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Entropy solutions for anisotropic nonlinear Dirichlet problems

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Abstract. We study in this paper nonlinear anisotropic problems with Dirichlet boundary value condition, L^1-data and variable exponent. We prove the existence and uniqueness of entropy solution under general conditions on the data.

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1. Introduction

We consider in this paper the following nonlinear anisotropic elliptic Dirichlet boundary value problem:

$$\begin{cases} b(u) - \displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i\left(x, \frac{\partial u}{\partial x_i}\right) = f & \text{in} \quad \Omega \\ \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where Ω is an open bounded domain of $\mathbb{R}^N$ $(N \geq 3)$, with smooth boundary, $b : \mathbb{R} \longrightarrow \mathbb{R}$ a continuous and non-decreasing function, with $b(0) = 0$ and $f \in L^1(\Omega)$.

All papers concerned by problems like (1) have considered particular cases of function b. Indeed, in [13], the authors considered $b \equiv 0$ which permit them to exploit minimization technics to prove the existence of weak solution and mini-max theory to prove that the weak solutions are multiple. Using their methods, Ouaro et als (see [11], [14]) studied the following problems:

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i\left(x, \frac{\partial u}{\partial x_i}\right) = f & \text{in} \quad \Omega \\ \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2}$$

where $f \in L^\infty(\Omega)$ (see [11]) and

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i\left(x, \frac{\partial u}{\partial x_i}\right) = f & \text{in} \quad \Omega \\ \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3}$$

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where $f \in L^1(\Omega)$ (see [14]).

In [11], the authors proved the existence and uniqueness of weak solution and in [14], Ouaro proved by using the results of [11], the existence and uniqueness of entropy solution.

In this paper, the function b is more general and the model contains both models studied in [11] and [14]. As b is general, it is not possible to use minimization tehnics used in [13] or [6] (see also [11], [14], [15], [16]) to get the existence of solution. Therefore, in this paper, we used the technic of monotone operators in Banach spaces (see [17]) to get the existence of entropy solutions of (1). For the uniqueness, since b is not necessarly invertible, then, we proved the uniqueness of the entropy solution in terms of $b(u)$ which is clearly equivalent to the uniqueness of u if and only if b is invertible.

The remaining part of the paper is the following: in Section 2, we introduce some preliminary results and in section 3, we study the existence and uniqueness of entropy solution.

2. Mathematical preliminaries

We study problem (1) under the following assumptions on the data.
Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary domain $\partial\Omega$ and $\vec{p}(.) = \big(p_1(.), \ldots, p_N(.)\big)$ such that for any $i = 1, \ldots, N$, $p_i(.) : \overline{\Omega} \longrightarrow \mathbb{R}$ is a continuous function with

$$1 < p_i^- := ess \inf_{x\in\Omega} p_i(x) \leq ess \sup_{x\in\Omega} p_i(x) := p_i^+ < \infty. \tag{4}$$

For any $i = 1, \ldots, N$, let $a_i : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be a Carathéodory function satisfying:
• there exists a positive constant C_1 such that

$$|a_i(x,\xi)| \leq C_1 \Bigg(j_i(x) + |\xi|^{p_i(x)-1} \Bigg) \tag{5}$$

for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}$, where j_i is a non-negative function in $L^{p_i'(.)}(\Omega)$, with $\frac{1}{p_i(x)} + \frac{1}{p_i'(x)} = 1$;
• for $\xi,\ \eta \in \mathbb{R}$ with $\xi \neq \eta$ and for almost every $x \in \Omega$, there exists a positive constant C_2 such that

$$\big(a_i(x,\xi) - a_i(x,\eta)\big)(\xi - \eta) \geq \begin{cases} C_2|\xi - \eta|^{p_i(x)} & \text{if } |\xi - \eta| \geq 1 \\ \\ C_2|\xi - \eta|^{p_i^-} & \text{if } |\xi - \eta| < 1 \end{cases} \tag{6}$$

and
• there exists a positive constant C_3 such that

$$a_i(x,\xi).\xi \geq C_3|\xi|^{p_i(x)}, \tag{7}$$

for $\xi \in \mathbb{R}$, for almost every $x \in \Omega$.
The hypotheses on a_i are classical in the study of nonlinear problems (see [5],[6]).
Throughout this paper, we assume that

$$\frac{\bar{p}(N-1)}{N(\bar{p}-1)} < p_i^- < \frac{\bar{p}(N-1)}{N-\bar{p}}, \qquad \frac{p_i^+ - p_i^- - 1}{p_i^-} < \frac{\bar{p}-N}{\bar{p}(N-1)}, \tag{8}$$

and
$$\sum_{i=1}^{N} \frac{1}{p_i^-} > 1, \tag{9}$$
where $\displaystyle\frac{N}{\bar{p}} = \sum_{i=1}^{N} \frac{1}{p_i^-}$.

A prototype example that is covered by our assumptions is the following anisotropic $\vec{p}(.)$-harmonic problem: Set
$$a_i(x,\xi) = |\xi|^{p_i(x)-2}\xi, \quad \text{where} \quad p_i(x) \geq 2 \quad \text{for} \quad \text{any} \quad i = 1, \dots, N.$$
Then, we obtain the problem
$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2} \frac{\partial u}{\partial x_i}\right) = f & \text{in} \quad \Omega \\ \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{10}$$
which, in the particular case when $p_i = p$ for any $i = 1, \dots, N$, is the p -Laplace equation.

We also recall in this section some definitions and basic properties of anisotropic Lebesgue and Sobolev spaces.
Set
$$C_+(\overline{\Omega}) = \left\{ p \in C(\overline{\Omega}) : \min_{x \in \overline{\Omega}} p(x) > 1 \quad a.e. \quad x \in \Omega \right\}$$
and we denotes by
$$p_M(x) := \max\big(p_1(x), \dots, p_N(x)\big) \quad \text{and} \quad p_m(x) := \min\big(p_1(x), \dots, p_N(x)\big).$$
For any $p \in C_+(\overline{\Omega})$, the variable exponent Lebesgue space is defined by
$$L^{p(.)}(\Omega) := \left\{ u : u \text{ is a measurable real valued function such that } \int_\Omega |u|^{p(x)} dx \ < \ \infty \right\},$$
endowed with the so-called Luxemburg norm
$$|u|_{p(.)} := \inf\left\{ \lambda > 0 : \ \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leq 1 \right\}.$$
The $p(.)$-*modular* of the $L^{p(.)}(\Omega)$ space is the mapping $\rho_{p(.)} : L^{p(.)}(\Omega) \longrightarrow \mathbb{R}$ defined by
$$\rho_{p(.)}(u) := \int_\Omega |u|^{p(x)} dx.$$
For any $u \in L^{p(.)}(\Omega)$, the following inequality (see [9], [10]) will be used later
$$\min\left\{ |u|_{p(.)}^{p^-};\ |u|_{p(.)}^{p^+} \right\} \leq \rho_{p(.)}(u) \leq \max\left\{ |u|_{p(.)}^{p^-};\ |u|_{p(.)}^{p^+} \right\}. \tag{11}$$
For any $u \in L^{p(.)}(\Omega)$ and $v \in L^{q(.)}(\Omega)$, with $\displaystyle\frac{1}{p(x)} + \frac{1}{q(x)} = 1$ in Ω, we have the Hölder type inequality
$$\left| \int_\Omega uv dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-} \right) |u|_{p(.)} |v|_{q(.)}. \tag{12}$$

If Ω is bounded and $p, q \in C_+(\overline{\Omega})$ such that $p(x) \leq q(x)$ for any $x \in \Omega$, then the embedding $L^{p(.)}(\Omega) \hookrightarrow L^{q(.)}(\Omega)$ is continuous (see [12], Theorem 2.8).

Herein we need the anisotropic Sobolev space

$$W_0^{1,\vec{p}(.)}(\Omega) := \left\{ u \in W_0^{1,1}(\Omega) \ : \ \frac{\partial u}{\partial x_i} \in L^{p_i(.)}(\Omega), \quad i = 1, \dots, N \right\},$$

which is a separable and reflexive Banach space (see [13]) under the norm

$$||u||_{\vec{p}(.)} = \sum_{i=1}^{N} \left| \frac{\partial u}{\partial x_i} \right|_{p_i(.)}.$$

We introduce the numbers

$$q = \frac{N(\bar{p}-1)}{N-1}; \qquad q^* = \frac{N(\bar{p}-1)}{N-\bar{p}} = \frac{Nq}{N-q},$$

and define $P_-^\star, \ P_-^+, \ P_{-,\infty} \ \in \mathbb{R}^+$ by

$$P_-^\star = \frac{N}{\displaystyle\sum_{i=1}^{N} \frac{1}{p_i^-} - 1}, \quad P_-^+ = \max\{p_1^-, \dots, p_N^-\} \quad \text{and} \quad P_{-,\infty} = \max\{P_-^+, \ P_-^\star\}.$$

We have the following embedding results (see [13], Theorem 1).

Theorem 2.1. *Assume that $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary. Assume also that relation (9) is fulfilled. For any $q \in C(\overline{\Omega})$ verifying*

$$1 < q(x) < P_{-,\infty} \quad \text{for any } x \in \overline{\Omega},$$

the embedding

$$W_0^{1,\vec{p}(.)}(\Omega) \hookrightarrow L^{q(.)}(\Omega)$$

is continuous and compact.

The following result is due to Troisi (see [19]).

Theorem 2.2. *Let $p_1, \dots, p_N \in [1, +\infty)$; $g \in W^{1,(p_1,\dots,p_N)}(\Omega)$ and*

$$q = \begin{cases} (\bar{p})^* & if \quad (\bar{p})^* < N \\ \in [1, +\infty) & if \quad (\bar{p})^* \geq N. \end{cases}$$

Then, there exists a constant $C_4 > 0$ depending on $N, p_1, \dots, p_N$ if $\bar{p} < N$ and also on q and $mes(\Omega)$ if $\bar{p} \geq N$ such that

$$||g||_{L^q(\Omega)} \leq C_4 \prod_{i=1}^{N} \left|\left| \frac{\partial g}{\partial x_i} \right|\right|_{L^{p_i}(\Omega)}^{1/N}. \tag{13}$$

In this paper, we will use the Marcinkiewicz space $\mathcal{M}^q(\Omega)$ $(1 < q < +\infty)$ as the set of measurable functions $g : \Omega \longrightarrow \mathbb{R}$ for which the distribution function

$$\lambda_g(k) = mes\big(\{x \in \Omega : \ |g(x)| > k\}\big), \quad k \geq 0 \tag{14}$$

satisfies an estimate of the form

$$\lambda_g(k) \leq Ck^{-q}, \text{ for some finite constant } \ C > 0. \tag{15}$$

We will use the following pseudo norm in $\mathcal{M}^q(\Omega)$:

$$||g||_{\mathcal{M}^q(\Omega)} := \inf\{C > 0 : \lambda_g(k) \leq Ck^{-q}, \ \ \forall \ k > 0\}. \tag{16}$$

Finally, we use through the paper, the truncation function T_k, $(k > 0)$, by

$$T_k(s) = \max\{-k; \min\{k; s\}\}. \tag{17}$$

It is clear that $\lim_{k\to\infty} T_k(s) = s$ and $|T_k(s)| = \min\{|s|; k\}$.

We define $\mathcal{T}_0^{1,\vec{p}(.)}(\Omega)$ as the set of the measurable functions $u : \Omega \longrightarrow \mathbb{R}$ such that $T_k(u) \in W_0^{1,\vec{p}(.)}(\Omega)$.

In the sequel we denote $W_0^{1,\vec{p}(.)}(\Omega) = E$ to simplify.

3. Existence and uniqueness result

Definition 3.1. *A measurable function $u \in \mathcal{T}_0^{1,\vec{p}(.)}(\Omega)$ is an entropy solution of (P) if $b(u) \in L^1(\Omega)$ and*

$$\sum_{i=1}^{N} \int_\Omega a_i\left(x, \frac{\partial u}{\partial x_i}\right) \frac{\partial}{\partial x_i} T_k(u-\varphi)dx + \int_\Omega b(u) T_k(u-\varphi)dx \leq \int_\Omega f(x) T_k(u-\varphi)dx, \tag{18}$$

for all $\varphi \in E \cap L^\infty(\Omega)$ and for every $k > 0$.

The existence result is the following theorem:

Theorem 3.1. *Assume (4)-(9). Then, there exists at least one entropy solution of the problem (P).*

Proof. The proof is done in tree steps.

Step 1. The approximate problem.
We consider the approximate problem

$$(P_n) \begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i\left(x, \frac{\partial u_n}{\partial x_i}\right) + T_n\big(b(u_n)\big) = f_n & \text{in} \quad \Omega \\ \\ u_n = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{19}$$

where $f_n = T_n(f) \in L^\infty(\Omega)$. Note that

$$f_n \underset{n\to+\infty}{\longrightarrow} f \quad \text{in} \quad L^1(\Omega) \quad \text{and} \quad ||f_n||_1 = \int_\Omega |f_n| dx \leq \int_\Omega |f| dx = ||f||_1. \tag{20}$$

Definition 3.2. *A measurable function $u_n \in E$ is a weak solution for the problem (P_n) if*

$$\sum_{i=1}^{N} \int_\Omega a_i\left(x, \frac{\partial u_n}{\partial x_i}\right) \frac{\partial \varphi}{\partial x_i} dx + \int_\Omega T_n\big(b(u_n)\big)\varphi dx = \int_\Omega f_n \varphi dx, \tag{21}$$

for every $\varphi \in E$.

Let us prove the following lemma.

Lemma 3.1. *There exists at least one weak solution u_n for the problem (P_n).*

Proof. We define the operator A_n as follow:

$$\langle A_n u, \varphi\rangle = \langle Au, \varphi\rangle + \int_\Omega T_n\big(b(u)\big)\varphi dx \quad \forall\ u, \varphi \in E \tag{22}$$

where

$$\langle Au,\varphi\rangle=\int_\Omega\sum_{i=1}^N a_i\Big(x,\frac{\partial u}{\partial x_i}\Big)\frac{\partial\varphi}{\partial x_i}dx. \tag{23}$$

Assertion 1. *The operator A_n is type M.*
• *The operator A is monotone.* Indeed, for $u,v\in E$, we have

$$\begin{aligned}\langle A(u)-A(v),u-v\rangle&=\langle A(u),u-v\rangle+\langle A(v),v-u\rangle\\&=\int_\Omega\sum_{i=1}^N a_i\Big(x,\frac{\partial u}{\partial x_i}\Big)\frac{\partial(u-v)}{\partial x_i}dx+\int_\Omega\sum_{i=1}^N a_i\Big(x,\frac{\partial v}{\partial x_i}\Big)\frac{\partial(v-u)}{\partial x_i}dx\\&=\int_\Omega\sum_{i=1}^N\Big[a_i\Big(x,\frac{\partial u}{\partial x_i}\Big)-a_i\Big(x,\frac{\partial v}{\partial x_i}\Big)\Big]\Big(\frac{\partial u}{\partial x_i}-\frac{\partial v}{\partial x_i}\Big)dx\end{aligned}$$

then

$$\langle A(u)-A(v),u-v\rangle\geq 0, \tag{24}$$

since for $i=1,\dots,N,$ for almost every $x\in\Omega$, $a_i(x,.)$ is monotone.

• *The operator A is hemicontinuous.* Indeed, let $\varphi: t\in\mathbb{R}\longmapsto\varphi(t)=\langle A(u+tv),v\rangle$ and let $t,\ t_0\in\mathbb{R}$ such that $t\longrightarrow t_0$. Put $w=u+tv\in E$ and $w_0=u+t_0v\in E$.

Therefore $||w-w_0||_{\vec{p}(.)}=||(t-t_0)v||_{\vec{p}(.)}=|t-t_0|.||v||_{\vec{p}(.)}\longrightarrow 0$, i.e $w\longrightarrow w_0$ in E.

We have

$$\begin{aligned}|\varphi(t)-\varphi(t_0)|&=|\langle A(u+tv),v\rangle-\langle A(u+t_0v),v\rangle|\\&\leq\sum_{i=1}^N\int_\Omega\Big|a_i\Big(x,\frac{\partial w}{\partial x_i}\Big)-a_i\Big(x,\frac{\partial w_0}{\partial x_i}\Big)\Big|\Big|\frac{\partial v}{\partial x_i}\Big|dx\\&\leq N\max_{1\leq i\leq N}\Big[\Big(\frac{1}{p_i^-}+\frac{1}{(p_i')^-}\Big)\Big|a_i\Big(x,\frac{\partial w}{\partial x_i}\Big)-a_i\Big(x,\frac{\partial w_0}{\partial x_i}\Big)\Big|_{p_i'(.)}\Big|\frac{\partial v}{\partial x_i}\Big|_{p_i(.)}\Big].\end{aligned}$$

Denote by $\psi_i(x,w)=a_i\big(x,\frac{\partial w}{\partial x_i}\big)$. Since $\psi_i(x,w)\longrightarrow\psi(x,w_0)$ in $L^{p_i'(.)}(\Omega)$ (see [11]) we deduce that φ is continuous. Then A is hemicontinuous.

Since the operator A is monotone and hemicontinuous, then according to the Lemma 2.1 in [17], A is of type M. Therefore, according to [1] the operator A_n is also of type M.

Assertion 2. *The operator A_n is coercive.*
Indeed, let $u\in E$. We have $T_n(b(u))u\geq 0$ for all $u\in E$.
Then

$$\begin{aligned}\langle A_n(u),u\rangle&\geq\int_\Omega\sum_{i=1}^N a_i\Big(x,\frac{\partial u}{\partial x_i}\Big)\frac{\partial u}{\partial x_i}dx\\&\geq C_3\int_\Omega\sum_{i=1}^N\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i(x)}dx.\end{aligned}$$

Denote

$$\mathcal{I}=\Big\{i\in\{1,\dots,N\}:\Big|\frac{\partial u}{\partial x_i}\Big|_{p_i(.)}\leq 1\Big\}\quad\text{and}\quad\mathcal{J}=\Big\{i\in\{1,\dots,N\}:\Big|\frac{\partial u}{\partial x_i}\Big|_{p_i(.)}>1\Big\}.$$

Then

$$\frac{1}{C_3}\langle A_n(u),u\rangle\geq\sum_{i\in\mathcal{I}}\int_\Omega\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i(x)}dx+\sum_{i\in\mathcal{J}}\int_\Omega\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i(x)}dx$$

$$
\begin{aligned}
&\geq \sum_{i\in\mathcal{I}}\left|\frac{\partial u}{\partial x_i}\right|_{p_i(.)}^{p_i^+}+\sum_{i\in\mathcal{J}}\left|\frac{\partial u}{\partial x_i}\right|_{p_i(.)}^{p_i^-}\\
&\geq \sum_{i\in\mathcal{J}}\left|\frac{\partial u}{\partial x_i}\right|_{p_i(.)}^{p_i^-}\\
&\geq \sum_{i\in\mathcal{J}}\left|\frac{\partial u}{\partial x_i}\right|_{p_i(.)}^{p_m^-}\\
&\geq \sum_{i=1}^{N}\left|\frac{\partial u}{\partial x_i}\right|_{p_i(.)}^{p_m^-}-\sum_{i\in\mathcal{I}}\left|\frac{\partial u}{\partial x_i}\right|_{p_i(.)}^{p_m^-}\\
&\geq \sum_{i=1}^{N}\left|\frac{\partial u}{\partial x_i}\right|_{p_i(.)}^{p_m^-}-N.
\end{aligned}
$$

Using the convexity of application $t\in\mathbb{R}^+\longmapsto t^{p_m^-}$, $p_m^->1$, we obtain

$$
\frac{1}{C_3}\langle A_n(u),u\rangle \geq \frac{1}{N^{p_m^- -1}}\left(\sum_{i=1}^{N}\left|\frac{\partial u}{\partial x_i}\right|_{p_i(.)}\right)^{p_m^-}-N.
$$

Then

$$
\langle A_n(u),u\rangle \geq \frac{C_3}{N^{p_m^- -1}}||u||_{\vec{p}(.)}^{p_m^-}-C_3N. \tag{25}
$$

Consequently the operator A_n is coercive.

Assertion 3. *The operator A_n is bounded.*
Indeed, let $u\in B\subset E$ a bounded space and $v\in E$. According to (5) and (12) and as b is onto, we have

$$
\begin{aligned}
|\langle A_n(u),v\rangle| &\leq \sum_{i=1}^{N}\int_\Omega\left|a_i\left(x,\frac{\partial u}{\partial x_i}\right)\right|\left|\frac{\partial v}{\partial x_i}\right|dx+\int_\Omega|T_n(b(u))v|dx\\
&\leq C_1\sum_{i=1}^{N}\left(\int_\Omega j_i(x)\left|\frac{\partial v}{\partial x_i}\right|dx+\int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-1}\left|\frac{\partial v}{\partial x_i}\right|dx\right)+\int_\Omega|b(u)||v|dx\\
&\leq C_1\sum_{i=1}^{N}\left(\frac{1}{p_i^-}+\frac{1}{(p_i')^-}\right)\left|\frac{\partial v}{\partial x_i}\right|_{p_i(.)}\left(|j_i|_{p_i'(.)}+\left||\frac{\partial u}{\partial x_i}|^{p_i(x)-1}\right|_{p_i'(.)}\right)+C\int_\Omega|v|dx
\end{aligned}
$$

where C is a positive constant.
Then the operator A_n is bounded.

The operator A_n is type M, bounded and coercive on E to its dual E^*, then A_n is surjective (see [17], Corollary 2.2). Therefore for $f_n\in E^*$, we can deduce the existence of a function $u_n\in E$ such that $\langle A_nu_n,\varphi\rangle=\langle f_n,\varphi\rangle$ for all $\varphi\in E$, namely

$$
\sum_{i=1}^{N}\int_\Omega a_i\left(x,\frac{\partial u_n}{\partial x_i}\right)\frac{\partial\varphi}{\partial x_i}dx+\int_\Omega T_n\big(b(u_n)\big)\varphi dx=\int_\Omega f_n\varphi dx \ \text{ for all } \ \varphi\in E. \qquad \square
$$

Our aim is to prove that these approximated solutions u_n tend, as n goes to infinity, to a measurable function u which is an entropy solution of the problem (P). To start with, we establish the priori estimates.

Step 2. A priori estimates.

Lemma 3.2. *There exists a positive constant C_5 which does not depends on n such that*

$$\sum_{i=1}^{N}\int_{\{|u_n|\leq k\}}\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i^-}dx \leq C_5(1+k) \tag{26}$$

for every $k>0$.

Proof. If we take $\varphi = T_k(u_n)$ as test function in (21) we obtain

$$\sum_{i=1}^{N}\int_{\{|u_n|\leq k\}} a_i\left(x,\frac{\partial u_n}{\partial x_i}\right)\frac{\partial u_n}{\partial x_i}dx + \int_\Omega T_n(b(u_n))T_k(u_n)dx = \int_\Omega f_n T_k(u_n)dx$$

and using relation (7) and the fact that $\displaystyle\int_\Omega T_n(b(u_n))T_k(u_n)dx \geq 0$, then

$$C_3\sum_{i=1}^{N}\int_{\{|u_n|\leq k\}}\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)}dx \leq k||f||_1. \tag{27}$$

We have

$$\begin{aligned}
\sum_{i=1}^{N}\int_{\{|u_n|\leq k\}}\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i^-}dx &= \sum_{i=1}^{N}\int_{\{|u_n|\leq k;\ |\frac{\partial u_n}{\partial x_i}|>1\}}\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i^-}dx + \sum_{i=1}^{N}\int_{\{|u_n|\leq k;\ |\frac{\partial u_n}{\partial x_i}|\leq 1\}}\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i^-}dx\\
&\leq \sum_{i=1}^{N}\int_{\{|u_n|\leq k\}}\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)}dx + N.mes(\Omega)\\
&\leq \frac{1}{C_3}k||f||_1 + N.mes(\Omega) \quad \text{due to relation} \quad (27)\\
&\leq C_5(1+k) \quad \text{with} \quad C_5 = \max\left\{\frac{1}{C_3}||f||_1;\ N.mes(\Omega)\right\}.
\end{aligned}$$

□

Lemma 3.3. *For any $k>0$, there exists some constants $C_6, C_7 > 0$ such that:*

(*i*) $||u_n||_{\mathcal{M}^{q^*}(\Omega)} \leq C_6$;

(*ii*) $\left|\left|\dfrac{\partial u_n}{\partial x_i}\right|\right|_{\mathcal{M}^{p_i^- q/\bar{p}}(\Omega)} \leq C_7, \qquad \forall\ i=1,\dots,N.$

Proof.

(*i*) According to Lemma 3.2 we have

$$\int_{\{|u_n|\leq k\}}\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i^-}dx \leq C_5(1+k) \ \ \forall\ k>0, \ \ \forall\ i=1,\dots,N.$$

• If $k \geq 1$, we have

$$\int_{\{|u_n|\leq k\}}\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i^-}dx \leq C_5'k$$

thus

$$\int_\Omega\left|\frac{\partial}{\partial x_i}T_k(u_n)\right|^{p_i^-}dx \leq C_5'k \quad \text{ i.e } \quad T_k(u_n)\in W_0^{1,(p_1^-,\dots,p_N^-)}(\Omega).$$

Using the Theorem 2.2, we deduce

$$||T_k(u_n)||_{L^{(\bar{p})^*}(\Omega)} \leq C_5''\prod_{i=1}^{N}\left|\left|\frac{\partial}{\partial x_i}T_k(u_n)\right|\right|_{L^{p_i^-}(\Omega)}^{1/N}.$$

Then

$$\int_\Omega \big|T_k(u_n)\big|^{(\bar p)^*}dx \le C_5'''\Big[\prod_{i=1}^N\Big(\int_\Omega \big|\frac{\partial}{\partial x_i}T_k(u_n)\big|^{p_i^-}dx\Big)^{\frac{1}{Np_i^-}}\Big]^{(\bar p)^*}$$

$$\le C\Big[\prod_{i=1}^N k^{\frac{1}{Np_i^-}}\Big]^{(\bar p)^*}$$

$$= C\Big[k^{\sum_{i=1}^N \frac{1}{Np_i^-}}\Big]^{(\bar p)^*} = Ck^{\frac{(\bar p)^*}{\bar p}}.$$

We have

$$\int_{\{|u_n|>k\}}\big|T_k(u_n)\big|^{(\bar p)^*}dx \le \int_\Omega \big|T_k(u_n)\big|^{(\bar p)^*}dx.$$

Therefore

$$k^{(\bar p)^*}mes(\{x\in\Omega : |u_n|>k\}) \le \int_\Omega \big|T_k(u_n)\big|^{(\bar p)^*}dx \le Ck^{\frac{(\bar p)^*}{\bar p}}.$$

Thus,

$$\lambda_{u_n}(k) \le Ck^{(\bar p)^*(\frac{1}{\bar p}-1)} = Ck^{-q^*}, \quad \forall\ k\ge 1.$$

• If $k<1$, we have

$$\lambda_{u_n}(k) = mes(\{x\in\Omega : |u_n|>k\}) \le mes(\Omega) \le mes(\Omega)k^{-q^*}, \ \ \forall\ k<1.$$

Consequently

$$\lambda_{u_n}(k) \le (C+mes(\Omega))k^{-q^*} = C_6k^{-q^*};$$

namely

$$||u_n||_{\mathcal{M}^{q^*}(\Omega)} \le C_6.$$

(ii) • Let $\alpha\ge 1$.

We have, $\forall\ k\ge 1$

$$\lambda_{\frac{\partial u_n}{\partial x_i}}(\alpha) = mes(\Big\{x\in\Omega : \Big|\frac{\partial u_n}{\partial x_i}\Big|>\alpha\Big\})$$

$$= mes(\Big\{\Big|\frac{\partial u_n}{\partial x_i}\Big|>\alpha; |u_n|\le k\Big\}) + mes(\Big\{\Big|\frac{\partial u_n}{\partial x_i}\Big|>\alpha; |u_n|>k\Big\})$$

$$\le \int_{\big\{|\frac{\partial u_n}{\partial x_i}|>\alpha;\ |u_n|\le k\big\}}dx + \lambda_{u_n}(k)$$

$$\le \int_{\{|u_n|\le k\}}\Big(\frac{1}{\alpha}\Big|\frac{\partial u_n}{\partial x_i}\Big|\Big)^{p_i^-}dx + \lambda_{u_n}(k)$$

$$\le \alpha^{-p_i^-}C'k + Ck^{-q^*}.$$

We obtain

$$\lambda_{\frac{\partial u_n}{\partial x_i}}(\alpha) \le B\big(k\alpha^{-p_i^-} + k^{-q^*}\big). \tag{28}$$

Let us consider the function $g : [1,+\infty[\longrightarrow \mathbb{R},\ x\longmapsto g(x) = \dfrac{x}{\alpha^{p_i^-}} + x^{-q^*}$.

We have $g'(x)=0$ for $x = \big(q^*\alpha^{p_i^-}\big)^{\frac{1}{q^*+1}}$.

If we take $k = \big(q^*\alpha^{p_i^-}\big)^{\frac{1}{q^*+1}} \ge 1$ in (28) we get

$$\lambda_{\frac{\partial u_n}{\partial x_i}}(\alpha) \le Bk\Big(\frac{q^*+1}{q^*}\frac{1}{\alpha^{p_i^-}}\Big)$$

$$\leq M\alpha^{-\frac{q^*}{q^*+1}p_i^-}$$

$$\leq M\alpha^{-p_i^- q/\bar{p}} \quad \forall\ \alpha \geq 1.$$

• If $0 \leq \alpha < 1$, we have

$$\lambda_{\frac{\partial u_n}{\partial x_i}}(\alpha) = mes(\left\{\left|\frac{\partial u_n}{\partial x_i}\right| > \alpha\right\}) \leq mes(\Omega) \leq mes(\Omega)\alpha^{-p_i^- q/\bar{p}}, \quad \forall\ 0 \leq \alpha < 1.$$

Then

$$\lambda_{\frac{\partial u_n}{\partial x_i}}(\alpha) \leq (M + mes(\Omega))\alpha^{-p_i^- q/\bar{p}} \quad \forall\ \alpha \geq 0$$

and we deduce

$$\left|\left|\frac{\partial u_n}{\partial x_i}\right|\right|_{\mathcal{M}^{p_i^- q/\bar{p}}(\Omega)} \leq C_7, \qquad \forall\ i = 1, \ldots, N. \qquad \square$$

Step 3. Existence of entropy solution

Using Lemma 3.3, we have the following useful lemma (see [5]).

Lemma 3.4. *For $i = 1, \ldots, N$, as $n \longrightarrow +\infty$, we have*

$$a_i\left(x, \frac{\partial u_n}{\partial x_i}\right) \longrightarrow a_i\left(x, \frac{\partial u}{\partial x_i}\right) \quad in \quad L^1(\Omega) \quad a.e \quad x \in \Omega. \tag{29}$$

In order to pass to the limit in relation (21), we need also the following convergence results as in [4]:

Proposition 3.1. *Assume (4)-(9). If $u_n \in E$ is a weak solution of (P_n) then the sequence $(u_n)_{n\in\mathbb{N}^*}$ is Cauchy in measure. In particular, there exists a measurable function u and a sub-sequence still denoted by u_n such that $u_n \longrightarrow u$ in measure.*

Proposition 3.2. *Assume (4)-(9). If $u_n \in E$ is a weak solution of (P_n) then*

(*i*) *for all $i = 1, \ldots, N$, $\dfrac{\partial u_n}{\partial x_i}$ converges in measure to the weak partial gradient of u;*

(*ii*) *for all $i = 1, \ldots, N$ and $k > 0$, $a_i\left(x, \dfrac{\partial}{\partial x_i}T_k(u_n)\right)$ converges to $a_i\left(x, \dfrac{\partial}{\partial x_i}T_k(u)\right)$ in $L^1(\Omega)$ strongly and in $L^{p_i'(\cdot)}(\Omega)$ weakly.*

We can now pass to the limit in relation (21).
Let $\varphi \in W_0^{1,\vec{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$ and choosing $T_k(u_n - \varphi)$ as test function in (21), we get

$$\begin{cases} \displaystyle\sum_{i=1}^{N} \int_\Omega a_i\left(x, \frac{\partial u_n}{\partial x_i}\right) \frac{\partial}{\partial x_i} T_k(u_n - \varphi) dx \\ \displaystyle + \int_\Omega T_n(b(u_n)) T_k(u_n - \varphi) dx = \int_\Omega f_n T_k(u_n - \varphi) dx. \end{cases} \tag{30}$$

For the right-hand side of (30) we have

$$\int_\Omega f_n(x) T_k(u_n - \varphi) dx \longrightarrow \int_\Omega f(x) T_k(u - \varphi) dx. \tag{31}$$

since f_n converges strongly to f in $L^1(\Omega)$, and $T_k(u_n - \varphi)$ converges weakly-* to $T_k(u - \varphi)$ in $L^\infty(\Omega)$ and a.e in Ω.

For the first term of (30) we have (see [5]):

$$\liminf_n \sum_{i=1}^{N}\int_\Omega a_i\left(x, \frac{\partial u_n}{\partial x_i}\right)\frac{\partial}{\partial x_i}T_k(u_n-\varphi)dx \geq \sum_{i=1}^{N}\int_\Omega a_i\left(x, \frac{\partial u}{\partial x_i}\right)\frac{\partial}{\partial x_i}T_k(u-\varphi)dx. \quad (32)$$

Finally we focus our attention on the second term of (30).
We have

$$T_n(b(u_n))T_k(u_n-\varphi) \longrightarrow b(u)T_k(u-\varphi) \ \text{ a.e. } \ x\in\Omega \quad (33)$$

and

$$|T_n(b(u_n))T_k(u_n-\varphi)| \leq k|b(u_n)|. \quad (34)$$

We show that $|b(u_n)| \leq ||f_n||_\infty$. Indeed, let us denote by

$$H_\epsilon(s) = \min\left(\frac{s^+}{\epsilon};1\right) \quad \text{and} \quad sign_0^+(s) = \begin{cases} 1 & \text{if } \ s>0 \\ 0 & \text{if } \ s\leq 0. \end{cases}$$

and if γ is a maximal monotone operator defined on $\mathbb{R}$, we denote by γ_0 the main section of γ; i.e.,

$$\gamma_0(s) = \begin{cases} \text{minimal absolute value of } \gamma(s) & \text{if } \ \gamma(s)\neq\emptyset \\ +\infty & \text{if } \ [s,+\infty)\cap D(\gamma)=\emptyset \\ -\infty & \text{if } \ (-\infty,s]\cap D(\gamma)=\emptyset. \end{cases}$$

Remark that if ϵ approach 0, $H_\epsilon(s) = sign_0^+(s)$.
We take $\varphi = H_\epsilon(u_n - M)$ as test function in (21), for the weak solution u_n and $M>0$ (a constant to be chosen later), to get

$$\begin{cases} \displaystyle\sum_{i=1}^{N}\int_\Omega a_i\left(x, \frac{\partial u_n}{\partial x_i}\right)\frac{\partial}{\partial x_i}H_\epsilon(u_n-M)dx \\ \displaystyle + \int_\Omega T_n\big(b(u_n)\big)H_\epsilon(u_n-M)dx = \int_\Omega f_n H_\epsilon(u_n-M)dx. \end{cases} \quad (35)$$

We have

$$\begin{aligned} \sum_{i=1}^{N}\int_\Omega a_i\left(x, \frac{\partial u_n}{\partial x_i}\right)\frac{\partial}{\partial x_i}H_\epsilon(u_n-M)dx &= \frac{1}{\epsilon}\sum_{i=1}^{N}\int_{\left\{\frac{(u_n-M)^+}{\epsilon}<1\right\}} a_i\left(x, \frac{\partial u_n}{\partial x_i}\right)\frac{\partial}{\partial x_i}(u_n-M)^+dx \\ &= \frac{1}{\epsilon}\sum_{i=1}^{N}\int_{\left\{0<u_n-M<\epsilon\right\}} a_i\left(x, \frac{\partial u_n}{\partial x_i}\right)\frac{\partial}{\partial x_i}u_n dx \\ &\geq 0 \ \text{ according to } \ (7). \end{aligned}$$

Then, (35) give

$$\int_\Omega T_n\big(b(u_n)\big)H_\epsilon(u_n-M)dx \leq \int_\Omega f_n H_\epsilon(u_n-M)dx,$$

which is equivalent to saying

$$\int_\Omega \Big(T_n\big(b(u_n)\big) - T_n\big(b(M)\big)\Big)H_\epsilon(u_n-M)dx \leq \int_\Omega \Big(f_n - T_n\big(b(M)\big)\Big)H_\epsilon(u_n-M)dx.$$

We now let ϵ goes to 0 in the above inequality to obtain

$$\int_\Omega \Big(T_n\big(b(u_n)\big) - T_n\big(b(M)\big)\Big)^+ dx \leq \int_\Omega \Big(f_n - T_n\big(b(M)\big)\Big)sign_0^+(u_n-M)dx. \quad (36)$$

Choosing $M = b_0^{-1}\big(||f_n||_\infty\big)$ in the above inequality (since b is surjective). We obtain

$$\int_\Omega \Big(T_n\big(b(u_n)\big) - T_n\big(||f_n||_\infty\big)\Big)^+ dx \le \int_\Omega \Big(f_n - T_n\big(||f_n||_\infty\big)\Big) sign_0^+\big(u_n - b_0^{-1}(||f_n||_\infty)\big)dx. \tag{37}$$

For any $n > ||f_n||_\infty$, we have

$$\int_\Omega \Big(f_n - T_n\big(||f_n||_\infty\big)\Big) sign_0^+\big(u_n - b_0^{-1}(||f_n||_\infty)\big)dx$$
$$= \int_\Omega \Big(f_n - ||f_n||_\infty\Big) sign_0^+\big(u_n - b_0^{-1}(||f_n||_\infty)\big)dx \le 0.$$

Then, (37) gives

$$\int_\Omega \Big(T_n\big(b(u_n)\big) - ||f_n||_\infty\Big)^+ dx \le 0.$$

Hence, for all $n > ||f_n||_\infty$, we have $\Big(T_n\big(b(u_n)\big) - ||f_n||_\infty\Big)^+ = 0$ a.e in Ω, which is equivalent to saying

$$T_n\big(b(u_n)\big) \le ||f_n||_\infty \ \text{ for all } \ n > ||f_n||_\infty. \tag{38}$$

Let us remark that as u_n is a weak solution of (21), then $(-u_n)$ is a weak solution to the following problem

$$(\tilde{P}_n)\begin{cases} -\displaystyle\sum_{i=1}^N \frac{\partial}{\partial x_i}\tilde{a}_i\Big(x, \frac{\partial u_n}{\partial x_i}\Big) + T_n\big(\tilde{b}(u_n)\big) = \tilde{f}_n & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega \end{cases} \tag{39}$$

where $\tilde{a}_i(x,\xi) = -a_i(x,-\xi)$, $\tilde{b}(s) = -b(-s)$ and $\tilde{f}_n = -f_n$.
According to (38) we deduce that

$$T_n\big(-b(u_n)\big) \le ||f_n||_\infty \ \text{ for all } \ n > ||f||_\infty.$$

Therefore

$$T_n\big(b(u_n)\big) \ge -||f_n||_\infty \ \text{ for all } \ n > ||f||_\infty. \tag{40}$$

It follows from (38) and (40) that for all $n > ||f_n||_\infty$, $\big|T_n\big(b(u_n)\big)\big| \le ||f_n||_\infty$ witch implies

$$|b(u_n)| \le ||f_n||_\infty \ \text{ a.e. in } \ \Omega.$$ □

We can now use the Lebesgue dominated convergence theorem to get

$$\lim_{n\to+\infty}\int_\Omega T_n(b(u_n))T_k(u_n - \varphi)dx = \int_\Omega b(u)T_k(u-\varphi)dx. \tag{41}$$

Combining (31), (32) and (41), we obtain

$$\sum_{i=1}^N \int_\Omega a_i\Big(x, \frac{\partial u}{\partial x_i}\Big)\frac{\partial}{\partial x_i}T_k(u-\varphi)dx + \int_\Omega b(u)T_k(u-\varphi)dx \le \int_\Omega f(x)T_k(u-\varphi)dx. \tag{42}$$

Then u is an entropy solution of (P).

■

Theorem 3.2. *Assume that (4)-(9) hold true and let u be an entropy solution of (P). Then, u is unique.*

Proof. The proof is done in two steps.

Step 1. A priori estimates

Lemma 3.5. *Assume (4)-(9) holds and $f \in L^1(\Omega)$. Let u be an entropy solution of (P). Then*

$$\sum_{i=1}^{N} \int_{\{|u|\leq k\}} \left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} dx \leq \frac{C_3}{k}||f||_1 \tag{43}$$

and there exists a positive constant C_8 such that

$$||b(u)||_1 \leq C_8.mes(\Omega) + ||f||_1. \tag{44}$$

Proof. Let us take $\varphi = 0$ in the entropy inequality (18).

• By the fact that $\int_\Omega b(u)T_k(u)dx \geq 0$ and using the relation (7), we get (43).

• Using the fact that $\sum_{i=1}^{N} \int_\Omega a_i\big(x, \frac{\partial u}{\partial x_i}\big)\frac{\partial}{\partial x_i}T_k(u)dx \geq 0$, relation (18) gives

$$\int_\Omega b(u)T_k(u)dx \leq \int_\Omega f(x)T_k(u)dx. \tag{45}$$

By (45), we deduce that

$$\int_{\{|u|\leq k\}} b(u)T_k(u)dx + \int_{\{|u|>k\}} b(u)T_k(u)dx \leq k||f||_1$$

which imply that

$$\int_{\{|u|>k\}} b(u)T_k(u)dx \leq k||f||_1$$

or

$$\int_{\{u>k\}} b(u)dx + \int_{\{u<-k\}} -b(u)dx \leq ||f||_1.$$

Therefore

$$\int_{\{|u|>k\}} |b(u)|dx \leq ||f||_1.$$

So, we obtain

$$\begin{aligned}\int_\Omega |b(u)|dx &= \int_{\{|u|\leq k\}} |b(u)|dx + \int_{\{|u|>k\}} |b(u)|dx \\ &\leq \int_{\{|u|\leq k\}} |b(u)|dx + ||f||_1.\end{aligned}$$

Since the function b is non-decreasing, then

$$\int_{\{|u|\leq k\}} |b(u)|dx \leq \max\{b(k); |b(-k)|\}.mes(\Omega).$$

Consequently, there exists a constant $C_8 = \max\{b(k); |b(-k)|\}$ such that

$$||b(u)||_1 \leq C_8.mes(\Omega) + ||f||_1.$$ □

Lemma 3.6. *Assume (4)-(9) holds and let $f \in L^1(\Omega)$. If u is an entropy solution of (P), then there exists a constant D which depends on f and Ω such that*

$$mes\{|u| > k\} \leq \frac{D}{\min(b(k), |b(-k)|)}, \quad \forall\ k > 0 \tag{46}$$

and a constant $D' > 0$ which depends on f and Ω such that

$$mes\left\{\left|\frac{\partial u}{\partial x_i}\right| > k\right\} \leq \frac{D'}{k^{\frac{1}{(p_M^-)'}}}, \quad \forall\ k \geq 1. \tag{47}$$

Proof. • For any $k > 0$, the relation (44) gives

$$\int_{\{|u|>k\}} \min(b(k), |b(-k)|)dx \leq \int_{\{|u|>k\}} |b(u)|dx \leq C_8.mes(\Omega) + ||f||_1.$$

Therefore,

$$\min(b(k), |b(-k)|).mes\{|u| > k\} \leq C_8.mes(\Omega) + ||f||_1 = D;$$

that is

$$mes\{|u| > k\} \leq \frac{D}{\min(b(k), |b(-k)|)}.$$

• See [4] for the proof of (47). □

Lemma 3.7. *Assume (4)-(9) holds and let $f \in L^1(\Omega)$. If u is an entropy solution of (P), then*

$$\lim_{h\longrightarrow+\infty} \int_\Omega |f|\chi_{\{|u|>h-t\}}dx = 0, \tag{48}$$

where $h > 0$ and $t > 0$.

Proof. Since the function b is surjective, according to Lemma 3.6-(46), we have $\lim\limits_{h\longrightarrow+\infty} mes\{|u| > h - t\} = 0$ and as $f \in L^1(\Omega)$, it follows by using the Lebesgue dominated convergence theorem that $\lim\limits_{h\longrightarrow+\infty} \int_\Omega |f|\chi_{\{|u|>h-t\}}dx = 0.$ □

Lemma 3.8. *Assume (4)-(9) holds and let $f \in L^1(\Omega)$. If u is an entropy solution of (P), then there exists a positive constant K such that*

$$\rho_{p_i'(.)}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-1}\chi_F\right) \leq K, \quad \forall\ i = 1, \dots, N \tag{49}$$

where $F = \{h < |u| \leq h + k\}$, $h > 0$, $k > 0$.

Proof. Let $\varphi = T_h(u)$ as test function in the entropy inequality (18). We get

$$\sum_{i=1}^N \int_\Omega a_i\left(x, \frac{\partial u}{\partial x_i}\right) \frac{\partial}{\partial x_i} T_k(u-T_h(u))dx + \int_\Omega b(u)T_k(u-T_h(u))dx \leq \int_\Omega f(x)T_k(u-T_h(u))dx.$$

Thus,

$$\sum_{i=1}^N \int_{\{h<|u|\leq h+k\}} a_i\left(x, \frac{\partial u}{\partial x_i}\right) \frac{\partial u}{\partial x_i}dx \leq k||f||_1$$

and using (7), we have

$$\int_F \left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} dx \leq \frac{C_3}{k}||f||_1, \quad \forall\ i = 1 \dots, N.$$

Consequently,

$$\rho_{p_i'(.)}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-1}\chi_F\right) \leq K, \quad \forall\ i = 1, \dots, N.$$

□

Step 2. **Uniqueness of entropy solution**.

Let $h > 0$ and u, v be two entropy solutions of (P). We write the entropy inequality corresponding to the solution u, with $T_h(v)$ as test function, and to the solution v, with $T_h(u)$ as test function:

$$\sum_{i=1}^{N}\int_\Omega a_i\left(x,\frac{\partial u}{\partial x_i}\right)\frac{\partial}{\partial x_i}T_k(u-T_h(v))dx+\int_\Omega b(u)T_k(u-T_h(v))dx \le \int_\Omega f(x)T_k(u-T_h(v))dx \tag{50}$$

and

$$\sum_{i=1}^{N}\int_\Omega a_i\left(x,\frac{\partial v}{\partial x_i}\right)\frac{\partial}{\partial x_i}T_k(v-T_h(u))dx+\int_\Omega b(v)T_k(v-T_h(u))dx \le \int_\Omega f(x)T_k(v-T_h(u))dx. \tag{51}$$

Upon addition, we get

$$\begin{cases} \displaystyle\sum_{i=1}^{N}\int_\Omega a_i\left(x,\frac{\partial u}{\partial x_i}\right)\frac{\partial}{\partial x_i}T_k(u-T_h(v))dx+\sum_{i=1}^{N}\int_\Omega a_i\left(x,\frac{\partial v}{\partial x_i}\right)\frac{\partial}{\partial x_i}T_k(v-T_h(u))dx \\ \displaystyle+\int_\Omega b(u)T_k(u-T_h(v))dx+\int_\Omega b(v)T_k(v-T_h(u))dx \\ \displaystyle\le \int_\Omega f(x)\big[T_k(u-T_h(v))+T_k(v-T_h(u))\big]dx. \end{cases} \tag{52}$$

Define

$$E_1 = \{|u-v| \le k;\ |v| \le h\};\quad E_2 = E_1 \cap \{|u| \le h\} \ \text{ and } \ E_3 = E_1 \cap \{|u| > h\}.$$

We start with the first integral in (52). We have

$$\begin{aligned} &\sum_{i=1}^{N}\int_{\{|u-T_h(v)|\le k\}} a_i\left(x,\frac{\partial u}{\partial x_i}\right)\frac{\partial}{\partial x_i}T_k(u-T_h(v))dx \\ &\quad= \sum_{i=1}^{N}\int_{\{|u-T_h(v)|\le k\}\cap\{|v|\le h\}} a_i\left(x,\frac{\partial u}{\partial x_i}\right)\frac{\partial}{\partial x_i}T_k(u-T_h(v))dx \\ &\qquad+\sum_{i=1}^{N}\int_{\{|u-T_h(v)|\le k\}\cap\{|v|>h\}} a_i\left(x,\frac{\partial u}{\partial x_i}\right)\frac{\partial}{\partial x_i}T_k(u-T_h(v))dx \\ &\quad= \sum_{i=1}^{N}\int_{\{|u-v|\le k\}\cap\{|v|\le h\}} a_i\left(x,\frac{\partial u}{\partial x_i}\right)\frac{\partial(u-v)}{\partial x_i}dx \\ &\qquad+\sum_{i=1}^{N}\int_{\{|u-hsign(v)|\le k\}\cap\{|v|>h\}} a_i\left(x,\frac{\partial u}{\partial x_i}\right)\frac{\partial u}{\partial x_i}dx \\ &\quad\ge \sum_{i=1}^{N}\int_{E_1} a_i\left(x,\frac{\partial u}{\partial x_i}\right)\frac{\partial}{\partial x_i}(u-v)dx \\ &\quad= \sum_{i=1}^{N}\int_{E_2} a_i\left(x,\frac{\partial u}{\partial x_i}\right)\frac{\partial}{\partial x_i}(u-v)dx+\sum_{i=1}^{N}\int_{E_3} a_i\left(x,\frac{\partial u}{\partial x_i}\right)\frac{\partial}{\partial x_i}(u-v)dx. \end{aligned}$$

Then, we obtain

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\{|u-T_h(v)|\leq k\}}a_i\Big(x,\frac{\partial u}{\partial x_i}\Big)\frac{\partial}{\partial x_i}T_k(u-T_h(v))dx\\&\qquad\geq\sum_{i=1}^{N}\int_{E_2}a_i\Big(x,\frac{\partial u}{\partial x_i}\Big)\frac{\partial}{\partial x_i}(u-v)dx-\sum_{i=1}^{N}\int_{E_3}a_i\Big(x,\frac{\partial u}{\partial x_i}\Big)\frac{\partial v}{\partial x_i}dx.\end{aligned}\tag{53}$$

According to (5), and the Hölder type inequality we have

$$\begin{aligned}\Big|\sum_{i=1}^{N}\int_{E_3}a_i\Big(x,\frac{\partial u}{\partial x_i}\Big)\frac{\partial v}{\partial x_i}dx\Big|&\leq C_1\sum_{i=1}^{N}\int_{E_3}\Big(j_i(x)+\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i(x)-1}\Big)\Big|\frac{\partial v}{\partial x_i}\Big|dx\\&\leq C_1\sum_{i=1}^{N}\Big(|j_i|_{p_i'(.)}+\Big||\frac{\partial u}{\partial x_i}|^{p_i(x)-1}\Big|_{p_i'(.),\{h<|u|\leq h+k\}}\Big)|\frac{\partial v}{\partial x_i}|_{p_i(.),\{h-k<|v|\leq h\}}\end{aligned}$$

where

$$\Big||\frac{\partial u}{\partial x_i}|^{p_i(x)-1}\Big|_{p_i'(.),\{h<|u|\leq h+k\}}=\Big\||\frac{\partial u}{\partial x_i}|^{p_i(x)-1}\Big\|_{L^{p_i'(.)}(\{h<|u|\leq h+k\})}.$$

For $i=1,\dots,N$, the quantity $\Big(|j_i|_{p_i'(.)}+\Big||\frac{\partial u}{\partial x_i}|^{p_i(x)-1}\Big|_{p_i'(.),\{h<|u|\leq h+k\}}\Big)$ is finite according to relation (11) and Lemma 3.8.

According to Lemma 3.7, the quantity $|\frac{\partial v}{\partial x_i}|_{p_i(.),\{h-k<|v|\leq h\}}$ converges to zero as h goes to infinity. Consequently the last integral of (53) converges to zero as h goes to infinity. Then

$$\sum_{i=1}^{N}\int_{\{|u-T_h(v)|\leq k\}}a_i\Big(x,\frac{\partial u}{\partial x_i}\Big)\frac{\partial}{\partial x_i}T_k(u-T_h(v))dx\geq I_h+\sum_{i=1}^{N}\int_{E_2}a_i\Big(x,\frac{\partial u}{\partial x_i}\Big)\frac{\partial}{\partial x_i}(u-v)dx\tag{54}$$

with $\lim_{h\longrightarrow+\infty}I_h=0$.

We may adopt the same procedure to treat the second term in (52) to obtain

$$\sum_{i=1}^{N}\int_{\{|v-T_h(u)|\leq k\}}a_i(x,\frac{\partial v}{\partial x_i})\frac{\partial}{\partial x_i}T_k(v-T_h(u))dx\geq J_h-\sum_{i=1}^{N}\int_{E_2}a_i(x,\frac{\partial v}{\partial x_i})\frac{\partial}{\partial x_i}(u-v)dx\tag{55}$$

with $\lim_{h\longrightarrow+\infty}J_h=0$.

For the two other terms in the left-hand side of (52), we denote

$$K_h=\int_\Omega b(u)T_k(u-T_h(v))dx+\int_\Omega b(v)T_k(v-T_h(u))dx.$$

We have

$$b(u)T_k(u-T_h(v))\longrightarrow b(u)T_k(u-v)\quad\text{a.e}\quad\text{since}\quad h\longrightarrow+\infty$$

and

$$\big|b(u)T_k(u-T_h(v))\big|\leq k|b(u)|\in L^1(\Omega).$$

Then, by the Lebesgue dominated convergence theorem, we obtain

$$\lim_{h\longrightarrow+\infty}\int_\Omega b(u)T_k(u-T_h(v))dx=\int_\Omega b(u)T_k(u-v)dx.$$

In the same way, we get

$$\lim_{h\longrightarrow+\infty}\int_\Omega b(v)T_k(v-T_h(u))dx=\int_\Omega b(v)T_k(v-u)dx.$$

Then

$$\lim_{h\longrightarrow+\infty}K_h=\int_\Omega \big(b(u)-b(v)\big)T_k(u-v)dx. \tag{56}$$

Now consider the right-hand side of inequality (52). We have

$$\lim_{h\longrightarrow+\infty}f(x)\bigg(T_k(u-T_h(v))+T_k(v-T_h(u))\bigg)=0 \quad \text{a.e}$$

and

$$\big|f(x)\big(T_k(u-T_h(v))+T_k(v-T_h(u))\big)\big|\le 2k|f|\in L^1(\Omega).$$

By the Lebesgue dominated convergence theorem, we obtain

$$\lim_{h\longrightarrow+\infty}\int_\Omega f(x)\bigg(T_k\big(u-T_h(v)\big)+T_k\big(v-T_h(u)\big)\bigg)dx=0. \tag{57}$$

After passing to the limit as h goes to $+\infty$ in (52) we get:

$$\sum_{i=1}^N\int_{\{|u-v|\le k\}}\bigg(a_i\big(x,\frac{\partial u}{\partial x_i}\big)-a_i\big(x,\frac{\partial v}{\partial x_i}\big)\bigg)\frac{\partial}{\partial x_i}(u-v)dx + \int_\Omega \big(b(u)-b(v)\big)T_k(u-v)dx\le 0. \tag{58}$$

Since b and $a_i(x,.)$ are monotone then

$$\int_\Omega \big(b(u)-b(v)\big)T_k(u-v)dx=0 \tag{59}$$

and

$$\int_{\{|u-v|\le k\}}\sum_{i=1}^N\bigg(a_i\big(x,\frac{\partial u}{\partial x_i}\big)-a_i\big(x,\frac{\partial v}{\partial x_i}\big)\bigg)\frac{\partial}{\partial x_i}(u-v)dx=0. \tag{60}$$

We deduce from (59) that

$$\lim_{k\longrightarrow 0}\int_\Omega \big(b(u)-b(v)\big)\frac{1}{k}T_k(u-v)dx=\int_\Omega |b(u)-b(v)|dx=0. \tag{61}$$

According to (6), we deduce from (60) that

$$\left|\frac{\partial}{\partial x_i}(u-v)\right|=0 \quad \text{a.e } x\in\Omega \quad \text{that is to say} \quad u-v=C \quad \text{a.e} \quad x\ \in\Omega,$$

where C is a positive constant. Therefore

$$u-v=C \quad \text{a.e } x\in\Omega$$

and

$$b(u)=b(v).$$

■

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