} > \frac{h_2^M}{e^M} = H_2. \]
From (8)-(11), we have
\[ \ln H_2 < u_2(\eta_2) < u_2(\xi_2) < \ln l_2^- \quad \text{or} \quad \ln l_2^+ < u_2(\eta_2) < u_2(\xi_2) < \ln H_1. \] (12)
According to the first equation of (4), we have
\[ \frac{a^M}{M} e^{u_1(\xi)} + \frac{h_1^M}{K} e^{-u_1(\xi)} + \frac{b^M h_1}{m^M} e^{-u_1(\xi)} > a(\xi) e^{u_1(\xi)} + \frac{b(\xi) e^{-u_1(\xi)}}{f(\xi) + n(\xi)} - h_1(\xi) e^{-u_1(\xi)} \]
\[ = a(\xi) \geq a', \]
and
\[ \frac{a^M}{K} e^{2u_1(\xi)} - a' e^{u_1(\xi)} + \left( \frac{h_1^M}{K} + \frac{b^M H_1}{m^M} \right) > 0, \]
which implies that
\[ u_1(\xi) > \ln A^+, \ u_1(\xi) < \ln A^- , \]
where
\[ A^\pm = \frac{a' \pm \sqrt{a'^2 - 4a^M (h_1^M + \frac{b^M H_1}{m^M})}}{2a^M/K}. \]
Similarly, by the first equation of (5), we obtain
\[ u_1(\eta_1) > \ln A^+, \ u_1(\eta_1) < \ln A^- . \]
We claim that \( \ln l_1^- < \ln A^- \), \( \ln A^+ < \ln l_1^+ \). In fact, employing Lemma 2.2, we have
\[ l_1^- = \frac{a^M - \sqrt{a'^2 - 4a^M h_1^M}}{2a^M/K} = f(a^M, a'/K, h_1^M) \]
\[ < f(a', \frac{a^M}{K}, h_1^M) = a' - \sqrt{a'^2 - 4a^M h_1^M} \]
\[ = A^- , \]
\[ l_1^+ = \frac{a^M + \sqrt{a'^2 - 4a^M h_1^M}}{2a^M/K} = g(a^M, a'/K, h_1^M) \]
\[ > g(a', \frac{a^M}{K}, h_1^M) = a' + \sqrt{a'^2 - 4a^M h_1^M} \]
\[ = A^+ . \]
From (6), (7), (13) and (14), we obtain
\[ \ln A^+ < u_1(\eta_1) < u_1(\xi_1) < \ln l_1^+ \text{ or } \ln l_1^- < u_1(\eta_1) < u_1(\xi_1) < \ln A^- . \]
By (12) and (15), we have for all \( t \in R \),
\[ \ln A^+ < u_1(t) < \ln l_1^+ \text{ or } \ln l_1^- < u_1(t) < \ln A^- , \]
and
\[ \ln H_2 < u_2(t) < \ln l_1^- \text{ or } \ln l_1^+ < u_2(t) < \ln H_1. \]
Clearly, \( \ln l_1^+ \), \( \ln A^\pm \), \( \ln H_1 \) and \( \ln H_2 \) are independent of \( \lambda \). Now, let
\[ \Omega_1 = \{ u = (u_1, u_2)^T \in X : \ln l_1^- < u_1(t) < \ln A^- , \ \ln H_2 < u_2(t) < \ln l_1^- \}, \]
\[ \Omega_2 = \{ u = (u_1, u_2)^T \in X : \ln A^+ < u_1(t) < \ln l_1^+ , \ \ln H_2 < u_2(t) < \ln l_1^- \}, \]
\[ \Omega_3 = \{ u = (u_1, u_2)^T \in X : \ln l_1^- < u_1(t) < \ln A^- , \ \ln l_1^+ < u_2(t) < \ln H_1 \}, \]
\[ \Omega_4 = \{ u = (u_1, u_2)^T \in X : \ln A^+ < u_1(t) < \ln l_1^+ , \ \ln l_1^+ < u_2(t) < \ln H_1 \}. \]
Then, $\Omega_i (i = 1, 2, 3, 4)$ are bounded open subsets of $X$, $\Omega_i \cap \Omega_j = \phi$ for $i \neq j$. Thus, $\Omega_i (i = 1, 2, 3, 4)$ satisfy the requirement (a) in Lemma 2.1.

Now, we show that (b) of Lemma 2.1 holds, i.e., we prove that when $u \in \partial \Omega_i \cap \ker L = \partial \Omega_i \cap R^2$, we have $QN(u, 0) \neq (0, 0)^T$, $i = 1, 2, 3, 4$. If it is not true, then when $u \in \partial \Omega_i \cap \ker L = \partial \Omega_i \cap R^2$, constant vector $u = (u_1, u_2)^T$ with $u \in \partial \Omega_i$, $i = 1, 2, 3, 4$, satisfies

$$\int_0^\infty a(t)(1 - \frac{e^{u_1}}{K})dt - \int_0^\infty h_1(t)e^{-u_1}dt = 0,$$

$$\int_0^\infty c(t)dt - \int_0^\infty d(t)e^{u_2}dt - \int_0^\infty h_2(t)e^{-u_2}dt = 0.$$  

In terms of differential mean value theorem, there exist two points $t_i$, $i = 1, 2$ such that

$$a(t_i)(1 - \frac{e^{u_1}}{K}) - h_1(t_i)e^{-u_1} = 0, \quad (18)$$

$$c(t_2) - d(t_2)e^{u_2} - h_2(t_2)e^{-u_2} = 0. \quad (19)$$

Following the arguments of (6)-(17), we have

$$\ln A^+ < u_1 < \ln l_1^+ \quad \text{or} \quad \ln l_1^- < u_1 < \ln A^-, \quad (20)$$

$$\ln H_2 < u_2 < \ln l_2^- \quad \text{or} \quad \ln l_2^+ < u_2 < \ln H_1. \quad (21)$$

Moreover, by (18), we have

$$u_1^\pm = \ln a(t_1) = \pm \sqrt{\frac{(a(t_1))^2 - 4h_1(t_1)a(t_1)K}{2a(t_1)K}}. \quad (22)$$

In the light of Lemma 2.2, we obtain

$$\ln l_1^- < u_1^- < \ln A^- < \ln A^+ < u_1^+ < \ln l_1^+. \quad (23)$$

Then, $u \in \Omega_1 \cap R^2$ or $u \in \Omega_2 \cap R^2$ or $u \in \Omega_3 \cap R^2$ or $u \in \Omega_4 \cap R^2$. This contradicts the fact that $u \in \partial \Omega_i \cap R^2$, $i = 1, 2, 3, 4$. This proves (b) in Lemma 2.1 holds.

Finally, we show that (c) in Lemma 2.1 holds. Note that the system of algebraic equations

$$a(t_1) - \frac{a(t_1)}{K}e^x - h_1(t_1)e^{-x} = 0,$$

$$c(t_2) - d(t_2)e^y - h_2(t_2)e^{-y} = 0$$

has four distinct solutions since (A1) and (A2) hold:

$$(x_1^*, y_1^*) = (\ln x_-, \ln y_-), \quad (x_2^*, y_2^*) = (\ln x_-, \ln y_+),$$

$$(x_3^*, y_3^*) = (\ln x_+, \ln y_-), \quad (x_4^*, y_4^*) = (\ln x_+, \ln y_+),$$

where

$$x_\pm = \frac{a(t_1) \pm \sqrt{(a(t_1))^2 - 4h_1(t_1)a(t_1)K}}{2a(t_1)K}, \quad y_\pm = \frac{c(t_2) \pm \sqrt{(c(t_2))^2 - 4d(t_2)h_2(t_2)}}{2d(t_2)}.$$  

It is easy to verify that

$$\ln l_1^- < \ln x_- < \ln A^- < \ln A^+ < \ln x_+ < \ln l_1^+. \quad (24)$$

and

$$\ln H_2 < \ln y_- < \ln l_2^- < \ln l_1^+ < \ln y_+ < \ln H_1.$$  

Therefore,

$$(x_1^*, y_1^*) \in \Omega_1, \quad (x_2^*, y_2^*) \in \Omega_2, \quad (x_3^*, y_3^*) \in \Omega_3, \quad (x_4^*, y_4^*) \in \Omega_4.$$
Since \( \text{Ker} L = \text{Im} Q \), we can take \( J = I \). In the light of the definition of the Leray-Schauder degree, a direct computation gives for \( i = 1, 2, 3, 4 \),

\[
\text{deg} \{ JQNu, 0, \Omega_i \cap \text{Ker} L, (0, 0)^T \} = \text{sign} \left[ -\frac{a(t_1)}{K} x^* + \frac{h_1(t_1)}{x^*} - d(t_2)y^* + \frac{h_2(t_2)}{y^*} \right] = \text{sign} \left[ \left( -\frac{a(t_1)}{K} x^* + \frac{h_1(t_1)}{x^*} \right)(-d(t_2)y^* + \frac{h_2(t_2)}{y^*}) \right].
\]

Since

\[
a(t_1) - \frac{a(t_1)}{K} x^* + \frac{h_1(t_1)}{x^*} = 0, \quad \text{and} \quad c(t_2) - d(t_2)y^* + \frac{h_2(t_2)}{y^*} = 0,
\]

then

\[
\text{deg} \{ JQNu, 0, \Omega_i \cap \text{Ker} L, (0, 0)^T \} = \text{sign} \left[ \left( a(t_1) - \frac{2a(t_1)}{K} x^* \right)(c(t_2) - d(t_2)y^*) \right],
\]

\( i = 1, 2, 3, 4 \). Thus

\[
\text{deg} \{ JQNu, 0, \Omega_1 \cap \text{Ker} L, (0, 0)^T \} = \text{sgn}[(a(t_1) - \frac{2a(t_1)}{K} x_-)(c(t_2) - d(t_2)y_-)] = 1,
\]

\[
\text{deg} \{ JQNu, 0, \Omega_2 \cap \text{Ker} L, (0, 0)^T \} = \text{sgn}[(a(t_1) - \frac{2a(t_1)}{K} x_-)(c(t_2) - d(t_2)y_+)] = -1,
\]

\[
\text{deg} \{ JQNu, 0, \Omega_3 \cap \text{Ker} L, (0, 0)^T \} = \text{sgn}[(a(t_1) - \frac{2a(t_1)}{K} x_+)(c(t_2) - d(t_2)y_-)] = -1,
\]

\[
\text{deg} \{ JQNu, 0, \Omega_4 \cap \text{Ker} L, (0, 0)^T \} = \text{sgn}[(a(t_1) - \frac{2a(t_1)}{K} x_+)(c(t_2) - d(t_2)y_+)] = 1.
\]

So far, we have proved that \( \Omega_i (i = 1, 2, 3, 4) \) satisfies all the assumptions in Lemma 2.1. Hence, system (2) has at least four different \( \omega \)-periodic solutions. Thus, system (1) has at least four different positive \( \omega \)-periodic solutions. This completes the proof of Theorem 2.3. \( \square \)

3. An example

Consider the following two species prey-predator system with harvesting terms and delays:

\[
\begin{align*}
\dot{x}(t) &= (4 + \sin t)x(t)(1 - \frac{9}{25}x(t)) - \frac{2 + \cos t}{200} x(t)y(t) - \frac{7 + 2 \cos t}{200}, \\
\dot{y}(t) &= y(t)(3 + \cos t - \frac{7 + \cos t}{30} y(t)) - \frac{2 + \cos t}{200} x(t)(t - \tau(t))y(y(t)) + \frac{7 + \cos t}{5} \tag{23}
\end{align*}
\]

In this case, \( a(t) = 4 + \sin t, \ K = \frac{5}{3}, \ b(t) = \frac{2 + \cos t}{200}, \ h_1(t) = \frac{7 + 3 \cos t}{200}, \ c(t) = 3 + \cos t, \ d(t) = \frac{7 + \cos t}{30}, \ f(t) = 4 + \cos t, \ r(t) = 2 + \sin t, \ m(t) = 5 + 2 \cos t, \ n(t) = 3 + 2 \sin t, \ h_2(t) = \frac{2 + \cos t}{5} \).

Since

\[
\begin{align*}
a^M = n^M = 5, \quad d^l = 3, \quad m^M = c^M = 4, \quad m^l = c^l = r^M = 2, \\
b^M = \frac{3}{200}, \quad b^l = \frac{1}{200}, \quad h_1^M = \frac{1}{20}, \quad h_1^l = \frac{1}{25}, \\
a^M = \frac{4}{15}, \quad d^l = h_2^l = \frac{1}{5}, \quad r^l = n^l = 1, \quad h_2^M = \frac{3}{5}, \quad H_1 = \frac{c^M}{d^l} = \frac{4}{1/5} = 20, \\
\text{I}^+_1 = \frac{a^M + \sqrt{a^M - \frac{4a^M h_1^l}{K}}}{2a^M/K} = \frac{25 + 5\sqrt{25 - \frac{108}{125}}}{54} < 25 \frac{27}{125}, \\
\frac{r^M I^+_1}{f^M + m^l I^+_1} = \frac{3 I^+_1}{3 + 3 I^+_1} = \frac{I^+_1}{1 + I^+_1} < 1,
\end{align*}
\]

MULTIPLE PERIODIC SOLUTIONS FOR A DELAYED PREDATOR-PREY SYSTEM 337
then
\[ m^M l_1^+ \frac{1}{1 + b l_1^+} + 2 \sqrt{d^M h_2^M} < 1 + \frac{4}{5} < 2 = c^l, \]
\[ 3 = a^l > 2 \sqrt{\frac{a^M}{K} (h_1^M + \frac{a^M H_1}{b'})} = 2 \times \frac{3}{\sqrt{5}}, \quad a^M > 2 \sqrt{\frac{a^H}{K}}, \]
which imply that (A1), (A2) hold. Hence, all conditions of Theorem 2.3 are satisfied. By Theorem 2.3, system (23) has at least four positive 2π-periodic solutions. □

References


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