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Local bifurcations of a ratio-dependent predator-prey system with Holling type III functional response

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Abstract. In this paper a ratio-dependent predator-prey system with Holling type III functional response is discussed. We give parameter conditions for exact number of equilibria. The conditions for pitchfork bifurcation are given by the center manifold theory. Hopf bifurcations are discussed by computing Lyapunov coefficients. It is shown that at most one stable limit cycle occurs.

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1. Introduction

The classical predator-prey system takes the following form [7]

$$\begin{cases} \dot{x} = xg(x, K) - yp(x), \\ \dot{y} = y(-d + cq(x)), \end{cases} \tag{1}$$

where x and y denote the densities of the prey and predators, respectively. Parameters c and d are both positive constants that represent the rate of predators in converting consumed prey into their growth and the death rate of the predator population, respectively. $g(x, K)$ is a continuous and differentiable function describing the specific growth rate of the prey in the absence of predators and satisfying $g(0, K) = r > 0$, $g(K, K) = 0$, $(\partial g)/(\partial x)(K, K) < 0$, $(\partial g)/(\partial x)(x, K) \leq 0$, $(\partial g)/(\partial K)(x, K) > 0$ for any $x > 0$. K, the carrying capacity of the prey, is also a positive constant. Usually, the function $g(x, K) = r(1 - x/K)$ is considered as a prototype and satisfies all assumptions. The functional response $p(x)$ of predators to the prey describes the change in the density of the prey attacked per unit time per predator as the prey density changes. It is continuous and differentiable and satisfies $p(0) = 0$. Many types of response functions have been used, for example, Lotka-Volterra type, Holling type II, Holling type III, Holling type IV, etc. The function $q(x)$ in system (1) describes how predators convert the consumed prey into the growth of predators. In most cases, $q(x) = p(x)$. There have been many works about the case $q(x) = p(x)$, see for example, Seo and DeAngelis [18] for Holling type I, Bazykin [4], Freedman [7], Kuang and Freedman [13], and May [17] for Holling type II, Lamontagne, Coutu and Rousseau [15] for generalized Holling type III, and Huang and Xiao [10] for Holling type IV. However, several biologists (Leslie [14], Leslie and Gower [16], Arditi and Ginzburg [2], Arditi, Ginzburg and Akcakaya [3], Akcakaya [1], Gutierrez [8], etc.) think that functional and numerical responses over typical ecological timescales

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depend on the densities of both prey and predators. Such a functional response is called a ratio-dependent response function. Based on Holling type II function, Arditi and Ginzburg [2] proposed a ratio-dependent function of the form $p(x/y) = e(x/y)/(m + x/y) = ex/(my + x)$ and the following ratio-dependent predator-prey model

$$\begin{cases} \dot{x} = rx(1 - \frac{x}{K}) - \frac{exy}{my+x}, \\ \dot{y} = y(-d + \frac{cx}{my+x}), \end{cases}$$

which has been studied by several researchers recently (see [5], [9], [12] and [19]). Holling type II functional response is usually used to describe the case that the predator is the invertebrate. Holling type III functional response is more suitable for vertebrates. Corresponding to Holling type III functional response,

$$p(\frac{x}{y}) = eq(\frac{x}{y}) = \frac{ex^2}{m^2y^2 + x^2}.$$

system (1) has the following form

$$\begin{cases} \dot{x} = rx(1 - \frac{x}{K}) - \frac{ex^2y}{m^2y^2+x^2}, \\ \dot{y} = y(-d + \frac{cx^2}{m^2y^2+x^2}). \end{cases} \tag{2}$$

By rescaling $x = K\tilde{x}, y = mK^{-1}\tilde{y}$ and $t \to r^{-1}t$, system (1.2) can be written as

$$\begin{cases} \dot{x} = x(1 - x) - \frac{\alpha x^2 y}{x^2+y^2}, \\ \dot{y} = y(-\beta + \frac{\gamma x^2}{x^2+y^2}), \end{cases} \tag{3}$$

where we still use x, y to present $\tilde{x}, \tilde{y}$ respectively and $\alpha := er^{-1}m^{-1}$, $\beta := dr^{-1}$, $\gamma := cr^{-1}$ are all positive constants.

In this paper, we qualitatively investigate system (3) for general $\alpha, \beta, \gamma > 0$. First, we give parameter conditions for exact number of equilibria. Further, we analytically give parameter conditions for pitchfork bifurcation. Finally, Hopf bifurcations are discussed by computing Lyapunov coefficients. It is shown that at most one limit cycle occurs.

2. Equilibria and their properties

In view of the biological sense, we only consider the equilibria of system (3) in the first quadrant $\mathcal{Q}_+ := \{(x, y) \neq (0, 0) | x \geq 0, y \geq 0\}$. Actually, system (3) has the same phase as the quartic differential system

$$\begin{cases} \dot{x} = x(1 - x)(x^2 + y^2) - \alpha x^2 y, \\ \dot{y} = y\big[- \beta(x^2 + y^2) + \gamma x^2\big], \end{cases} \tag{4}$$

in the first quadrant. In fact, by rescaling the time $t \to t/(x^2 + y^2)$, system (3) can be transformed into system (4). Therefore, system (3) is orbitally equivalent to system (4) in the first quadrant. Then, for the equilibria of system (3), from the orbitally equivalent system (4), we consider the algebraic equations

$$\begin{cases} P(x, y) := x[(1 - x)(x^2 + y^2) - \alpha xy] = 0, \\ Q(x, y) := y[-\beta(x^2 + y^2) + \gamma x^2] = 0. \end{cases} \tag{5}$$

First, the origin $O : (0, 0)$ is excluded because it is meaningless in system (3). For $y = 0$, we can find the equilibrium $E_0 : (1, 0)$. For $y \neq 0$, from the second equation of (5), we have that $y = \sqrt{(\gamma - \beta)/\beta}x$. Combining with the first equation, we obtain $x = 1/\gamma(\gamma - \alpha\sqrt{(\gamma - \beta)\beta}) := x_*$. In fact, equations (5) have another solution (x_{**}, y_{**}), where $x_{**} := 1/\gamma(\gamma - \alpha\sqrt{(\gamma+\beta)\beta})$ and $y_{**} := -\sqrt{(\gamma-\beta)/\beta}x_{**}$. However, E_{**} : (x_{**}, y_{**}) is outside $\mathcal{Q}_+$. Therefore, when $\gamma > \beta$ and $\alpha < \gamma/\sqrt{(\gamma-\beta)\beta}$, there is a equilibrium $E_* : (x_*, \sqrt{(\gamma-\beta)/\beta}x_*)$. Further, we obtain the following lemma.

Lemma 2.1. *System (3) has at most two equilibria in the first quadrant $\mathcal{Q}_+$. The number of equilibria and their qualitative properties are described in Table 1, where*

$$\alpha_0 := \frac{\gamma(-2\beta^2 + 2\beta\gamma + \gamma)}{2\beta\sqrt{(\gamma-\beta)\beta}}. \tag{6}$$

TABLE 1. Qualitative properties for various parameters

Parameters	*Number*	*Equilibria*
1. $0<\gamma<\beta$	*1*	E_0 *(stable node or focus)*
2. $\gamma=\beta$	*1*	E_0 *(nonhyperbolic)*
3. $0<\beta<\gamma<\frac{2\beta(\beta+1)}{2\beta+1},\ 0<\alpha<\alpha_0$	*2*	E_0*(saddle),* E_* *(stable node or focus)*
4. $0<\beta<\gamma<\frac{2\beta(\beta+1)}{2\beta+1},\ \alpha=\alpha_0$	*2*	E_0*(saddle),* E_* *(nonhyperbolic)*
5. $0<\beta<\gamma<\frac{2\beta(\beta+1)}{2\beta+1},\ \alpha_0<\alpha<\frac{\gamma}{\sqrt{(\gamma-\beta)\beta}}$	*2*	E_0 *(saddle),* E_* *(unstable node or focus)*
6. $0<\beta<\gamma<\frac{2\beta(\beta+1)}{2\beta+1},\ \alpha\geq\frac{\gamma}{\sqrt{(\gamma-\beta)\beta}}$	*1*	E_0 *(saddle)*
7. $\gamma\geq\frac{2\beta(\beta+1)}{2\beta+1},\ 0<\alpha<\frac{\gamma}{\sqrt{(\gamma-\beta)\beta}}$	*2*	E_0 *(saddle),* E_* *(stable node or focus)*
8. $\gamma\geq\frac{2\beta(\beta+1)}{2\beta+1},\ \alpha\geq\frac{\gamma}{\sqrt{(\gamma-\beta)\beta}}$	*1*	E_0 *(saddle)*

Proof. Compute the Jacobian matrix of the vector field (4)

$$\begin{aligned} A &:= \begin{pmatrix} P_x(x,y) & P_y(x,y) \\ Q_x(x,y) & Q_y(x,y) \end{pmatrix} \\ &= \begin{pmatrix} -4x^3 - 2xy^2 + 3x^2 - 2\alpha xy + y^2 & -x(2xy + \alpha x - 2y) \\ 2(\gamma-\beta)xy & (\gamma-\beta)x^2 - 3\beta y^2 \end{pmatrix} \end{aligned}$$

and let T and D denote its trace and determinant respectively. At E_0,

$$D|_{E_0} = -(\gamma-\beta),\ T|_{E_0} = -1 + (\gamma-\beta).$$

Then, E_0 is a saddle if and only if $\gamma > \beta$, degenerate if and only if $\gamma = \beta$, stable node if and only if $\gamma < \beta$. At E_*, we have

$$D|_{E_*} = -2(\gamma-\beta)(\alpha\sqrt{(\gamma-\beta)\beta} - \gamma)^5/(\gamma^3\beta) > 0$$

when $\gamma > \beta$ and $\alpha < \gamma/\sqrt{(\beta-\beta)\beta}$.

It immediately shows that E_* is either a node, a focus or of center type. Further, for $\alpha, \beta, \gamma > 0$ and $\gamma > \beta$,

$$T|_{E_*} = \frac{(\alpha\sqrt{(\gamma-\beta)\beta} - \gamma)^2}{\gamma^3\beta}\Big\{2\beta\sqrt{(\gamma-\beta)\beta}\alpha - \gamma\big[2\beta(\gamma-\beta) + \gamma\big]\Big\} = 0$$

if and only if $\alpha = \alpha_0$, the constant defined in (6). It follows that E_* is either a stable node or focus if $0 < \alpha < \alpha_0$ or an unstable node or focus if $\alpha > \alpha_0$. E_* is of center type if $\alpha = \alpha_0$. Moreover, for $\alpha, \beta, \gamma > 0$, $\alpha_0 < \gamma/\sqrt{(\gamma-\beta)\beta}$ if and only if $\gamma < 2\beta(\beta+1)/(2\beta+1)$. Therefore, when $\gamma \geq 2\beta(\beta+1)/(2\beta+1)$ and $0 < \alpha < \gamma/\sqrt{(\beta-\beta)\beta}$, we have that E_* is a stable node or focus. In this way, we complete the proof. $\square$

For those nonhyperbolic cases mentioned in Lemma 2.1, we will further discussed in section 3 for their qualitative properties and bifurcations: For $\gamma = \beta$, E_0 is a nonhyperbolic equilibrium point with eigenvalues 0 and -1 and will be discussed; for $\beta < \gamma < 2\beta(\beta+1)/(2\beta+1)$ and $\alpha = \alpha_0$, E_* is of center type with eigenvalues $\pm i\ \Omega$, where

$$\Omega := \big[(2\beta+1)\gamma - 2\beta(\beta+1)\big]^2 \sqrt{-\gamma(\gamma-\beta)\big[(2\beta+1)\gamma - 2\beta(\beta+1)\big]}/(4\beta^3), \qquad (7)$$

and will be discussed.

3. Bifurcations

In this section we show that a pitchfork bifurcation may occur at E_0 and Hopf bifurcation may occur at E_*.

Table 1 of Lemma 2.1 show that system (3) has a nonhyperbolic equilibrium E_0 : $(1,0)$ with eigenvalues 0 and -1 if $\gamma = \beta$.

Theorem 3.1. *For $\gamma = \gamma_0 := \beta$, E_0 is an unstable node of system (3) and as γ crosses $\gamma = \gamma_0$, i.e., γ varies from $\gamma < \gamma_0$ to $\gamma > \gamma_0$, a pitchfork bifurcation happens at E_0 such that the stable node E_0 changes into three equilibria: a saddle E_0 and two stable nodes E_* and E_{**}.*

Proof. Let $\varepsilon := \gamma - \gamma_0$. For sufficiently small $|\varepsilon|$, system (4) can be expanded as

$$\begin{cases} \dot{x} = -x - \alpha y + O(\|(x,y)\|^2), \\ \dot{y} = \varepsilon y + O(\|(x,y)\|^2), \end{cases} \qquad (8)$$

where E_0 is translated to the origin $O : (0,0)$. With the change of variables

$$x = -\alpha z_1 + z_2,\ y = z_1,$$

and the time rescaling $d\varsigma = -dt$, system (8) is normalized linearly to the form

$$\begin{cases} \dot{z}_1 = \quad g_1(z_1, z_2, \varepsilon), \\ \dot{z}_2 = z_2 + g_2(z_1, z_2, \varepsilon), \end{cases} \qquad (9)$$

where

$$\begin{aligned} g_1(z_1,z_2,\varepsilon) \quad &= \quad -\varepsilon z_1 + 2\alpha\varepsilon z_1^2 - 2\varepsilon z_1 z_2 + (\beta - \alpha^2\varepsilon) z_1^3 + 2\alpha\varepsilon z_1^2 z_2 - \varepsilon z_1 z_2^2, \\ g_2(z_1,z_2,\varepsilon) \quad &= \quad -\alpha\varepsilon z_1 + \alpha^2(1+2\varepsilon) z_1^2 - \alpha(4+\varepsilon) z_1 z_2 + 3z_2^2 - \alpha(2\alpha^2 - \beta + 1 \\ &\quad +\alpha^2\varepsilon) z_1^3 + (7\alpha^2 + 1 + 2\alpha^2\varepsilon) z_1^2 z_2 - \alpha(8+\varepsilon) z_1 z_2^2 + 3z_2^3 \\ &\quad +\alpha^2(\alpha^2+1) z_1^4 - 2\alpha(2\alpha^2+1) z_1^3 z_2 + (6\alpha^2+1) z_1^2 z_2^2 - 4\alpha z_1 z_2^3 + z_2^4. \end{aligned}$$

Suspended with the parameter ε, system (9) can be regarded as a 3-dimensional one. The center manifold theory ([6]) shows that the suspended system has a smooth 2-dimensional center manifold $\mathcal{W}_\varepsilon^c = \{(z_1, z_2, \varepsilon) \mid z_2 = \varpi(z_1, \varepsilon),\ \varpi(0,0) = 0,\ D\varpi(0,0) = 0\}$ near the origin and the smooth function h can be approximated as $\varpi(z_1,\varepsilon) := \phi_2(z_1,\varepsilon) + O(\|(z_1,\varepsilon)\|^3)$ where the second order approximation ϕ_2, by Theorem 3 in [6], satisfies

$$\begin{aligned} (M\phi_2)(z_1,\varepsilon) \quad &:= \quad (\partial\phi_2)/(\partial z_1) g_1(z_1, \phi_2(z_1,\varepsilon), \varepsilon) - z_2 - g_2(z_1, \phi_2(z_1,\varepsilon), \varepsilon) \\ &= \quad O(\|(z_1,\varepsilon)\|^3). \end{aligned} \qquad (10)$$

Comparing the coefficients in (10), we obtain

$$\phi_2(z_1,\varepsilon) \quad = \quad -\alpha^2 z_1^2 + \alpha\varepsilon z_1.$$

Thus we obtain the restricted equation of system (9) to the center manifold $\mathcal{W}_\varepsilon^c$, i.e.,

$$\dot{z}_1 = G(z_1,\varepsilon) := -\varepsilon z_1 + \big(2\alpha\varepsilon + O(|\varepsilon|^2)\big)z_1^2 + (\beta + O(|\varepsilon|)z_1^3 + O(\|(z_1,\varepsilon)\|^3).$$

Then, for $\varepsilon = 0$ it shows that $(\partial G)/(\partial z_1)(0,0) = 0, (\partial^2 G)/(\partial z_1^2)(0,0) = 0$ and $(\partial G^3)/(\partial z_1^3)(0,0) = \beta \neq 0$ in (8) the origin O is the unique equilibrium and two equilibria arise from O as ε varies from 0 to positive. Therefore, E_0 is an unstable node at $\varepsilon = 0$ and system (3) undergoes a pitchfork bifurcation at E_0 for $\gamma = \beta$.

□

In the following, we consider Hopf bifurcations, proving that in the nonhyperbolic case 4 from Table 1 E_* is a weak focus of multiplicity 1.

Theorem 3.2. *For $\beta < \gamma < \frac{2\beta(\beta+1)}{2\beta+1}$ and $\alpha = \alpha_0$, equilibrium E_* of system (3) is a stable weak focus of order 1 and at most one stable limit cycle arises as $\alpha > \alpha_0$ from the supercritical Hopf bifurcation.*

Proof. Table 1 of Lemma 2.1 show that equilibrium $E_* : (x_*, \sqrt{(\gamma-\beta)/\beta}x_*)$ is of center type with eigenvalues $\pm\mathbf{i}\Omega$ if $\alpha = \alpha_0$ and $\beta < \gamma < 2\beta(\beta+1)/(2\beta+1)$, where α_0 and Ω are defined in (6) and (7) respectively. Consider $\beta < \gamma < 2\beta(\beta+1)/(2\beta+1)$ and let $\epsilon := \alpha - \alpha_0$. For sufficient small $|\epsilon|$, the linearization of system (4) at E_* has a pair of conjugate complex eigenvalues $\lambda_{1,2} = \sigma(\epsilon) \pm \mathbf{i}\,\omega(\epsilon)$ such that

$$\sigma(0) = 0,\ \omega(0) = \Omega,\ \frac{d\sigma}{d\epsilon}|_{\epsilon=0} = -\frac{\sqrt{(\gamma-\beta)\beta}}{4\beta^2\gamma}\big[(2\beta+1)\gamma - 2\beta(\beta+1)\big]^2 \neq 0.$$

Further, for sufficiently small $|\epsilon|$, translating E_* to the origin $O : (0,0)$ and applying the linear transformation

$$\begin{aligned} x &= -\frac{\gamma^2\beta}{2\delta(\gamma-\beta)(\alpha\beta\delta-\gamma)^2}z_1 - \frac{\gamma\big[(2\beta+1)\gamma - 2\beta(\beta+1)\big]^2}{4\beta\delta^2\Omega^2(\alpha\beta\delta-\gamma)^2}\big[K_1(\epsilon)z_1 + K_2(\epsilon)z_2\big],\\ y &= -\frac{1}{\Omega}\Big[K_1(\epsilon)z_1 + K_2(\epsilon)z_2\Big],\end{aligned}$$

and the time rescaling $d\varsigma = K_2(\epsilon)\Big\{1 - 6(\gamma-\beta)/\big\{\gamma\delta[(2\beta+1)\gamma - 2\beta(\beta+1)]\big\}\Big\}dt$, we reduce (4) to the form

$$\begin{cases} \dot{z_1} = \tau(\epsilon)z_1 - z_2 \ \ + W_1(z_1,z_2,\epsilon),\\ \dot{z_2} = z_1 + \tau(\epsilon)z_2 + W_2(z_1,z_2,\epsilon),\end{cases} \tag{11}$$

where the linear part is standardized,

$$\tau(\epsilon) := \frac{(\gamma-\beta)\big[(2\beta+1)\gamma - 2\beta(\beta+1)\big]^2\epsilon}{4\beta^2\gamma\delta\Omega} + O(|\epsilon|^2)$$

and

$$\begin{aligned} K_1(\epsilon) &:= \frac{a_{11}(\epsilon) + a_{22}(\epsilon)}{2(1+a_{12}(\epsilon))},\\ K_2(\epsilon) &:= \frac{\sqrt{4(a_{12}(\epsilon)a_{21}(\epsilon) + a_{12}(\epsilon) + a_{21}(\epsilon) + 1) - (a_{11}(\epsilon) + a_{22}(\epsilon))^2}}{2(1+a_{12}(\epsilon))},\end{aligned}$$

Let $z = z_1 + \mathbf{i}z_2$ and $\varphi = W_1 + \mathbf{i}W_2$. Then (11) can be represented as the complex form

$$\dot{z} = (\mathbf{i} + \tau(\epsilon))z + \varphi(z,\bar{z},\epsilon). \tag{12}$$

Applying the near-identity transformation

$$z = w + \sum\nolimits_{r+j=n} p_{rj}(\epsilon)\, w^r\overline{w}^j,$$

for $n = 2$ and $n = 3$ separately, where

$$p_{rj} = \begin{cases} 0, & r = j+1, \\ ((r+j-1)\tau + \mathbf{i}(r-j-1))^{-1} g_{rj}, & r \neq j+1, \end{cases}$$

and

$$g_{rj} = (\partial^{r+j}/\partial z^r \partial \bar{z}^j)\varphi(0,0,\epsilon)/(r!j!), \ r, j = 1, 2, 3 \cdots$$

as done in [11], we normalize the second degree terms and the third degree terms separately and reduce system (12) to the normal form

$$\dot{w} = (\mathbf{i} + \tau(\epsilon))w + \sum_{i=1}^{2} C_i(\epsilon) w^{i+1} \bar{w}^i + O(\|(w, \bar{w})\|^7),$$

where C_i s are complex coefficients. Thus, the first Lyapunov quantities are given by

$$L_1 \quad := \quad \mathrm{Re}(C_1(0)) = \frac{\beta^4 f_1(\gamma, \beta)}{4\gamma\Omega(\gamma-\beta)^3 \left[(2\beta+1)\gamma - 2\beta(\beta+1)\right]^4}, \tag{13}$$

where $f_1(\gamma, \beta) = (6\beta+3)\gamma^2 - 2\beta(7\beta+6)\gamma + 8\beta^2(\beta+1)$ and it can shown that $f_1(\gamma, \beta) < 0$ when $0 < \beta < \gamma < 2\beta(\beta+1)/(2\beta+1)$. Therefore, when $\epsilon = 0$, from (13), we have that $L_1 < 0$, implying that the equilibrium O of system (11) is a locally stable weak focus of multiplicity 1 and at most one stable limit cycle arises around the unstable focus from a supercritical Hopf bifurcation. In this way, we complete the proof.

□

At the end of this section, we scan those bifurcations as parameter (α, β, γ) varies. However, it is hard to plot the bifurcation curve in the 3-dimensional (α, β, γ)-space. We project it on the (γ, α)-plane as shown in Figure 1, where the projected curve of the pitchfork bifurcation surface $\gamma = \beta$ is the dashed line l_1. The solid line

$$l_2 : \alpha = \frac{\gamma(-2\beta^2 + 2\beta\gamma + \gamma)}{2\beta\sqrt{(\gamma-\beta)\beta}}, \ \beta < \gamma < 2\beta(\beta+1)/(2\beta+1)$$

is the projection of the Hopf bifurcation surface. Besides, as shown in Figure 1, by Table 1, system (3) has one equilibrium E_0 when (α, β, γ) lies on the region $\gamma < \beta$. As (α, β, γ) crossing $\gamma = \beta$ from $\gamma < \beta$ to $\gamma > \beta$, a stable node E_* and another equilibrium(but it always be outside $\mathcal{Q}_+$) arise. As (α, β, γ) crossing $\alpha = \gamma/\sqrt{(\gamma-\beta)\beta}$ from $\alpha < \gamma/\sqrt{(\gamma-\beta)\beta}$ to $\alpha > \gamma/\sqrt{(\gamma-\beta)\beta}$, E_* disappears outside $\mathcal{Q}_+$.

Remark 3.1. By the results of Lemma 2.1, Theorem 3.1 and Theorem 3.2, noticing the relationship between the original parameters r, e, m, c, d and the new parameters α, β, γ, we can give the explanation of the dynamical properties of system (3). For $\gamma < \beta$, i.e., $c < d$, system (3) has only one stable node or focus E_0, which implies that the predator will become extinct. For $\gamma = \beta$, i.e., $c = d$, system (3) has only one unstable degenerate node E_0. As c varies from $c < d$ to $c > d$, a pitchfork bifurcation happens at E_0 such that a interior equilibrium E_* appears. When $d < c < 2d(d+r)/(2d+r)$ and $e < cm(-2d^2 + 2cd + cr)/(2d\sqrt{(c-d)d})$, E_* is a stable node, which means that the predator and the prey will coexist and reach a steady state. When $d < c < 2d(d+r)/(2d+r)$ and $e = cm(-2d^2 + 2cd + cr)/(2d\sqrt{(c-d)d})$, E_* is a stable weak focus of order 1 and at most one limit cycle arises from Hopf bifurcation , i.e. when $d < c < 2d(d+r)/(2d+r)$ and $cm(-2d^2+2cd+cr)/(2d\sqrt{(c-d)d}) < e < cmr/\sqrt{(c-d)d}$, one limit cycle arises, which means the predator and the prey will oscillate periodically. When $d < c < 2d(d+r)/(2d+r)$ and $e \geq cmr/\sqrt{(c-d)d}$, the interior equilibrium

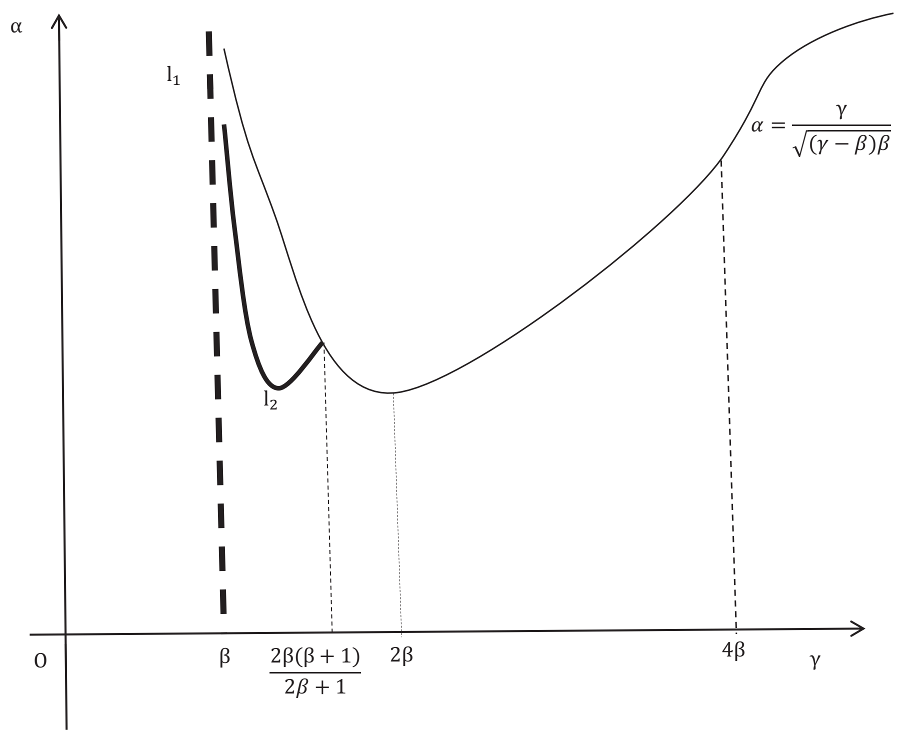

Fig. 1. Bifurcation diagram.

E_* disappear. When $c \geq 2d(d+r)/(2d+r)$ and $e < cmr/\sqrt{(c-d)d}$, E_* emerges and it is stable, which implies that the predator and the prey will coexist and reach a steady state. When $c \geq 2d(d+r)/(2d+r)$ and $e \geq cmr/\sqrt{(c-d)d}$, the interior equilibrium E_* disappear.

4. Conclusion

In this paper, we studied the dynamical behavior of a ratio-dependent predator-prey system with Holling type III functional response. By the linear transformation and the time rescaling, the original five parameters can be reduced to three parameters. For the reduced system, we give parameter conditions for exact number of equilibria as shown in Table 1. Further, we analytically give parameter conditions for pitchfork bifurcation (see Theorem 3.1). Finally, we give the conditions when a supercritical Hopf bifurcation occurs in Theorem 3.2, which implies at most one stable limit cycle arises.

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