On some $I$-convergent sequence spaces defined by a compact operator

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Abstract. In this article we introduce and study $I$-convergent sequence spaces $S_I$, $S^I_0$ and $S^I_\infty$ with the help of a compact operator $T$ on the real space $\mathbb{R}$. We study some topological and algebraic properties, prove the decomposition theorem and study some inclusion relations on these spaces.

Key words and phrases. Compact operator, Ideal, filter, $I$-convergent sequence, solid and monotone space, Banach space, Lipschitz function.

1. Introduction and preliminaries

Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ be the sets of all natural, real and complex numbers respectively. We denote

$$\omega = \{x = (x_k) : x_k \in \mathbb{R} \text{ or } \mathbb{C}\}$$

the space of all real or complex sequences.

**Definition 1.1.** Let $K$ be a non-trivial scalar valued field and $X$ be a vector space over $K$. Then a real valued mapping $\| \cdot \|$ on $X$ is said to be a norm on or over $X$ if it satisfies the following properties:

1. $\| x \| \geq 0$, $\| x \| = 0 \iff x = 0$,
2. $\| \alpha x \| = |\alpha| \| x \|$, 
3. $\| x + y \| \leq \| x \| + \| y \|$, for all $\alpha \in K$, $x, y \in X$.

The pair $(X, \| \cdot \|)$ is called a normed linear space over $K$.

**Remark 1.1.** If $K = \mathbb{R}$ (field of reals) or $K = \mathbb{C}$ (field of complex), then $X$ is called a real/complex normed linear space respectively.

**Definition 1.2.** A normed linear space $X$ is said to be a Banach space if it is complete. That is, if every Cauchy sequence in $X$ is convergent in $X$.

**Definition 1.3.** A linear operator $T$ is an operator such that

1. the domain $\mathcal{D}(T)$ of $T$ is a vector space and the range $\mathcal{R}(T)$ lies in a vector space over the same field,
2. $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$, for all $x, y \in \mathcal{D}(T)$.

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Definition 1.4. Let $X$ and $Y$ be two normed linear spaces and $T : \mathcal{D}(T) \to Y$ be a linear operator, where $\mathcal{D}(T) \subset X$. Then, the operator $T$ is said to be bounded if there exists a real $k > 0$ such that

$$\|Tx\| \leq k \|x\|,$$

for all $x \in \mathcal{D}(T)$.

The set of all bounded linear operators $\mathcal{B}(X,Y)$ is a normed linear space normed by

$$\|T\| = \sup_{x \in X, \|x\|=1} \|Tx\| \quad \text{ (see \cite{9,10})}$$

and $\mathcal{B}(X,Y)$ is a Banach space if $Y$ is Banach space.

Definition 1.5. Let $X$ and $Y$ be two normed linear spaces. An operator $T : X \to Y$ is said to be a compact linear operator (or completely continuous linear operator) if

1. $T$ is linear,
2. $T$ maps every bounded sequence $(x_k)$ in $X$ onto a sequence $T(x_k)$ in $Y$ which has a convergent subsequence.

The set of all compact linear operators $\mathcal{C}(X,Y)$ is closed subspace of $\mathcal{B}(X,Y)$ and $\mathcal{C}(X,Y)$ is a Banach space if $Y$ is Banach space.

Throughout the paper, we denote $\ell_\infty$, $c$ and $c_0$ as the Banach spaces of bounded, convergent and null sequences of reals respectively with norm

$$\|x\| = \sup_k |x_k| .$$

Following Basar and Altay \cite{1} and Sengönül \cite{15}, we introduce the sequence spaces $\mathcal{S}$ and $\mathcal{S}_0$ with the help of compact operator $T$ on $\mathbb{R}$ as follows.

$$\mathcal{S} = \{ x = (x_k) \in \ell_\infty : Tx \in c \}$$

and

$$\mathcal{S}_0 = \{ x = (x_k) \in \ell_\infty : Tx \in c_0 \} .$$

As a generalisation of usual convergence, the concept of statistical convergent was first introduced by Fast \cite{3} and also independently by Buck \cite{2} and Schoenberg \cite{14} for real and complex sequences. Later on, it was further investigated from a sequence space point of view and linked with the Summability Theory by Fridy \cite{4}, Šalát\cite{11}, Tripathy \cite{16} and many others.

Definition 1.6. A sequence $x=(x_k) \in \omega$ is said to be statistically convergent to a limit $L$ if for every $\epsilon > 0$, we have

$$\lim_{k} \frac{1}{k} |\{ n \in \mathbb{N} : |x_n - L| \geq \epsilon, \ n \leq k \}| = 0 .$$

where vertical lines denote the cardinality of the enclosed set.

That is, if $\delta(A(\epsilon)) = 0$, where

$$A(\epsilon) = \left\{ k \in \mathbb{N} : |x_k - L| \geq \epsilon \right\} .$$

The notion of ideal convergence (I-convergence) was introduced and studied by Kostyrko, Mačaj, Salát and Wilczyński \cite{7,8}. Later on, it was studied by Šalát, Tripathy and Ziman \cite{12,13}, Tripathy and Hazarika \cite{17,18}, Khan and Ebadullah \cite{5,6} and many others.
Here we give some preliminaries about the notion of I-convergence.

**Definition 1.7.** Let \( \mathbb{N} \) be a non empty set. Then a family of sets \( I \subseteq 2^\mathbb{N} \) (power set of \( \mathbb{N} \)) is said to be an ideal if
\[
\begin{align*}
(1) & \text{ } I \text{ is additive i.e } \forall A, B \in I \Rightarrow A \cup B \in I, \\
(2) & \text{ } I \text{ is hereditary i.e } \forall A \in I \text{ and } B \subseteq A \Rightarrow B \in I.
\end{align*}
\]

**Definition 1.8.** A non-empty family of sets \( \mathcal{J} \subseteq 2^\mathbb{N} \) is said to be filter on \( \mathbb{N} \) if and only if
\[
\begin{align*}
(1) & \text{ } \Phi \notin \mathcal{J}, \\
(2) & \forall A, B \in \mathcal{J} \Rightarrow A \cap B \in \mathcal{J}, \\
(3) & \forall A \in \mathcal{J} \text{ and } A \subseteq B \Rightarrow B \in \mathcal{J}.
\end{align*}
\]

**Definition 1.9.** An Ideal \( I \subseteq 2^\mathbb{N} \) is called non-trivial, if \( I \neq 2^\mathbb{N} \).

**Definition 1.10.** A non-trivial ideal \( I \subseteq 2^\mathbb{N} \) is called admissible, if
\[
\{\{x\} : x \in \mathbb{N}\} \subseteq I.
\]

**Definition 1.11.** A non-trivial ideal I is maximal, if there cannot exist any non-trivial ideal \( J \neq I \) containing \( I \) as a subset.

**Remark 1.2.** For each ideal \( I \), there is a filter \( \mathcal{L}(I) \) corresponding to \( I \).
That is \( \mathcal{L}(I) = \{K \subseteq \mathbb{N} : K^c \in I\} \), where \( K^c = \mathbb{N} \setminus K \).

**Definition 1.12.** A sequence \( x = (x_k) \in \omega \) is said to be I-convergent to a number \( L \), if for every \( \varepsilon > 0 \), the set \( \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\} \in I \).
In this case, we write \( I - \lim x_k = L \).

**Definition 1.13.** A sequence \( x = (x_k) \in \omega \) is said to be I-null, if \( L = 0 \). In this case, we write \( I - \lim x_k = 0 \).

**Definition 1.14.** A sequence \( x = (x_k) \in \omega \) is said to be I-cauchy, if for every \( \varepsilon > 0 \) there exists a number \( m = m(\varepsilon) \) such that \( \{k \in \mathbb{N} : |x_k - x_m| \geq \varepsilon\} \in I \).

**Definition 1.15.** A sequence \( x = (x_k) \in \omega \) is said to be I-bounded, if there exists some \( M > 0 \) such that \( \{k \in \mathbb{N} : |x_k| \geq M\} \in I \).

**Definition 1.16.** A sequence space \( E \) is said to be solid(normal), if \( (\alpha_k x_k) \in E \) whenever \( (x_k) \in E \) and for any sequence \( (\alpha_k) \) of scalars with \( |\alpha_k| \leq 1 \), for all \( k \in \mathbb{N} \).

**Definition 1.17.** A sequence space \( E \) is said to be symmetric, if \( (x_{\pi(k)}) \in E \) whenever \( x_k \in E \), where \( \pi \) is a permutation on \( \mathbb{N} \).

**Definition 1.18.** A sequence space \( E \) is said to be sequence algebra, if \( (x_k) * (y_k) = (x_k, y_k) \in E \) whenever \( (x_k), (y_k) \in E \).

**Definition 1.19.** A sequence space \( E \) is said to be convergence free, if \( (y_k) \in E \) whenever \( (x_k) \in E \) and \( x_k = 0 \) implies \( y_k = 0 \), for all \( k \).

**Definition 1.20.** Let \( K = \{k_1 < k_2 < k_3 < k_4 < k_5 \ldots\} \subseteq \mathbb{N} \) and \( E \) be a Sequence space. A \( K \)-step space of \( E \) is a sequence space \( \lambda^E_K = \{x_{k_n} \in \omega : (x_k) \in E\} \).

**Definition 1.21.** A canonical pre-image of a sequence \( (x_{k_n}) \in \lambda^E_K \) is a sequence \( (y_k) \in \omega \) defined by
\[
y_k = \begin{cases} 
   x_k, & \text{if } k \in K, \\
   0, & \text{otherwise.}
\end{cases}
\]
A canonical preimage of a step space $\lambda^E_K$ is a set of preimages of all elements in $\lambda^E_K$, i.e., $y$ is in the canonical preimage of $\lambda^E_K$, iff $y$ is the canonical preimage of some $x \in \lambda^E_K$.

**Definition 1.22.** A sequence space $E$ is said to be monotone, if it contains the canonical preimages of its step space.

**Definition 1.23.** If $I = I_f$, the class of all finite subsets of $\mathbb{N}$. Then, $I$ is an admissible ideal in $\mathbb{N}$ and $I_f$ convergence coincides with the usual convergence.

**Definition 1.24.** If $I = I_\delta = \{A \subseteq \mathbb{N} : \delta(A) = 0\}$. Then, $I$ is an admissible ideal in $\mathbb{N}$ and we call the $I_\delta$-convergence as the logarithmic statistical convergence.

**Definition 1.25.** If $I = I_d = \{A \subseteq \mathbb{N} : d(A) = 0\}$. Then, $I$ is an admissible ideal in $\mathbb{N}$ and we call the $I_d$-convergence as the asymptotic statistical convergence.

**Remark 1.3.** If $I_\delta - \lim x_n = l$, then $I_d - \lim x_n = l$.

**Definition 1.26.** A map $h$ defined on a domain $D \subset X$ i.e. $h : D \subset X \rightarrow \mathbb{R}$ is said to satisfy Lipschitz condition if $|h(x) - h(y)| \leq K|x - y|$ where $K$ is known as the Lipschitz constant.

**Definition 1.27.** A convergence field of $I$-convergence is a set

$$F(I) = \{x = (x_k) \in l_\infty : \text{there exists } I - \lim x \in \mathbb{R}\}.$$

The convergence field $F(I)$ is a closed linear subspace of $l_\infty$ with respect to the supremum norm, $F(I) = l_\infty \cap c^I$ (see [12]).

The function $h : F(I) \rightarrow \mathbb{R}$ defined by $h(x) = I - \lim x$, for all $x \in F(I)$ is a Lipschitz function (see [12]).

We used the following lemmas to establish some results of this article.

**Lemma(I).** Every solid space is monotone.

**Lemma(II).** Let $K \in \mathcal{L}(I)$ and $M \subseteq \mathbb{N}$. If $M \notin I$, then $M \cap K \notin I$.

**Lemma(III).** If $I \subseteq 2^\mathbb{N}$ and $M \subseteq \mathbb{N}$. If $M \notin I$, then $M \cap N \notin I$.

Throughout the article, $T$ is considered as a compact operator on the real space $\mathbb{R}$.

### 2. Main Results

In this article we introduce and study the following classes of sequence.

**S**$^I_\epsilon$ = \(\left\{ x = (x_k) \in \ell_\infty : \{k \in \mathbb{N} : |T(x_k) - L| \geq \epsilon \} \in I, \text{for some } L \in \mathbb{R} \right\}\); \hspace{1cm} (2.1)

**S**$^I_0$ = \(\left\{ x = (x_k) \in \ell_\infty : \{k \in \mathbb{N} : |T(x_k)| \geq \epsilon \} \in I \right\}\); \hspace{1cm} (2.2)

**S**$^I_\infty$ = \(\left\{ x = (x_k) \in \ell_\infty : \exists M > 0 \text{ s.t. } \{k \in \mathbb{N} : |T(x_k)| \geq M \} \in I \right\}\). \hspace{1cm} (2.3)

**Theorem 2.1.** The classes of sequences $S^I$, $S^I_0$ and $S^I_\infty$ are linear spaces.
Proof. We shall prove the result for the space $S^I$. Rests will follow similarly. For, let $x = (x_k)$, $y = (y_k)$ be two elements of $S^I$ and $\alpha$, $\beta$ be scalars. Now, since $(x_k)$, $(y_k) \in S^I$, then, for given $\epsilon > 0$, there exists $L_1$, $L_2 \in \mathbb{R}$ such that the sets
\[
\left\{ k \in \mathbb{N} : | T(x_k) - L_1 | < \frac{\epsilon}{2 | \alpha |} \right\} \in \mathcal{L}(I) \quad (2.4)
\]
and
\[
\left\{ k \in \mathbb{N} : | T(y_k) - L_2 | < \frac{\epsilon}{2 | \beta |} \right\} \in \mathcal{L}(I). \quad (2.5)
\]
Therefore,
\[
| T(\alpha(x_k) + \beta(y_k)) - (\alpha L_1 + \beta L_2) | = | \alpha T(x_k) + \beta T(y_k) - (\alpha L_1 + \beta L_2) | \\
= | \alpha T(x_k) - \alpha L_1 + \beta T(y_k) - \beta L_2 | \leq | \alpha | | T(x_k) - L_1 | + | \beta | | T(y_k) - L_2 | \\
< | \alpha | \frac{\epsilon}{2 | \alpha |} + | \beta | \frac{\epsilon}{2 | \beta |} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]
Thus the set
\[
A_3 = \left\{ k \in \mathbb{N} : | T(\alpha(x_k) + \beta(y_k)) - (\alpha L_1 + \beta L_2) | < \epsilon \right\} \in \mathcal{L}(I).
\]
Therefore the set
\[
A_3^c = \left\{ k \in \mathbb{N} : | T(\alpha(x_k) + \beta(y_k)) - (\alpha L_1 + \beta L_2) | \geq \epsilon \right\} \in I
\]
implies that $\alpha(x_k) + \beta(y_k) \in S^I$, for all scalars $\alpha$, $\beta$ and $(x_k)$, $(y_k) \in S^I$. Hence $S^I$ is linear. \(
\)

Theorem 2.2. The spaces $S^I$ and $S_0^I$ are normed spaces normed by
\[
\| x \|_* = \sup_{k} | T(x_k) | .
\]

Proof. The proof of the result is easy in view of existing techniques and hence omitted. \(\)

Theorem 2.3. A sequence $x = (x_k) \in \ell_\infty$ I-converges if and only if for every $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that
\[
\left\{ k \in \mathbb{N} : | T(x_k) - T(x_{N_\epsilon}) | < \epsilon \right\} \in \mathcal{L}(I). \quad (2.6)
\]

Proof. Let $x = (x_k) \in \ell_\infty$. Suppose that $L = I - \lim x$. Then, the set
\[
B_\epsilon = \left\{ k \in \mathbb{N} : | T(x_k) - L | < \frac{\epsilon}{2} \right\} \in \mathcal{L}(I) \quad \text{for all } \epsilon > 0.
\]
Fix an $N_\epsilon \in B_\epsilon$. Then, we have
\[
| T(x_k) - T(x_{N_\epsilon}) | \leq | T(x_k) - L | + | T(x_{N_\epsilon}) - L | < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]
which holds for all \( k \in B \).

Hence \( \{ k \in \mathbb{N} : |T(x_k) - T(x_{N_k})| < \epsilon \} \in \mathcal{L}(I) \).

Conversely, suppose that

\[
\left\{ k \in \mathbb{N} : |T(x_k) - T(x_{N_k})| < \epsilon \right\} \in \mathcal{L}(I).
\]

That is \( \{ k \in \mathbb{N} : |T(x_k) - T(x_{N_k})| < \epsilon \} \in \mathcal{L}(I) \), for all \( \epsilon > 0 \). Then, the set

\[ C_\epsilon = \left\{ k \in \mathbb{N} : T(x_k) \in [T(x_{N_k}) - \epsilon, T(x_{N_k}) + \epsilon] \right\} \in \mathcal{L}(I) \text{ for all } \epsilon > 0. \]

Let \( J_\epsilon = \left[T(x_{N_k}) - \epsilon, T(x_{N_k}) + \epsilon\right] \). If we fix an \( \epsilon > 0 \) then we have \( C_\epsilon \in \mathcal{L}(I) \) as well as \( C_{\frac{\epsilon}{2}} \in \mathcal{L}(I) \). Hence \( C_\epsilon \cap C_{\frac{\epsilon}{2}} \in \mathcal{L}(I) \). This implies that

\[ J = J_\epsilon \cap J_{\frac{\epsilon}{2}} \neq \phi. \]

That is

\[ \{ k \in \mathbb{N} : T(x_k) \in J \} \in \mathcal{L}(I). \]

That is

\[ \text{diam} J \leq \text{diam} J_\epsilon \]

where the diam of \( J \) denotes the length of interval \( J \).

In this way, by induction, we get the sequence of closed intervals

\[ J_k = I_0 \supseteq I_1 \supseteq \ldots \supseteq I_k \supseteq \ldots \]

with the property that \( \text{diam} I_k \leq \frac{1}{2} \text{diam} I_{k-1} \) for \( k = 2, 3, 4, \ldots \) and \( \{ k \in \mathbb{N} : T(x_k) \in I_k \} \in \mathcal{L}(I) \) for \( k = 1, 2, 3, 4, \ldots \).

Then there exists a \( \xi \in \cap I_k \) where \( k \in \mathbb{N} \) such that \( \xi = I - \lim T(x_k) \).

Hence the result. \( \square \)

**Theorem 2.4.** Let \( I \) be an admissible ideal. Then, the following are equivalent.

(a) \( (x_k) \in \mathcal{S}^I \);

(b) there exists \( (y_k) \in \mathcal{S} \) such that \( x_k = y_k \), for a.a.k.r.I;

(c) there exists \( (y_k) \in \mathcal{S} \) and \( (z_k) \in \mathcal{S}_I \) such that \( x_k = y_k + z_k \) for all \( k \in \mathbb{N} \) and \( \{ k \in \mathbb{N} : |T(y_k) - L| \geq \epsilon \} \in I \);

(d) there exists a subset \( K = \{ k_1 < k_2 < k_3 \ldots \} \) of \( \mathbb{N} \) such that \( K \in \mathcal{L}(I) \) and \( \lim_{n \to \infty} |T(x_{k_n}) - L| = 0 \).

**Proof.** (a) implies (b). Let \( (x_k) \in \mathcal{S}^I \). Then, for any \( \epsilon > 0 \), there exists \( L \in \mathbb{R} \) such that the set

\[ \{ k \in \mathbb{N} : |T(x_k) - L| \geq \epsilon \} \in I. \]

Let \( (m_t) \) be an increasing sequence with \( m_t \in \mathbb{N} \) such that

\[ \{ k \leq m_t : |T(x_k) - L| \geq t^{-1} \} \in I. \]

Define a sequence \( (y_k) \) as

\[ y_k = x_k, \text{ for all } k \leq m_1. \]

For \( m_t < k \leq m_{t+1}, t \in \mathbb{N} \),

\[ y_k = \begin{cases} x_k, & \text{if } |T(x_k) - L| < t^{-1}, \\ L, & \text{otherwise}. \end{cases} \]
Then, \((y_k) \in S\) and from the following inclusion
\[
\{ k \leq m_t : x_k \neq y_k \} \subseteq \{ k \in \mathbb{N} : |T(x_k) - L| \geq \epsilon \} \in I.
\]

We get \(x_k = y_k\), for a.a.k.r.\(I\).

(b) implies (c). For \((x_k) \in S^I\). Then, there exists \((y_k) \in S\) such that \(x_k = y_k\), for a.a.k.r.\(I\). Let \(K = \{ k \in \mathbb{N} : x_k \neq y_k \}\), then \(K \in I\).

Define a sequence \((z_k)\) as
\[
z_k = \begin{cases} 
x_k - y_k, & \text{if } k \in K, \\
0, & \text{otherwise.}
\end{cases}
\]

Then \((z_k) \in S^I\) and \((y_k) \in S\).

(c) implies (d). Let \(P_1 = \{ k \in \mathbb{N} : |T(x_k)| \geq \epsilon \} \in I\) and
\[
K = P_1^c = \{ k_1 < k_2 < k_3 < \ldots \} \in \mathcal{L}(I).
\]

Then, we have \(\lim_{n \to \infty} |T(x_{k_n}) - L| = 0\).

(d) implies (a). Let \(K = \{ k_1 < k_2 < k_3 < \ldots \} \in \mathcal{L}(I)\) and
\[
\lim_{n \to \infty} |T(x_{k_n}) - L| = 0.
\]

Then, for any \(\epsilon > 0\), and by Lemma (II), we have
\[
\{ k \in \mathbb{N} : |T(x_k) - L| \geq \epsilon \} \subseteq K^c \cup \{ k \in K : |T(x_k) - L| \geq \epsilon \}.
\]

Thus, \((x_k) \in S^I\). \(\square\)

**Theorem 2.5.** The function \(h : S^I \to \mathbb{R}\) defined by \(h(x) = I - \lim T(x)\), for all \(x \in S^I\) is a Lipschitz function and hence uniformly continuous.

**Proof.** Clearly the function \(h\) is well defined. Let \(x = (x_k), y = (y_k) \in S^I, x \neq y\). Then, the sets
\[
A_x = \{ k \in \mathbb{N} : |T(x) - h(x)| \geq \|x - y\|_* \} \in I.
\]
\[
A_y = \{ k \in \mathbb{N} : |T(y) - h(y)| \geq \|x - y\|_* \} \in I.
\]

where \(\|x - y\|_* = \sup_k |T(x_k - y_k)|\). Thus, the sets
\[
B_x = \{ k \in \mathbb{N} : |T(x) - h(x)| < \|x - y\|_* \} \in \mathcal{L}(I).
\]
\[
B_y = \{ k \in \mathbb{N} : |T(y) - h(y)| < \|x - y\|_* \} \in \mathcal{L}(I).
\]

Hence, \(B = B_x \cap B_y \in \mathcal{L}(I)\), so that \(B \neq \emptyset\). Now taking \(k \in B\), we have
\[
|h(x) - h(y)| \leq |h(x) - T(x)| + |T(x) - T(y)| + |T(y) - h(y)| \leq 3 \|x - y\|_*.
\]

Thus, \(h\) is Lipschitz function and hence uniformly continuous. \(\square\)

**Theorem 2.6.** If \(T\) is an identity operator and \(h : S^I \to \mathbb{R}\) is a function defined by \(h(x) = I - \lim T(x)\), for all \(x \in S^I\) and if \(x = (x_k), y = (y_k) \in S^I\), then, \((x, y) \in S^I\) and \(h(x, y) = h(x)h(y)\).

**Proof.** For \(\epsilon > 0\), the sets
\[
B_x = \{ k \in \mathbb{N} : |T(x) - h(x)| < \epsilon \} \in \mathcal{L}(I), \quad (2.7)
\]
\[
B_y = \{ k \in \mathbb{N} : |T(y) - h(y)| < \epsilon \} \in \mathcal{L}(I). \quad (2.8)
\]

where \(\|x - y\|_* = \epsilon\). Now, since \(T\) is an identity operator, we have
\[
|T(xy) - h(x)h(y)| = |T(x_ky_k - h(x)h(y))| = |T(x_ky_k - T(x_k)h(y) + T(x_k)h(y) - h(x)h(y))|
\]
\[ |x_k y_k - x_k h(y) + x_k h(y) - h(x) h(y)| \leq |x_k||y_k - h(y)| + |h(y)||x_k - h(x)|. \]  
(2.9)

As \( S^I \subseteq \ell_\infty \), there exists an \( M \in \mathbb{R} \) such that \( |x_k| < M \) and \( |h(y)| < M \). Therefore, from (2.7), (2.8) and (2.9), we have

\[ |T(xy) - h(x) h(y)| = |T(x_k y_k) - h(x) h(y)| \leq M\epsilon + M\epsilon = 2M\epsilon \]

for all \( k \in B_x \cap B_y \in \ell(L) \). Hence \((x,y) \in S^I\) and \(h(x,y) = h(x)h(y)\).

\[ \square \]

**Theorem 2.7.** The space \( S_\ell^I \) is solid and monotone.

**Proof.** For, let \((x_k) \in S_\ell^I\). Then, the set

\[ \{ k \in \mathbb{N} : |T(x_k)| \geq \epsilon \} \in I. \]

(2.10)

Let \((\alpha_k)\) be a sequence of scalars with \(|\alpha_k| \leq 1\), for all, \(k \in \mathbb{N}\). Therefore,

\[ |T(\alpha_k x_k)| = |\alpha_k T(x_k)| \leq |\alpha_k| |T x_k| \leq |T x_k|, \text{ for all } k \in \mathbb{N}. \]

Thus, from the above inequality and (2.10), we have

\[ \{ k \in \mathbb{N} : |T(\alpha_k x_k)| \geq \epsilon \} \subseteq \{ k \in \mathbb{N} : |T(x_k)| \geq \epsilon \} \in I. \]

implies that

\[ \{ k \in \mathbb{N} : |T(\alpha_k x_k)| \geq \epsilon \} \in I. \]

Therefore, \( \alpha_k x_k \in S_\ell^I \). Hence the space \( S_\ell^I \) is solid.

That the space is monotone follows from lemma (I).

\[ \square \]

**Theorem 2.8.** The inclusions \( S_0^I \subset S^I \subset S_\ell^I \) hold.

**Proof.** Let \((x_k) \in S^I\). Then, there exists some \(L\) such that

\[ I - \lim_k |T(x_k) - L| = 0. \]

That is, the set

\[ \{ k \in \mathbb{N} : |T(x_k) - L| \geq \epsilon \} \in I. \]

We have

\[ |T(x_k)| = |T(x_k) - L + L| \leq |T(x_k) - L| + |L|. \]

Taking supremum over \( k \) on both sides, we get \((x_k) \in S_\ell^I\).

The inclusion \( S_0^I \subset S^I \subset S_\ell^I \) is obvious. Hence \( S_0^I \subset S^I \subset S_\ell^I \).

\[ \square \]

**Theorem 2.9.** The set \( S^I \) is closed subspace of \( \ell_\infty \).

**Proof.** Let \((x_k^{(n)})\) be a Cauchy sequence in \( S^I \) such that \( x_k^{(n)} \rightarrow x \).

We show that \( x \in S^I \). Since \((x_k^{(n)}) \in S^I\), then there exists \(a_n\) such that \( \{ k \in \mathbb{N} : |T(x_k^{(n)}) - a_n| \geq \epsilon \} \in I. \)

We need to show that

1. \((a_n)\) converges to \(a\).
2. If \(U = \{ k \in \mathbb{N} : |T(x_k) - a| < \epsilon \}\), then \(U^c \in I. \)

(1) Since \((x_k^{(n)})\) is Cauchy sequence in \( S^I \) \( \Rightarrow \) for a given \( \epsilon > 0\), there exists \(k_0 \in \mathbb{N}\) such that \( \sup_{k \geq k_0} |T(x_k^{(n)}) - T(x_k^{(q)})| < \frac{\epsilon}{3}\), for all \(n, q \geq k_0. \)

For a given \( \epsilon > 0\), we have

\[ B_{\epsilon} = \{ k \in \mathbb{N} : |T x_k^{(n)} - T x_k^{(q)}| < \frac{\epsilon}{3} \}. \]
Theorem 2.10. The space $S^I$ is nowhere dense subsets of $\ell_\infty$.

References


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