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Multiple positive periodic solutions for a delayed predator-prey system with Beddington-DeAngelis functional response and harvesting terms

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Abstract. In this paper, a delayed predator-prey system with Beddington-DeAngelis functional response and harvesting terms is studied. By using Mawhin's continuation theorem, the sufficient conditions are established for the existence of at least four positive periodic solutions. Finally, an example is presented to illustrate the effectiveness of the results.

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1. Introduction

In the past decades, various mathematical models with delays have been proposed in the study of population dynamics([1-20]). Among these models, predator-prey systems play an important role in population theory. One of the most popular predator-prey models is the one with a Beddington-DeAngelis functional response which was originally proposed by Beddington [1] and DeAngelis et al. [2], independently. The dynamics of this model is described by differential equations in the form

$$\begin{aligned} \dot{x} &= rx(t)\left(1-\frac{x(t)}{K}\right)-\frac{bx(t)y(t)}{1+nx(t)+my(t)}, \\ \dot{y} &= y(t)\left(-d+\frac{fbx(t)}{1+nx(t)+my(t)}\right). \end{aligned}$$

Recently, predator-prey systems with a Beddington-DeAngelis functional response were widely investigated ([3,4,10,12,14,15,17]). For example, in [3], the authors studied the following nonautonomous delayed predator-prey model with the Beddington-DeAngelis functional response

$$\begin{aligned} \dot{x} &= x(t)[a(t)-b(t)x(t-\tau(t,x(t),y(t))]-\frac{c(t)x(t)y(t)}{1+nx(t)+my(t)}, \\ \dot{y} &= y(t)\left[\frac{f(t)x(t-\sigma(t,x(t),y(t)))}{1+nx(t-\sigma(t,x(t),y(t)))+my(t-\sigma(t,x(t),y(t)))}-d(t)\right]. \end{aligned}$$

In addition, since the exploitation of biological resources and the harvest of population species are commonly practiced in fishery, forestry, and wildlife management, the study of population dynamics with harvesting is an important subject in mathematical bioeconomics (see [5-7,11,13,16,18-20], for example). This motivates us to consider the

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following nonautonomous delayed predatory-prey system with Beddington-DeAngelis functional response and harvesting terms:

$$\begin{aligned}\dot{x}(t) &= a(t)x(t)\left(1-\frac{x(t)}{K}\right)-\frac{b(t)x(t)y(t)}{f(t)+m(t)x(t)+n(t)y(t)}-h_1(t),\\ \dot{y}(t) &= y(t)(c(t)-d(t)y(t))-\frac{r(t)x(t-\tau(t))y(t)}{f(t)+m(t)x(t-\tau(t))+n(t)y(t-\tau(t))}-h_2(t),\end{aligned} \tag{1}$$

where $x(t)$ and $y(t)$ denote prey and predator population, respectively, $a(t)$, $b(t)$, $c(t)$, $d(t)$, $f(t)$, $m(t)$, $n(t)$, $h_i(t)$ $(i=1,2)$, $\tau(t)$ are all positive continuous ω-periodic functions, K is a positive constant. Here $a(t)$, $c(t)$ represent the intrinsic growth rate and K the carrying capacity of the prey, $d(t)$ is the death rate of the predator, $m(t)$, $n(t)$ is the conversion factor denoting the number of newly born predators for each captured prey. $h_i(t)$, $i=1,2$ is the i-th species harvesting terms standing for the harvests, $\tau(t)$ is the state dependent delay.

The rest of the paper is arranged as follows. In Section 2, we present Mawhin's continuation theorem and establish existence of four positive periodic solutions. In section 3, we give an example to illustrate the effectiveness of the results.

2. Existence of four positive periodic solutions

In this section, by using Mawhin's continuation theorem, we shall show the existence of positive periodic solutions of (1). To do so, we need to make some preparations.

Let X and Z be real normed vector spaces. Let $L : DomL \subset X \to Z$ be a linear mapping and $N : X \times [0,1] \to Z$ be a continuous mapping.

The mapping L will be called a Fredholm mapping of index zero if $dim\, Ker\, L = codim\, Im\, L < \infty$ and $Im\, L$ is closed in Z.

If L is a Fredholm mapping of index zero, then there exist two continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that $Im\, P = Ker\, L$ and $Ker\, Q = Im\, L = Im\,(I-Q)$, and $X = Ker\, L \oplus Ker\, P$ and $Z = Im\, L \oplus Im\, Q$. It follows that $L|_{DomL\cap KerP} : (I-P)X \to Im\, L$ is invertible and its inverse is denoted by K_P.

Set Ω is a bounded open subset of X, if $QN\bar{\Omega}\times[0,1]$ is bounded and $K_P(I-Q)N : \bar{\Omega}\times[0,1] \to X$ is compact, then the mapping N is called L-compact on $\bar{\Omega}\times[0,1]$.

Because $Im\, Q$ is isomorphic to $Ker\, L$, there exists an isomorphism $J : Im\, Q \to Ker\, L$.

The Mawhin's continuous theorem [8, p. 40] is given as follows.

Lemma 2.1. ([8]) *Let L be a Fredholm mapping of index zero and let N be L-compact on $\bar{\Omega}\times[0,1]$. Assume*
(a) for each $\lambda \in (0,1)$, every solution x of $Lx = \lambda N(x,\lambda)$ is such that $x \notin \partial\Omega \cap DomL$;
(b) $QN(x,0) \neq 0$ for each $x \in \partial\Omega \cap \ker L$;
(c) $deg(JQN(x,0), \Omega\cap\ker L, 0) \neq 0$.
Then, $Lx = N(x,1)$ has at least one solution in $\overline{\Omega}\cap DomL$.

Lemma 2.2. *Let $x>0, y>0, z>0$ and $x > 2\sqrt{yz}$. For the functions $f(x,y,z) = \frac{x-\sqrt{x^2-4yz}}{2y}$ and $g(x,y,z) = \frac{x+\sqrt{x^2-4yz}}{2y}$, the following assertions hold.*
(i) $f(x,y,z)$ is monotonically decreasing on the variable $x \in (0,\infty)$, monotonically increasing on the variable $y \in (0,\infty)$, monotonically increasing on the variable $z \in (0,\infty)$, respectively.
(ii) $g(x,y,z)$ is monotonically increasing on the variable $x \in (0,\infty)$, monotonically

decreasing on the variable $y \in (0,\infty)$, monotonically decreasing on the variable $z \in (0,\infty)$, respectively.

Proof. By the relationship of the derivative and the monotonicity, the above assertions are easily proved, and we omit them. $\square$

For the sake of convenience, we denote

$$f^l = \min_{t\in[0,\omega]} f(t), \quad f^M = \max_{t\in[0,\omega]} f(t), \quad \bar{f} = \frac{1}{\omega}\int_0^{\omega} f(t)\mathrm{d}t,$$

here $f(t)$ is a continuous ω-periodic function.

Throughout this paper, we need the following assumptions.

(A1) $\quad a^l > 2\sqrt{\frac{a^M}{K}\left(h_1^M + \frac{b^M H_1}{m^l}\right)}, \quad$ and $\quad a^M > 2\sqrt{\frac{a^l h_1^l}{K}},$

(A2) $\quad c^l > \frac{r^M l_1^+}{f_1^l + m^l l_1^-} + 2\sqrt{d^M h_2^M}.$

Our main result is stated as follows.

Theorem 2.3. *Assume that (A1) and (A2) hold. Then, system (1) has at least four positive ω-periodic solutions.*

Proof. By making the substitution $x(t) = \exp\{u_1(t)\}, \quad y(t) = \exp\{u_2(t)\}$, system (1) can be reformulated as

$$\begin{aligned}
\dot{u}_1(t) &= a(t)\left(1 - \tfrac{e^{u_1(t)}}{K}\right) - \tfrac{b(t)e^{u_2(t)}}{f(t)+m(t)e^{u_1(t)}+n(t)e^{u_2(t)}} - h_1(t)e^{-u_1(t)},\\
\dot{u}_2(t) &= c(t) - d(t)e^{u_2(t)} - \tfrac{r(t)e^{u_1(t-\tau(t))}}{f(t)+m(t)e^{u_1(t-\tau(t))}+n(t)e^{u_2(t-\tau(t))}} - h_2(t)e^{-u_2(t)}.
\end{aligned} \tag{2}$$

Let

$$X = Z = \{u = (u_1,u_2)^T \in C(\mathbb{R},\mathbb{R}^2) : u(t+\omega) = u(t)\},$$

and define

$$\| u \| = \sum_{i=1}^{2} \max_{t\in[0,\omega]} |u_i(t)|, \quad u \in X \text{ or } Z.$$

Equipped with the above norm $\|\cdot\|$, X and Z are Banach spaces. For $u \in X$, let

$$N(u,\lambda) = \begin{pmatrix} a(t)(1 - \frac{e^{u_1(t)}}{K}) - \lambda\frac{b(t)e^{u_2(t)}}{f(t)+m(t)e^{u_1(t)}+n(t)e^{u_2(t)}} - h_1(t)e^{-u_1(t)} \\ \\ c(t) - d(t)e^{u_2(t)} - \lambda\frac{r(t)e^{u_1(t-\tau(t))}}{f(t)+m(t)e^{u_1(t-\tau(t))}+n(t)e^{u_2(t-\tau(t))}} - h_2(t)e^{-u_2(t)} \end{pmatrix},$$

$$Lu = \dot{u} = \frac{\mathrm{d}u(t)}{\mathrm{d}t}.$$

We put

$$Pu = \frac{1}{\omega}\int_0^{\omega} u(t)\mathrm{d}t,\ u \in X; \quad Qz = \frac{1}{\omega}\int_0^{\omega} z(t)\mathrm{d}t,\ z \in Z.$$

Thus, it follows that $Ker\,L = \mathbb{R}^2$, $Im\,L = \{z \in Z : \int_0^{\omega} z(t)\mathrm{d}t = 0\}$ is closed in Z, $dim\,Ker\,L = 2 = codim\,Im\,L$, and P, Q are continuous projectors such that

$$Im\,P = Ker\,L, \quad KerQ = Im\,L = Im\,(I-Q).$$

Hence, L is a Fredholm mapping of index zero. Furthermore, the generalized inverse (to L) $K_P : Im\,L \to Ker\,P \cap Dom\,L$ is given by

$$K_P(z) = \int_0^t z(s)\mathrm{d}s - \frac{1}{\omega}\int_0^{\omega}\int_0^t z(s)\mathrm{d}s dt.$$

Then

$$QN(u,\lambda) = \begin{pmatrix} \frac{1}{\omega}\int_0^{\omega} F_1(s,\lambda)\mathrm{d}s \\ \\ \frac{1}{\omega}\int_0^{\omega} F_2(s,\lambda)\mathrm{d}s \end{pmatrix}$$

and

$$K_P(I-Q)N(u,\lambda) = \begin{pmatrix} \int_0^t F_1(s,\lambda)\mathrm{d}s - \frac{1}{\omega}\int_0^{\omega}\int_0^t F_1(s,\lambda)\mathrm{d}s\mathrm{d}t + (\frac{1}{2}-\frac{t}{\omega})\int_0^{\omega} F_1(s,\lambda)\mathrm{d}s \\ \\ \int_0^t F_2(s,\lambda)\mathrm{d}s - \frac{1}{\omega}\int_0^{\omega}\int_0^t F_2(s,\lambda)\mathrm{d}s\mathrm{d}t + (\frac{1}{2}-\frac{t}{\omega})\int_0^{\omega} F_2(s,\lambda)\mathrm{d}s \end{pmatrix},$$

where

$$\begin{aligned} F_1(s,\lambda) &= a(s)(1-\tfrac{e^{u_1(s)}}{K}) - \lambda\tfrac{b(s)e^{u_2(s)}}{f(s)+m(s)e^{u_1(s)}+n(s)e^{u_2(s)}} - h_1(s)e^{-u_1(s)}, \\ F_2(s,\lambda) &= c(s) - d(s)e^{u_2(s)} - \lambda\tfrac{r(s)e^{u_1(s-\tau(s))}}{f(s)+m(s)e^{u_1(s-\tau(s))}+n(s)e^{u_2(s-\tau(s))}} - h_2(s)e^{-u_2(s)}. \end{aligned}$$

Obviously, QN and $K_P(I-Q)N$ are continuous and, moreover, $QN(\overline{\Omega}\times[0,1])$, $K_P(I-Q)N(\overline{\Omega}\times[0,1])$ are relatively compact for any open bounded set $\Omega\subset X$ by using the Arzela-Ascoli theorem. Hence, N is L-compact on $\overline{\Omega}\times[0,1]$, with any open bounded set $\Omega\subset X$.

In order to use Lemma 2.1, we have to find at least four appropriate open bounded subsets in X. Corresponding to the operator equation $Lu=\lambda N(u,\lambda)$, $\lambda\in(0,1)$, we have

$$\begin{aligned} \dot{u}_1(t) &= \lambda\left(a(t)(1-\tfrac{e^{u_1(t)}}{K}) - \lambda\tfrac{b(t)e^{u_2(t)}}{f(t)+m(t)e^{u_1(t)}+n(t)e^{u_2(t)}} - h_1(t)e^{-u_1(t)}\right), \\ \dot{u}_2(t) &= \lambda\left(c(t) - d(t)e^{u_2(t)} - \lambda\tfrac{r(t)e^{u_1(t-\tau(t))}}{f(t)+m(t)e^{u_1(t-\tau(t))}+n(t)e^{u_2(t-\tau(t))}} - h_2(t)e^{-u_2(t)}\right). \end{aligned} \tag{3}$$

Assume that $u\in X$ is an ω-periodic solution of system (3) for some $\lambda\in(0,1)$. Then, there exist $\xi_i,\ \eta_i\in[0,\omega]$ such that

$$u_i(\xi_i) = \max_{t\in[0,\omega]} u_i(t), \quad u_i(\eta_i) = \min_{t\in[0,\omega]} u_i(t), \quad i=1,2.$$

It is clear that $\dot{u}_i(\xi_i)=0$, $\dot{u}_i(\eta_i)=0$, $i=1,2$. From this and (3), we have

$$\begin{aligned} &a(\xi_1)\left(1-\tfrac{e^{u_1(\xi_1)}}{K}\right) - \lambda\tfrac{b(\xi_1)e^{u_1(\xi_1)}}{f(\xi_1)+m(\xi_1)e^{u_1(\xi_1)}+n(\xi_1)e^{u_2(\xi_1)}} - h_1(\xi_1)e^{-u_1(\xi_1)} = 0, \\ &c(\xi_2) - d(\xi_2)e^{u_2(\xi_2)} - \lambda\tfrac{r(\xi_2)e^{u_1(\xi_2-\tau(\xi_2))}}{f(\xi_2)+m(\xi_2)e^{u_1(\xi_2-\tau(\xi_2))}+n(\xi_2)e^{u_2(\xi_2-\tau(\xi_2))}} - h_2(\xi_2)e^{-u_2(\xi_2)} = 0, \end{aligned} \tag{4}$$

and

$$\begin{aligned} &a(\eta_1)\left(1-\tfrac{e^{u_1(\eta_1)}}{K}\right) - \lambda\tfrac{b(\eta_1)e^{u_2(\eta_1)}}{f(\eta_1)+m(\eta_1)e^{u_1(\eta_1)}+n(\eta_1)e^{u_2(\eta_1)}} - h_1(\eta_1)e^{-u_1(\eta_1)} = 0, \\ &c(\eta_2) - d(\eta_2)e^{u_2(\eta_2)} - \lambda\tfrac{r(\eta_2)e^{u_1(\eta_2-\tau(\eta_2))}}{f(\eta_2)+m(\eta_2)e^{u_1(\eta_2-\tau(\eta_2))}+n(\eta_2)e^{u_2(\eta_2-\tau(\eta_2))}} - h_2(\eta_2)e^{-u_2(\eta_2)} = 0. \end{aligned} \tag{5}$$

According to the first equation of (4), we have

$$\begin{aligned}\tfrac{a^l}{K}e^{u_1(\xi_1)}+h_1^l e^{-u_1(\xi_1)} &\le \tfrac{a(\xi_1)e^{u_1(\xi_1)}}{K}+\lambda\tfrac{b(\xi_1)e^{u_1(\xi_1)}}{f(\xi_1)+m(\xi_1)e^{u_1(\xi_1)}+n(\xi_1)e^{u_2(\xi_1)}}+h_1(\xi_1)e^{-u_1(\xi_1)}\\ &= a(\xi_1)\le a^M,\end{aligned}$$

namely,

$$\frac{a^l}{K}e^{2u_1(\xi_1)}-a^M e^{u_1(\xi_1)}+h_1^l<0,$$

which implies that

$$\ln l_1^- < u_1(\xi_1) < \ln l_1^+, \tag{6}$$

where

$$l_1^\pm=\frac{a^M\pm\sqrt{a^{2M}-\frac{4a^l h_1^l}{K}}}{2a^l/K}.$$

Similarly, by the first equation of (5), we obtain

$$\ln l_1^- < u_1(\eta_1) < \ln l_1^+. \tag{7}$$

Denote $c_1^+=c^l-\frac{r^M l_1^+}{f^l+m^l l_1^-}$. The second equation of (4) gives

$$d^M e^{2u_2(\xi_2)}+h_2^M > d(\xi_2)e^{2u_2(\xi_2)}+h_2(\xi_2) > c_1^+ e^{u_2(\xi_2)},$$

that is,

$$d^M e^{2u_2(\xi_2)}-c_1^+e^{u_2(\xi_2)}+h_2^M>0,$$

which implies that

$$u_2(\xi_2)>\ln l_2^+, \quad \text{or} \quad u_2(\xi_2)<\ln l_2^-, \tag{8}$$

where

$$l_2^\pm=\frac{c_1^+\pm\sqrt{(c_1^+)^2-4d^M h_2^M}}{2d^M}.$$

Similarly, by the second equation of (5), we get

$$u_2(\eta_2)>\ln l_2^+ \quad \text{or} \quad u_2(\eta_2)<\ln l_2^-. \tag{9}$$

Denote $H_1=\frac{c^M}{d^l}$. Then from the second equation of (4), we have

$$d^l e^{u_2(\xi_2)} \le d(\xi_2)e^{u_2(\xi_2)} < d(\xi_2)e^{u_2(\xi_2)}+h_2(\xi_2)e^{-u_2(\xi_2)} < c^M,$$

that is,

$$u_2(\xi_2)<\ln\frac{c^M}{d^l}=\ln H_1. \tag{10}$$

Similarly, denote $H_2=\frac{h_2^l}{c^M}$, then from the second equation of (5), we obtain

$$h_2^l e^{-u_2(\eta_2)} \le h_2(\eta_2)e^{-u_2(\eta_2)} < h_2(\eta_2)e^{-u_2(\eta_2)}+d(\eta_2)e^{u_2(\eta_2)} < c^M,$$

which implies that

$$u_2(\eta_2)>\ln H_2. \tag{11}$$

We claim that $\ln l_2^+<\ln H_1,\ \ln H_2<\ln l_2^-$. In fact,

$$l_2^+=\tfrac{c_1^+ +\sqrt{(c_1^+)^2-4d^M h_2^M}}{2d^M}<\tfrac{c_1^+}{d^M}<\tfrac{c^l}{d^M}<\tfrac{c^M}{d^l}=H_1,$$

$$l_2^-=\tfrac{c_1^+ -\sqrt{(c_1^+)^2-4d^M h_2^M}}{2d^M}=\tfrac{2h_2^M}{c_1^+ +\sqrt{(c_1^+)^2-4d^M h_2^M}}>\tfrac{h_2^M}{c^+}>\tfrac{h_2^l}{c^l}>\tfrac{h_2^l}{c^M}=H_2.$$

From (8)-(11), we have

$$\ln H_2<u_2(\eta_2)<u_2(\xi_2)<\ln l_2^- \quad \text{or} \quad \ln l_2^+<u_2(\eta_2)<u_2(\xi_2)<\ln H_1. \tag{12}$$

According to the first equation of (4), we have

$$\begin{aligned}
&\tfrac{a^M}{K}e^{u_1(\xi_1)} + h_1^M e^{-u_1(\xi_1)} + \tfrac{b^M H_1}{m^l}e^{-u_1(\xi_1)}\\
&\quad > \tfrac{a(\xi_1)e^{u_1(\xi_1)}}{K} + h_1(\xi_1)e^{-u_1(\xi_1)} + \lambda\tfrac{b(\xi_1)e^{u(\xi_1)}}{f(\xi_1)+m(\xi_1)e^{u_1(\xi_1)}+n(\xi_1)e^{u_2(\xi_1)}} - h_1(\xi_1)e^{-u_1(\xi_1)}\\
&\quad = a(\xi_1) \ge a^l,
\end{aligned}$$

namely,

$$\frac{a^M}{K}e^{2u_1(\xi_1)} - a^l e^{u_1(\xi_1)} + \left(h_1^M + \frac{b^M H_1}{m^l}\right) > 0,$$

which implies that

$$u_1(\xi_1) > \ln A^+, \quad \mathrm{u}_1(\xi_1) < \ln A^-, \tag{13}$$

where

$$A^{\pm} = \frac{a^l \pm \sqrt{a^{2l} - 4\frac{a^M}{K}(h_1^M + \frac{b^M H_1}{m^l})}}{2a^M/K}.$$

Similarly, by the first equation of (5), we obtain

$$u_1(\eta_1) > \ln A^+, \quad \mathrm{u}_1(\eta_1) < \ln A^-. \tag{14}$$

We claim that $\ln l_1^- < \ln A^-,\ \ln A^+ < \ln l_1^+$. In fact, employing Lemma 2.2, we have

$$\begin{aligned}
l_1^- &= \frac{a^M - \sqrt{a^{2M} - \frac{4a^l h_1^l}{K}}}{2a^l/K} = f(a^M, \frac{a^l}{K}, h_1^l)\\
&< f(a^l, \frac{a^M}{K}, h_1^M) = \frac{a^l - \sqrt{a^{2l} - \frac{4a^M h_1^M}{K}}}{2a^M/K} < \frac{a^l - \sqrt{a^{2l} - \frac{4a^M}{K}(h_1^M + \frac{b^M H_1}{m^l})}}{2a^M/K} = A^-,\\
l_1^+ &= \frac{a^M + \sqrt{a^{2M} - \frac{4a^l h_1^l}{K}}}{2a^l/K} = g(a^M, \frac{a^l}{K}, h_1^l)\\
&> g(a^l, \frac{a^M}{K}, h_1^M) = \frac{a^l + \sqrt{a^{2l} - \frac{4a^M h_1^M}{K}}}{2a^M/K} > \frac{(a^l + \sqrt{a^{2l} - 4\frac{a^M}{K}(h_1^M + \frac{b^M H_1}{m^l})}}{2a^M/K} = A^+.
\end{aligned}$$

From (6), (7), (13) and (14), we obtain

$$\ln A^+ < u_1(\eta_1) < u_1(\xi_1) < \ln l_1^+ \text{ or } \ln l_1^- < u_1(\eta_1) < u_1(\xi_1) < \ln A^-. \tag{15}$$

By (12) and (15), we have for all $t \in R$,

$$\ln A^+ < u_1(t) < \ln l_1^+ \quad \text{or} \quad \ln l_1^- < u_1(t) < \ln A^-, \tag{16}$$

and

$$\ln H_2 < u_2(t) < \ln l_2^- \quad \text{or} \quad \ln l_2^+ < u_2(t) < \ln H_1. \tag{17}$$

Clearly, $\ln l_1^{\pm},\ \ln A^{\pm},\ \ln H_1$ and $\ln H_2$ are independent of λ. Now, let

$$\begin{aligned}
\Omega_1 &= \{u = (u_1, u_2)^T \in X : \ln l_1^- < u_1(t) < \ln A^-,\ \ln H_2 < u_2(t) < \ln l_2^-\},\\
\Omega_2 &= \{u = (u_1, u_2)^T \in X : \ln A^+ < u_1(t) < \ln l_1^+,\ \ln H_2 < u_2(t) < \ln l_2^-\},\\
\Omega_3 &= \{u = (u_1, u_2)^T \in X : \ln l_1^- < u_1(t) < \ln A^-,\ \ln l_2^+ < u_2(t) < \ln H_1\},\\
\Omega_4 &= \{u = (u_1, u_2)^T \in X : \ln A^+ < u_1(t) < \ln l_1^+,\ \ln l_2^+ < u_2(t) < \ln H_1\}.
\end{aligned}$$

Then, $\Omega_i (i = 1, 2, 3, 4)$ are bounded open subsets of X, $\Omega_i \cap \Omega_j = \phi$ for $i \neq j$. Thus, $\Omega_i (i = 1, 2, 3, 4)$ satisfy the requirement (a) in Lemma 2.1.

Now, we show that (b) of Lemma 2.1 holds, i.e., we prove that when $u \in \partial\Omega_i \cap \ker L = \partial\Omega_i \cap R^2$, we have $QN(u, 0) \neq (0, 0)^T$, $i = 1, 2, 3, 4$. If it is not true, then when $u \in \partial\Omega_i \cap Ker\, L = \partial\Omega_i \cap R^2$, constant vector $u = (u_1, u_2)^T$ with $u \in \partial\Omega_i, i = 1, 2, 3, 4$, satisfies

$$\begin{aligned} \int_0^\omega a(t)(1 - \tfrac{e^{u_1}}{K})\mathrm{d}t - \int_0^\omega h_1(t)e^{-u_1}\mathrm{d}t &= 0, \\ \int_0^\omega c(t)\mathrm{d}t - \int_0^\omega d(t)e^{u_2}\mathrm{d}t - \int_0^\omega h_2(t)e^{-u_2}\mathrm{d}t &= 0. \end{aligned}$$

In terms of differential mean value theorem, there exist two points t_i, $i = 1, 2$ such that

$$a(t_1)(1 - \frac{e^{u_1}}{K}) - h_1(t_1)e^{-u_1} = 0, \tag{18}$$

$$c(t_2) - d(t_2)e^{u_2} - h_2(t_2)e^{-u_2} = 0. \tag{19}$$

Following the arguments of (6)-(17), we have

$$\ln A^+ < u_1 < \ln l_1^+ \quad \text{or} \quad \ln l_1^- < u_1 < \ln A^-, \tag{20}$$

$$\ln H_2 < u_2 < \ln l_2^- \quad \text{or} \quad \ln l_2^+ < u_2 < \ln H_1. \tag{21}$$

Moreover, by (18), we have

$$u_1^\pm = \ln \frac{a(t_1) \pm \sqrt{(a(t_1))^2 - 4h_1(t_1)a(t_1)/K}}{2a(t_1)/K}.$$

In the light of Lemma 2.2, we obtain

$$\ln l_1^- < u_1^- < \ln A^- < \ln A^+ < u_1^+ < \ln l_1^+. \tag{22}$$

Then, $u \in \Omega_1 \cap R^2$ or $u \in \Omega_2 \cap R^2$ or $u \in \Omega_3 \cap R^2$ or $u \in \Omega_2 \cap R^2$. This contradicts the fact that $u \in \partial\Omega_i \cap R^2$, $i = 1, 2, 3, 4$. This proves (b) in Lemma 2.1 holds.

Finally, we show that (c) in Lemma 2.1 holds. Note that the system of algebraic equations

$$\begin{aligned} a(t_1) - \tfrac{a(t_1)}{K}e^x - h_1(t_1)e^{-x} &= 0, \\ c(t_2) - d(t_2)e^y - h_2(t_2)e^{-y} &= 0 \end{aligned}$$

has four distinct solutions since (A1)and (A2) hold:

$$\begin{aligned} (x_1^*, y_1^*) = (\ln x_-, \ln y_-), \quad (x_2^*, y_2^*) = (\ln x_-, \ln y_+), \\ (x_3^*, y_3^*) = (\ln x_+, \ln y_-), \quad (x_4^*, y_4^*) = (\ln x_+, \ln y_+), \end{aligned}$$

where

$$x_\pm = \frac{a(t_1) \pm \sqrt{(a(t_1))^2 - 4h_1(t_1)a(t_1)/K}}{2a(t_1)/K}, \quad y_\pm = \frac{c(t_2) \pm \sqrt{(c(t_2))^2 - 4d(t_2)h_2(t_2)}}{2d(t_2)}.$$

It is easy to verify that

$$\ln l_1^- < \ln x_- < \ln A^- < \ln A^+ < \ln x_+ < \ln l_1^+.$$

and

$$\ln H_2 < \ln y_- < \ln l_2^- < \ln l_1^+ < \ln y_+ < \ln H_1.$$

Therefore,

$$(x_1^*, y_1^*) \in \Omega_1, \quad (x_2^*, y_2^*) \in \Omega_2, \quad (x_3^*, y_3^*) \in \Omega_3, \quad (x_4^*, y_4^*) \in \Omega_4.$$

Since $Ker\,L = Im\,Q$, we can take $J = I$. In the light of the definition of the Leray-Schauder degree, a direct computation gives for i =1,2,3,4,

$$\deg\{JQN(u,0), \Omega_i \cap KerL, (0,0)^T\} = \text{sign} \begin{vmatrix} -\frac{a(t_1)}{K}x^* + \frac{h_1(t_1)}{x^*} & 0 \\ 0 & -d(t_2)y^* + \frac{h_2(t_2)}{y^*} \end{vmatrix}$$

$$= \text{sign}\left[\left(-\tfrac{a(t_1)}{K}x^* + \tfrac{h_1(t_1)}{x^*})(-d(t_2)y^* + \tfrac{h_2(t_2)}{y^*}\right)\right].$$

Since

$$a(t_1) - \frac{a(t_1)}{K}x^* + \frac{h_1(t_1)}{x^*} = 0, \quad \text{and} \quad c(t_2) - d(t_2)y^* + \frac{h_2(t_2)}{y^*} = 0,$$

then

$$\deg\{JQN(u,0), \Omega_i \cap KerL, (0,0)^T\} = \text{sgn}\left[\left(a(t_1) - \frac{2a(t_1)}{K}x^*\right)(c(t_2) - d(t_2)y^*)\right],$$

$i = 1, 2, 3, 4$. Thus

$$\begin{aligned}
\deg\{JQN(u,0), \Omega_1 \cap KerL, (0,0)^T\} &= \text{sgn}[(a(t_1) - \tfrac{2a(t_1)}{K}x_-)(c(t_2) - d(t_2)y_-)] = 1,\\
\deg\{JQN(u,0), \Omega_2 \cap KerL, (0,0)^T\} &= \text{sgn}[(a(t_1) - \tfrac{2a(t_1)}{K}x_-)(c(t_2) - d(t_2)y_+)] = -1,\\
\deg\{JQN(u,0), \Omega_3 \cap KerL, (0,0)^T\} &= \text{sgn}[(a(t_1) - \tfrac{2a(t_1)}{K}x_+)(c(t_2) - d(t_2)y_-)] = -1,\\
\deg\{JQN(u,0), \Omega_4 \cap KerL, (0,0)^T\} &= \text{sgn}[(a(t_1) - \tfrac{2a(t_1)}{K}x_+)(c(t_2) - d(t_2)y_-)] = 1.
\end{aligned}$$

So far, we have proved that $\Omega_i (i = 1, 2, 3, 4)$ satisfies all the assumptions in Lemma 2.1. Hence, system (2) has at least four different ω-periodic solutions. Thus, system (1) has at least four different positive ω-periodic solutions. This completes the proof of Theorem 2.3. □

3. An example

Consider the following two species prey-predator system with harvesting terms and delays:

$$\begin{aligned}
\dot{x}(t) &= (4 + \sin t)x(t)(1 - \tfrac{9}{5}x(t)) - \frac{\frac{2+\cos t}{200}x(t)y(t)}{1+(5+2\cos t)x(t)+(3+2\sin t)y(t)} - \frac{7+2\cos t}{200},\\
\dot{y}(t) &= y(t)(3 + \cos t - \tfrac{7+\cos t}{30}y(t)) - \frac{(2+\sin t)x(t-\tau(t))y(t)}{1+(5+2\cos t)x(t-\tau(t))+(3+2\sin t)y(t-\tau(t))} - \frac{2+\cos t}{5}.
\end{aligned} \tag{23}$$

In this case, $a(t) = 4 + \sin t$, $K = \frac{5}{9}$, $b(t) = \frac{2+\cos t}{200}$, $h_1(t) = \frac{7+3\cos t}{200}$, $c(t) = 3 + \cos t$, $d(t) = \frac{7+\cos t}{30}$, $f(t) = 4 + \cos t$, $r(t) = 2 + \sin t$, $m(t) = 5 + 2\cos t$, $n(t) = 3 + 2\sin t$, $h_2(t) = \frac{2+\cos t}{5}$. Since

$$a^M = n^M = 5, \quad a^l = 3, \quad m^M = c^M = 4, \quad m^l = c^l = r^M = 2,$$

$$b^M = \frac{3}{200}, \quad b^l = \frac{1}{200}, \quad h_1^M = \frac{1}{20}, \quad h_1^l = \frac{1}{25},$$

$$d^M = \frac{4}{15}, \quad d^l = h_2^l = \frac{1}{5}, \quad r^l = n^l = 1, \quad h_2^M = \frac{3}{5}, \quad H_1 = \frac{c^M}{d^l} = \frac{4}{1/5} = 20,$$

$$l_1^+ = \frac{a^M + \sqrt{a^{2M} - \frac{4a^l h_1^l}{K}}}{2a^l/K} = \frac{25 + 5\sqrt{25 - \frac{108}{125}}}{54} < \frac{25}{27},$$

$$\frac{r^M l_1^+}{f^M + m^l l_1^-} = \frac{3l_1^+}{3 + 3l_1^-} = \frac{l_1^+}{1 + l_1^-} < 1,$$

then

$$\frac{m^M l_1^+}{1+b^l l_1^+} + 2\sqrt{d^M h_2^M} < 1 + \frac{4}{5} < 2 = c^l,$$

$$3 = a^l > 2\sqrt{\frac{a^M}{K}(h_1^M + \frac{n^M H_1}{b^l})} = 2 \times \frac{3}{\sqrt{5}}, \quad a^M > 2\sqrt{\frac{a^l h_1^l}{K}},$$

which imply that (A1), (A2) hold. Hence, all conditions of Theorem 2.3 are satisfied. By Theorem 2.3, system (23) has at least four positive 2π-periodic solutions. □

References

[1] J.R. Beddington, Mutual interference between parasites or predators and its effect on searching efficiency, *J. Animal Ecol.* **44** (1975), 331–340.

[2] D.L. DeAngelis, R.A. Goldstein, and R.V. ONeill, A model for trophic interaction, *Ecology* **56** (1975), 881-892.

[3] H.F. Huo, W.T. Li, and J.J. Nieto, Periodic solutions of delayed predator-prey model with the Beddington-DeAngelis functional response, *Chaos, Solitons and Fractals* **33** (2007), 505–512.

[4] Z.J. Dua and Z.S. Feng, Periodic solutions of a neutral impulsive predator-prey model with Beddington-DeAngelis functional response with delays, *J. Comput. Appl. Math.* **258** (2014), 87–98.

[5] D. Hu and Z. Zhang, Four positive periodic solutions to a Lotka-Volterra cooperative system with harvesting terms, *Nonlinear Analysis R.W.A.* **11** (2010), 1115–1121.

[6] K. Zhao and Y. Ye, Four positive periodic solutions to a periodic Lotka-Volterra predatory-prey system with harvesting terms, *Nonlinear Analysis R.W.A.* **11** (2010), 2448–2455.

[7] K. Zhao and Y. Li, Four positive periodic solutions to two species parasitical system with harvesting terms, *Comput. Math. Appl.* **59** (2010), 2703–2710.

[8] R. Gaines and J. Mawhin, *Coincidence Degree and Nonlinear Differetial Equitions*, Springer Verlag, Berlin, 1977.

[9] W. Wang, Positive periodic solutions of delayed Nicholsons blowflies models with a nonlinear density-dependent mortality term, *Appl. Math. Model.* **36** (2012), 4708–4713.

[10] J.A. Cui and Y. Takeuchi, Permanence, extinction and periodic solution of predator-prey system with Beddington-DeAngelis functional response, *J. Math. Anal. Appl.* **317** (2006) 464–474.

[11] H. Fang, Existence of eight positive periodic solutions for a food-limited two-species cooperative patch system with harvesting terms, *Commun. Nonlinear Sci. Numer. Simulat.* **18** (2013), 1857–1869.

[12] H. Qiu, M. Liu, K. Wang, and Y. Wang, Dynamics of a stochastic predator-prey system with Beddington-DeAngelis functional response, *Appl. Math. Comput.* **219** (2012), 2303–2312.

[13] Y. Li, K. Zhao, and Y. Ye, Multiple positive periodic solutions of n species delay competition systems with harvesting terms, *Nonlinear Analysis R.W.A.* **12** (2011), 1013–1022.

[14] H. Baek, Qualitative analysis of Beddington-DeAngelis type impulsive predator-prey models, *Nonlinear Analysis R.W.A.* **11** (2010), 1312–1322.

[15] M. Fazly and M. Hesaaraki, Periodic solutions for predatorCprey systems with Beddington-DeAngelis functional response on time scales, *Nonlinear Analysis R.W.A.* **9** (2008), 1224–1235.

[16] Y. Li, K. Zhao, and Y. Ye, Multiple positive periodic solutions of n species delay competition systems with harvesting terms, *Nonlinear Analysis R.W.A.* **12** (2011), 1013–1022.

[17] H.Y. Li and Y. Takeuchi, Dynamics of the density dependent predator-prey system with Beddington-DeAngelis functional response, *J. Math. Anal. Appl.* **374** (2011), 644–654.

[18] Y.H. Fan and L.L. Wang, Multiplicity of periodic solutions for a delayed ratio-dependent predator-prey model with monotonic functional response and harvesting terms, *Appl. Math. Compu.* **244** (2014), 878–894.

[19] Y. Li, Periodic solutions of a periodic delay predator-prey system, *Proc. Amer. Math. Soc.* **127** (1999), 1331–1335.

[20] D. Xiao, W. Li, and M. Han, Dynamics in a ratio-dependent predator-prey model with predator harvesting, *J. Math. Anal. Appl.* **324**(1) (2006), 14–29.

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