Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 43(2), 2016, Pages 141–150
ISSN: 1223-6934

On some I-convergent sequence spaces defined by a compact operator

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Abstract. In this article we introduce and study I-convergent sequence spaces $\mathcal{S}^I$, $\mathcal{S}_0^I$ and $\mathcal{S}_\infty^I$ with the help of a compact operator T on the real space $\mathbb{R}$. We study some topological and algebraic properties, prove the decomposition theorem and study some inclusion relations on these spaces.

Key words and phrases. Compact operator, Ideal, filter, I-convergent sequence, solid and monotone space, Banach space, Lipschitz function.

1. Introduction and preliminaries

Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ be the sets of all natural, real and complex numbers respectively. We denote

$$\omega = \{x = (x_k) : x_k \in \mathbb{R} \text{ or } \mathbb{C}\}$$

the space of all real or complex sequences.

Definition 1.1. Let K be a non-trivial scalar valued field and X be a vector space over K. Then a real valued mapping $\| . \|$ on X is said to be a norm on or over X if it satisfies the following properties:
(1) $\| x \| \geq 0, \ \| x \| = 0 \Leftrightarrow x = 0,$
(2) $\| \alpha x \| = | \alpha | \| x \|,$
(3) $\| x + y \| \leq \| x \| + \| y \|,$ for all $\alpha \in K, \ x, \ y \in X.$

The pair $(X, \| . \|)$ is called a normed linear space over K.

Remark 1.1. If $K = \mathbb{R}$ (field of reals) or $K = \mathbb{C}$ (field of complex), then X is called a real/complex normed linear space respectively.

Definition 1.2. A normed linear space X is said to be a Banach space if it is complete. That is, if every Cauchy sequence in X is convergent in X.

Definition 1.3. A linear operator T is an operator such that
(1) the domain $\mathcal{D}(T)$ of T is a vector space and the range $\mathcal{R}(T)$ lies in a vector space over the same field,
(2) $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y),$ for all, $x, y \in \mathcal{D}(T).$

Received December 15, 2014.

Definition 1.4. Let X and Y be two normed linear spaces and $T : \mathcal{D}(T) \to Y$ be a linear operator, where $\mathcal{D}(T) \subset X$. Then, the operator T is said to be bounded if there exists a real $k > 0$ such that

$$\| Tx \| \leq k \| x \|, \text{ for all, } x \in \mathcal{D}(T).$$

The set of all bounded linear operators $\mathcal{B}(X,Y)$ is a normed linear space normed by

$$\| T \| = \sup_{x \in X, \|x\|=1} \| Tx \| \quad (\text{see } [9,10])$$

and $\mathcal{B}(X,Y)$ is a Banach space if Y is Banach space.

Definition 1.5. Let X and Y be two normed linear spaces. An operator $T : X \to Y$ is said to be a compact linear operator (or completely continuous linear operator) if
(1) T is linear,
(2) T maps every bounded sequence (x_k) in X onto a sequence $T(x_k)$ in Y which has a convergent subsequence.

The set of all compact linear operators $\mathcal{C}(X,Y)$ is closed subspace of $\mathcal{B}(X,Y)$ and $\mathcal{C}(X,Y)$ is a Banach space if Y is Banach space.

Throughout the paper, we denote ℓ_∞, c and c_0 as the Banach spaces of bounded, convergent and null sequences of reals respectively with norm

$$\|x\| = \sup_k | x_k | .$$

Following Basar and Altay [1] and Sengönül [15], we introduce the sequence spaces $\mathcal{S}$ and $\mathcal{S}_0$ with the help of compact operator T on $\mathbb{R}$ as follows.

$$\mathcal{S} = \{x = (x_k) \in \ell_\infty : Tx \in c\}$$

and

$$\mathcal{S}_0 = \{x = (x_k) \in \ell_\infty : Tx \in c_0\}.$$

As a generalisation of usual convergence, the concept of statistical convergent was first introduced by Fast [3] and also independently by Buck [2] and Schoenberg [14] for real and complex sequences. Later on, it was further investigated from a sequence space point of view and linked with the Summability Theory by Fridy 4], Šalát[11], Tripathy [16] and many others.

Definition 1.6. A sequence x=$(x_k) \in \omega$ is said to be statistically convergent to a limit L if for every $\epsilon > 0$, we have

$$\lim_k \frac{1}{k} |\{n \in \mathbb{N} : |x_n - L| \geq \epsilon, \ n \leq k\}| = 0.$$

where vertical lines denote the cardinality of the enclosed set.

That is, if $\delta(A(\epsilon)) = 0$, where

$$A(\epsilon) = \left\{ k \in \mathbb{N} : | x_k - L | \geq \epsilon \right\}.$$

The notion of ideal convergence (I-convergence) was introduced and studied by Kostyrko, Mačaj, Salăt and Wilczyński [7,8]. Later on, it was studied by Šalát, Tripathy and Ziman [12,13], Tripathy and Hazarika [17,18], Khan and Ebadullah [5,6] and many others.

Here we give some preliminaries about the notion of I-convergence.

Definition 1.7. Let $\mathbb{N}$ be a non empty set. Then a family of sets $I \subseteq 2^{\mathbb{N}}$ (power set of $\mathbb{N}$) is said to be an ideal if
(1) I is additive i.e $\forall A, B \in I \Rightarrow A \cup B \in I$,
(2) I is hereditary i.e $\forall A \in I$and $B \subseteq A \Rightarrow B \in I$.

Definition 1.8. A non-empty family of sets $\mathcal{J} \subseteq 2^{\mathbb{N}}$ is said to be filter on $\mathbb{N}$ if and only if
(1) $\Phi \notin \mathcal{J}$,
(2) $\forall\ A,\ B \in \mathcal{J} \Rightarrow A \cap B \in \mathcal{J}$,
(3) $\forall\ A \in \mathcal{J}$ and $A \subseteq B \Rightarrow B \in \mathcal{J}$.

Definition 1.9. An Ideal $I \subseteq 2^{\mathbb{N}}$ is called non-trivial, if $I \neq 2^{\mathbb{N}}$.

Definition 1.10. A non-trivial ideal $I \subseteq 2^{N}$ is called admissible, if

$$\{\{x\} : x \in \mathbb{N}\} \subseteq I.$$

Definition 1.11. A non-trivial ideal I is maximal, if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset.

Remark 1.2. For each ideal I, there is a filter $\pounds(I)$ corresponding to I.
That is $\pounds(I) = \{K \subseteq \mathbb{N} : K^c \in I\}$, where $K^c = \mathbb{N} \setminus K$.

Definition 1.12. A sequence $x = (x_k) \in \omega$ is said to be I-convergent to a number L, if for every $\epsilon > 0$, the set $\{k \in \mathbb{N} : |x_k - L| \geq \epsilon\} \in\ I$.
In this case, we write $I - \lim x_k = L$.

Definition 1.13. A sequence $x = (x_k) \in \omega$ is said to be I-null, if $L = 0$. In this case, we write $I - \lim x_k = 0$.

Definition 1.14. A sequence $x = (x_k) \in \omega$ is said to be I-cauchy, if for every $\epsilon > 0$ there exists a number $m = m(\epsilon)$ such that $\{k \in \mathbb{N} : |x_k - x_m| \geq \epsilon\} \in\ I$.

Definition 1.15. A sequence $x = (x_k) \in \omega$ is said to be I-bounded, if there exists some $M > 0$ such that $\{k \in \mathbb{N} : |x_k| \geq M\} \in\ I$.

Definition 1.16. A sequence space E is said to be solid(normal), if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ and for any sequence(α_k) of scalars with $|\ \alpha_k\ | \leq 1$, for all $k \in \mathbb{N}$.

Definition 1.17. A sequence space E is said to be symmetric, if $(x_{\pi(k)}) \in E$ whenever $x_k \in E$, where π is a permutation on $\mathbb{N}$.

Definition 1.18. A sequence space E is said to be sequence algebra, if $(x_k) * (y_k) = (x_k.y_k) \in E$ whenever $(x_k), (y_k) \in E$.

Definition 1.19. A sequence space E is said to be convergence free, if $(y_k) \in E$ whenever $(x_k) \in E$ and $x_k = 0$ implies $y_k = 0$, for all k.

Definition 1.20. Let $K = \{k_1 < k_2 < k_3 < k_4 < k_5 ...\} \subset \mathbb{N}$ and E be a Sequence space. A K-step space of E is a sequence space $\lambda_K^E = \{(x_{k_n}) \in \omega : (x_k) \in E\}$.

Definition 1.21. A canonical pre-image of a sequence $(x_{k_n}) \in \lambda_K^E$ is a sequence $(y_k) \in \omega$ defined by

$$y_k = \begin{cases} x_k, \text{ if } k \in K, \\ 0, \text{ otherwise.} \end{cases}$$

A canonical preimage of a step space λ_K^E is a set of preimages of all elements in λ_K^E.i.e. y is in the canonical preimage of λ_K^E, iff y is the canonical preimage of some $x \in \lambda_K^E$.

Definition 1.22. A sequence space E is said to be monotone, if it contains the canonical preimages of its step space.

Definition 1.23. If $I = I_f$, the class of all finite subsets of $\mathbb{N}$. Then, I is an admissible ideal in $\mathbb{N}$ and I_f convergence coincides with the usual convergence.

Definition 1.24. If $I = I_\delta = \{A \subseteq \mathbb{N} : \delta(A) = 0\}$. Then, I is an admissible ideal in $\mathbb{N}$ and we call the I_δ-convergence as the logarithmic statistical convergence.

Definition 1.25. If $I = I_d = \{A \subseteq \mathbb{N} : d(A) = 0\}$. Then, I is an admissible ideal in $\mathbb{N}$ and we call the I_d-convergence as the asymptotic statistical convergence.

Remark 1.3. If $I_\delta - \lim x_n = l$, then $I_d - \lim x_n = l$.

Definition 1.26. A map $\hbar$ defined on a domain $D \subset X$ i.e $\hbar : D \subset X \to \boldsymbol{R}$ is said to satisfy Lipschitz condition if $|\hbar(x) - \hbar(y)| \leq K|x - y|$ where K is known as the Lipschitz constant.

Definition 1.27. A convergence field of I-covergence is a set

$$F(I) = \{x = (x_k) \in l_\infty : \text{there exists } I - \lim x \in \boldsymbol{R}\}.$$

The convergence field $F(I)$ is a closed linear subspace of l_∞ with respect to the supremum norm, $F(I) = l_\infty \cap c^I (see[12])$.

The function $\hbar : F(I) \to \mathbb{R}$ defined by $\hbar(x) = I - \lim x,$ for all x $\in F(I)$ is a lipschitz function (see[12]).

We used the following lemmas to establish some results of this article.

Lemma(I). Every solid space is monotone.
Lemma(II). Let $K \in \pounds(I)$ and $M \subseteq \mathbb{N}$. If $M \notin I$, then $M \cap K \notin I$.
Lemma(III). If $I \subseteq 2^{\mathbb{N}}$ and $M \subseteq \mathbb{N}$. If $M \notin I$, then $M \cap \mathbb{N} \notin I$.

Throughout the article, T is considered as a compact operator on the real space $\mathbb{R}$.

2. Main Results

In this article we introduce and study the following classes of sequence.

$$\mathcal{S}^I = \Big\{x = (x_k) \in \ell_\infty : \big\{k \in \mathbb{N} : \mid T(x_k) - L \mid \geq \epsilon\big\} \in I, \text{for some } L \in \mathbb{R}\Big\}; \tag{2.1}$$

$$\mathcal{S}_0^I = \Big\{x = (x_k) \in \ell_\infty : \big\{k \in \mathbb{N} : \mid T(x_k) \mid \geq \epsilon\big\} \in I\Big\}; \tag{2.2}$$

$$\mathcal{S}_\infty^I = \Big\{x = (x_k) \in \ell_\infty : \exists M > 0 \ s.t. \big\{k \in \mathbb{N} : \mid T(x_k) \mid \geq M\big\} \in I\Big\}. \tag{2.3}$$

Theorem 2.1. *The classes of sequences $\mathcal{S}^I$, $\mathcal{S}_0^I$ and $\mathcal{S}_\infty^I$ are linear spaces.*

Proof. We shall prove the result for the space $\mathcal{S}^I$. Rests will follow similarly.
For, let $x = (x_k),\ y = (y_k)$ be two elements of $\mathcal{S}^I$ and $\alpha,\ \beta$ be scalars.
Now, since $(x_k),\ (y_k) \in \mathcal{S}^I$, then, for given $\epsilon > 0$, there exists $L_1,\ L_2 \in \mathbb{R}$ such that the sets

$$\left\{k \in \mathbb{N} :\mid T(x_k) - L_1 \mid < \frac{\epsilon}{2 \mid \alpha \mid}\right\} \in \pounds(I) \tag{2.4}$$

and

$$\left\{k \in \mathbb{N} :\mid T(y_k) - L_2 \mid < \frac{\epsilon}{2 \mid \beta \mid}\right\} \in \pounds(I). \tag{2.5}$$

Therefore,

$$\mid T\big(\alpha(x_k) + \beta(y_k)\big) - \big(\alpha L_1 + \beta L_2\big) \mid = \mid \alpha T(x_k) + \beta T(y_k) - \big(\alpha L_1 + \beta L_2\big) \mid$$

$$= \mid \alpha T(x_k) - \alpha L_1 + \beta T(y_k) - \beta L_2 \mid \leq \mid \alpha \mid\mid T(x_k) - L_1 \mid + \mid \beta \mid\mid T(y_k) - L_2 \mid$$

$$< \mid \alpha \mid \frac{\epsilon}{2 \mid \alpha \mid} + \mid \beta \mid \frac{\epsilon}{2 \mid \beta \mid} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus the set

$$A_3 = \left\{k \in \mathbb{N} :\mid T\big(\alpha(x_k) + \beta(y_k)\big) - \big(\alpha L_1 + \beta L_2\big) \mid < \epsilon\right\} \in \pounds(I).$$

Therefore the set

$$A_3^c = \left\{k \in \mathbb{N} :\mid T\big(\alpha(x_k) + \beta(y_k)\big) - \big(\alpha L_1 + \beta L_2\big) \mid \geq \epsilon\right\} \in I$$

implies that $\alpha(x_k) + \beta(y_k) \in \mathcal{S}^I$, for all scalars $\alpha,\ \beta$ and $(x_k),\ (y_k) \in \mathcal{S}^I$.
Hence $\mathcal{S}^I$ is linear. □

Theorem 2.2. *The spaces $\mathcal{S}^I$ and $\mathcal{S}_0^I$ are normed spaces normed by*

$$\|x\|_* = \sup_k \mid T(x_k) \mid .$$

Proof. The proof of the result is easy in view of existing techniques and hence omitted. □

Theorem 2.3. *A sequence $x = (x_k) \in \ell_\infty$ I-converges if and only if for every $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that*

$$\left\{k \in \mathbb{N} :\mid T(x_k) - T(x_{N_\epsilon}) \mid < \epsilon\right\} \in \pounds(I). \tag{2.6}$$

Proof. Let $x = (x_k) \in \ell_\infty$.
Suppose that $L = I - \lim x$. Then, the set

$$B_\epsilon = \left\{k \in \mathbb{N} :\mid T(x_k) - L \mid < \frac{\epsilon}{2}\right\} \in \pounds(I) \ \text{ for all } \epsilon > 0.$$

Fix an $N_\epsilon \in B_\epsilon$. Then, we have

$$\mid T(x_k) - T(x_{N_\epsilon}) \mid \leq \mid T(x_k) - L \mid + \mid T(x_{N_\epsilon}) - L \mid < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

which holds for all $k \in B_\epsilon$.
Hence $\left\{k \in \mathbb{N} :| T(x_k) - T(x_{N_\epsilon}) |< \epsilon\right\} \in £(I)$.
Conversely, suppose that

$$\left\{k \in \mathbb{N} :| T(x_k) - T(x_{N_\epsilon}) |< \epsilon\right\} \in £(I).$$

That is $\left\{k \in \mathbb{N} :| T(x_k) - T(x_{N_\epsilon}) |< \epsilon\right\} \in £(I)$, for all $\epsilon > 0$. Then, the set

$$C_\epsilon = \left\{k \in \mathbb{N} : T(x_k) \in [T(x_{N_\epsilon}) - \epsilon, T(x_{N_\epsilon}) + \epsilon]\right\} \in £(I) \text{ for all } \epsilon > 0.$$

Let $J_\epsilon = \left[T(x_{N_\epsilon}) - \epsilon, T(x_{N_\epsilon}) + \epsilon\right]$. If we fix an $\epsilon > 0$ then we have $C_\epsilon \in £(I)$ as well as $C_{\frac{\epsilon}{2}} \in £(I)$. Hence $C_\epsilon \cap C_{\frac{\epsilon}{2}} \in £(I)$. This implies that

$$J = J_\epsilon \cap J_{\frac{\epsilon}{2}} \neq \phi.$$

That is

$$\{k \in \mathbb{N} : T(x_k) \in J\} \in £(I).$$

That is

$$diamJ \leq diamJ_\epsilon$$

where the diam of J denotes the length of interval J.
In this way, by induction, we get the sequence of closed intervals

$$J_\epsilon = I_0 \supseteq I_1 \supseteq ..... \supseteq I_k \supseteq ....$$

with the property that $diamI_k \leq \frac{1}{2}diamI_{k-1}$ for (k=2,3,4,.....) and $\{k \in \mathbb{N} : T(x_k) \in I_k\} \in £(I)$ for (k=1,2,3,4,......).
Then there exists a $\xi \in \cap I_k$ where $k \in \mathbb{N}$ such that $\xi = I - \lim T(x_k)$.
Hence the result. □

Theorem 2.4. *Let I be an admissible ideal. Then, the following are equivalent.*
(a) $(x_k) \in \mathcal{S}^I$;
(b) there exists $(y_k) \in \mathcal{S}$ such that $x_k = y_k$, for a.a.k.r.I;
(c) there exists $(y_k) \in \mathcal{S}$ and $(z_k) \in \mathcal{S}^I_\circ$ such that $x_k = y_k + z_k$ for all $k \in \mathbb{N}$ and $\{k \in \mathbb{N} :| T(y_k) - L |\geq \epsilon\} \in I$;
(d) there exists a subset $K = \{k_1 < k_2 < k_3...\}$ of $\mathbb{N}$ such that $K \in £(I)$ and $\lim_{n\to\infty} | T(x_{k_n}) - L |= 0$.

Proof. (a) implies (b). Let $(x_k) \in \mathcal{S}^I$. Then, for any $\epsilon > 0$, there exists $L \in \mathbb{R}$ such that the set

$$\{k \in \mathbb{N} :| T(x_k) - L |\geq \epsilon\} \in I.$$

Let (m_t) be an increasing sequence with $m_t \in \mathbb{N}$ such that

$$\{k \leq m_t : |T(x_k) - L| \geq t^{-1}\} \in I.$$

Define a sequence (y_k) as

$$y_k = x_k, \text{ for all } k \leq m_1.$$

For $m_t < k \leq m_{t+1}, t \in \mathbb{N}$.

$$y_k = \begin{cases} x_k, \text{ if } |T(x_k) - L| < t^{-1}, \\ L, \text{otherwise.} \end{cases}$$

Then, $(y_k) \in \mathcal{S}$ and from the following inclusion

$$\{k \leq m_t : x_k \neq y_k\} \subseteq \{k \in \mathbb{N} : |T(x_k) - L| \geq \epsilon\} \in I.$$

We get $x_k = y_k$, for a.a.k.r.I.
(b) implies (c). For $(x_k) \in \mathcal{S}^I$. Then, there exists $(y_k) \in \mathcal{S}$ such that $x_k = y_k$, for a.a.k.r.I. Let $K = \{k \in \mathbb{N} : x_k \neq y_k\}$, then $K \in I$.
Define a sequence (z_k) as

$$z_k = \begin{cases} x_k - y_k, & \text{if } k \in K, \\ 0, & \text{otherwise.} \end{cases}$$

Then $(z_k) \in \mathcal{S}_\circ^I$ and $(y_k) \in \mathcal{S}$.
(c) implies (d). Let $P_1 = \{k \in \mathbb{N} : |T(x_k)| \geq \epsilon\} \in I$ and

$$K = P_1^c = \{k_1 < k_2 < k_3 < ...\} \in \pounds(I).$$

Then, we have $\lim\limits_{n\to\infty} |T(x_{k_n}) - L| = 0$.
(d) implies (a). Let $K = \{k_1 < k_2 < k_3 < ...\} \in \pounds(I)$ and

$$\lim_{n\to\infty} |T(x_{k_n}) - L| = 0.$$

Then, for any $\epsilon > 0$, and by Lemma (II), we have

$$\{k \in \mathbb{N} : |T(x_k) - L| \geq \epsilon\} \subseteq K^c \cup \{k \in K : |T(x_k) - L| \geq \epsilon\}.$$

Thus, $(x_k) \in \mathcal{S}^I$. □

Theorem 2.5. *The function $\hbar : \mathcal{S}^I \to \mathbb{R}$ defined by $\hbar(x) = I - \lim T(x)$, for all $x \in \mathcal{S}^I$ is a Lipschitz function and hence uniformly continuous.*

Proof. Clearly the function $\hbar$ is well defined. Let $x = (x_k),\ y = (y_k) \in \mathcal{S}^I,\ x \neq y$. Then, the sets

$$A_x = \{k \in \mathbb{N} :| T(x) - \hbar(x) |\geq\| x - y \|_* \} \in I.$$
$$A_y = \{k \in \mathbb{N} :| T(y) - \hbar(y) |\geq\| x - y \|_* \} \in I.$$

where $\|x - y\|_* = \sup\limits_k | T(x_k - y_k) |$. Thus, the sets

$$B_x = \{k \in \mathbb{N} :| T(x) - \hbar(x) |<\| x - y \|_* \} \in \pounds(I).$$
$$B_y = \{k \in \mathbb{N} :| T(y) - \hbar(y) |<\| x - y \|_* \} \in \pounds(I).$$

Hence, $B = B_x \cap B_y \in \pounds(I)$, so that $B \neq \emptyset$. Now taking $k \in B$, we have

$$| \hbar(x) - \hbar(y) |\leq| \hbar(x) - T(x) | + | T(x) - T(y) | + | T(y) - \hbar(y) |\leq 3 \| x - y \|_* .$$

Thus, $\hbar$ is Lipschitz function and hence uniformly continuous. □

Theorem 2.6. *If T is an identity operator and $\hbar : \mathcal{S}^I \to \mathbb{R}$ is a function defined by $\hbar(x) = I - \lim T(x)$, for all $x \in \mathcal{S}^I$ and if $x = (x_k),\ y = (y_k) \in \mathcal{S}^I$, then, $(x.y) \in \mathcal{S}^I$ and $\hbar(x.y) = \hbar(x)\hbar(y)$.*

Proof. For $\epsilon > 0$, the sets

$$B_x = \{k \in \mathbb{N} : |T(x) - \hbar(x)| < \epsilon\} \in \pounds(I), \tag{2.7}$$
$$B_y = \{k \in \mathbb{N} : |T(y) - \hbar(y)| < \epsilon\} \in \pounds(I). \tag{2.8}$$

where $\|x - y\|_* = \epsilon$. Now, since T is an identity operator, we have

$$|T(xy)-\hbar(x)\hbar(y)| = |T(x_k y_k)-\hbar(x)\hbar(y)| = |T(x_k y_k)-T(x_k)\hbar(y)+T(x_k)\hbar(y)-\hbar(x)\hbar(y)|$$

$$= |x_k y_k - x_k\hbar(y) + x_k\hbar(y) - \hbar(x)\hbar(y)| \le |x_k||y_k - \hbar(y)| + |\hbar(y)||x_k - \hbar(x)|. \quad (2.9)$$

As $\mathcal{S}^I \subseteq \ell_\infty$, there exists an $M \in \mathbb{R}$ such that $|x_k| < M$ and $|\hbar(y)| < M$.
Therefore, from (2.7), (2.8) and (2.9), we have

$$|T(xy) - \hbar(x)\hbar(y)| = |T(x_k y_k) - \hbar(x)\hbar(y)| \le M\epsilon + M\epsilon = 2M\epsilon$$

for all $k \in B_x \cap B_y \in \pounds(I)$. Hence $(x.y) \in \mathcal{S}^I$ and $\hbar(x.y) = \hbar(x)\hbar(y)$. □

Theorem 2.7. *The space $\mathcal{S}^I_\circ$ is solid and monotone.*

Proof. For, let $(x_k) \in \mathcal{S}^I_\circ$. Then, the set

$$\Big\{k \in \mathbb{N} : \mid T(x_k) \mid \ge \epsilon\Big\} \in I. \quad (2.10)$$

Let (α_k) be a sequence of scalars with $\mid \alpha_k \mid \le 1$, for all, $k \in \mathbb{N}$.
Therefore,

$$\mid T(\alpha_k x_k) \mid = \mid \alpha_k T(x_k) \mid \le \mid \alpha_k \mid\mid Tx_k \mid \le \mid Tx_k \mid, \text{for all k} \in \mathbb{N}.$$

Thus, from the above inequality and (2.10), we have

$$\Big\{k \in \mathbb{N} : \mid T(\alpha_k x_k) \mid \ge \epsilon\Big\} \subseteq \Big\{k \in \mathbb{N} : \mid T(x_k) \mid \ge \epsilon\Big\} \in I$$

implies that

$$\Big\{k \in \mathbb{N} : \mid T(\alpha_k x_k) \mid \ge \epsilon\Big\} \in I.$$

Therefore, $\alpha_k x_k \in \mathcal{S}^I_\circ$. Hence the space $\mathcal{S}^I_\circ$ is solid.
That the space is monotone follows from lemma (I). □

Theorem 2.8. *The inclusions $\mathcal{S}^I_0 \subset \mathcal{S}^I \subset \mathcal{S}^I_\infty$ hold.*

Proof. Let $(x_k) \in \mathcal{S}^I$. Then, there exists some L such that

$$I - \lim_k \mid T(x_k) - L \mid = 0.$$

That is, the set

$$\{k \in \mathbb{N} : \mid T(x_k) - L \mid \ge \epsilon\} \in I.$$

We have

$$\mid T(x_k) \mid = \mid T(x_k) - L + L \mid \le \mid T(x_k) - L \mid + \mid L \mid .$$

Taking supremum over k on both sides, we get $(x_k) \in \mathcal{S}^I_\infty$.
The inclusion $\mathcal{S}^I_0 \subset \mathcal{S}^I$ is obvious. Hence $\mathcal{S}^I_\circ \subset \mathcal{S}^I \subset \mathcal{S}^I_\infty$. □

Theorem 2.9. *The set $\mathcal{S}^I$ is closed subspace of $\ell\infty$.*

Proof. Let $(x_k^{(n)})$ be a Cauchy sequence in $\mathcal{S}^I$ such that $x_k^{(n)} \to x$.
We show that $x \in \mathcal{S}^I$. Since $(x_k^{(n)}) \in \mathcal{S}^I$, then there exists a_n such that $\Big\{k \in \mathbb{N} : \mid T(x_k^{(n)}) - a_n \mid \ge \epsilon\Big\} \in I$.
We need to show that
(1) (a_n) converges to a.
(2) If $U = \Big\{k \in \mathbb{N} : \mid T(x_k) - a \mid < \epsilon\Big\}$, then $U^c \in I$.
(1) Since $(x_k^{(n)})$ is Cauchy sequence in $\mathcal{S}^I \Rightarrow$ for a given $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that $\sup_k |T(x_k^{(n)}) - T(x_k^{(q)})| < \frac{\epsilon}{3}$, for all $n,\ q \ge k_0$.
For a given $\epsilon > 0$, we have

$$B_{nq} = \Big\{k \in \mathbb{N} : \mid Tx_k^{(n)} - Tx_k^{(q)} \mid < \frac{\epsilon}{3}\Big\}$$

$$B_q = \Big\{k \in \mathbb{N} :| Tx_k^{(q)} - a_q |< \frac{\epsilon}{3}\Big\}$$
$$B_n = \Big\{k \in \mathbb{N} :| Tx_k^{(n)} - a_n |< \frac{\epsilon}{3}\Big\}$$

Then, B_{nq}^c, $B_q^c, B_n^c \in I$. Let $B^c = B_{nq}^c \cup B_q^c \cup B_n^c$, where $B = \{k \in \mathbb{N} :| a_q - a_\delta n |< \epsilon\}$. Then, $B^c \in I$.
We choose $k_0 \in B^c$. Then for each $n, q \geq k_0$, we have

$$\{k \in \mathbb{N} : | a_q - a^n |< \epsilon\} \supseteq \Big[\{k \in \mathbb{N} :| a_q - T(x_k^{(q)}) |< \frac{\epsilon}{3}\} \cap \{k \in \mathbb{N} :| T(x_k^{(q)}) - T(x_k^{(n)}) |< \frac{\epsilon}{3}\} \cap \{k \in \mathbb{N} :| T(x_k^{(n)}) - a_n |< \frac{\epsilon}{3}\}\Big].$$

Then (a_n) is a Cauchy sequence of scalars in $\mathbb{R}$, so there exists a scalar a in $\mathbb{R}$ such that $a_n \to a$ as $n \to \infty$.

(2) Let $0 < \delta < 1$ be given. Then we show that if $U = \Bigg\{k \in \mathbb{N} :| T(x_k) - a |\leq \delta\Bigg\}$, then $U^c \in I$.
Since $x_k^{(n)} \to x$, then there exists $q_0 \in \mathbb{N}$ such that

$$P = \big\{k \in \mathbb{N} :| T(x_k^{(q_0)}) - T(x_k) |< \frac{\delta}{3}\big\} \tag{2.11}$$

implies $P^c \in I$. The number q_0 can be chosen that together with (2.11), we have

$$Q = \{k \in \mathbb{N} :| a_{q_0} - a |< \frac{\delta}{3}\}$$

such that $Q^c \in I$. Since $\{k \in \mathbb{N} :| T(x_k^{(q_0)}) - a_{q_0} |\geq \delta\} \in I$. Then we have a subset S of $\mathbb{N}$ such that $S^c \in I$, where $S = \{k \in \mathbb{N} :| T(x_k^{(q_0)}) - a_{q_0} |< \frac{\delta}{3}\}$.
Let $U^c = P^c \cup Q^c \cup S^c$, where $U = \big\{k \in \mathbb{N} :| T(x_k) - a |< \delta\big\}$.
Therefore, for each $k \in U^c$, we have

$$\{k \in \mathbb{N} : | T(x_k) - a |< \delta\} \supseteq \Bigg[\{k \in \mathbb{N} :| T(x_k) - T(x_k^{(q_0)}) |< \frac{\delta}{3}\} \cap \{k \in \mathbb{N} :| T(x_k^{(q_0)}) - a_{q_0} |< \frac{\delta}{3}\} \cap \{k \in \mathbb{N} :| a_{q_0} - a |^{p_k} < \frac{\delta}{3}\}\Bigg].$$

Then the result follows. □

Since the inclusions $\mathcal{S}^I \subset \ell_\infty$ and is strict so in view of Theorem (2.9.) we have the following result.

Theorem 2.10. *The space $\mathcal{S}^I$ is nowhere dense subsets of ℓ_∞.*

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