Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 43(2), 2016, Pages 151–163
ISSN: 1223-6934

Competition phenomena and weak homoclinic solutions to anisotropic difference equations with variable exponent

Aboudramane Guiro, Blaise Koné, and Stanislas Ouaro

Abstract. In this paper, we prove the existence of weak homoclinic solutions for a family of second order difference equations under competition phenomena between parameters.

Key words and phrases. Anisotropic difference equations, homoclinic solutions, discrete Hölder type inequality, competition phenomena.

1. Introduction

In this paper, we study the following nonlinear anisotropic discrete problem with an homoclinic condition at the boundary

$$\begin{cases} -\Delta(a(k-1,\Delta u(k-1))) + \alpha(k)|u(k)|^{p(k)-2}u(k) = \delta f(k,u(k)), \ k \in \mathbb{Z} \\ \lim\limits_{|k|\to\infty} u(k) = 0, \end{cases} \tag{1}$$

where $\Delta u(k) = u(k+1) - u(k)$ is the forward difference operator.
Difference equations can be seen as a discrete computation of PDEs and are usually studied in connection with numerical analysis. Many authors studied anisotropic PDEs with as main operators, the following :

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a\left(x, \frac{\partial}{\partial x_i}\right),$$

under Leray–Lions type conditions [9] in the context of variable exponent (see[1, 2, 6, 7, 10, 11, 14, 15]).
The problem (1) can then be seen as a discrete counterpart of such PDEs under a homogeneous Dirichlet boundary condition.
In this paper, we adapt the classical minimization methods used for the study of anisotropic PDEs to prove the existence of solutions to problem (1).
Anisitropic discrete problems on a bounded interval was studied by many authors (see [4, 7, 8, 12]). Note that, in this paper, we examine anisotropic difference equations on an unbounded discrete interval, typically, on the whole set $\mathbb{Z}$, with asymptotic conditions of homoclinic type. The first study in that direction was done by Guiro *et al* (see [5]). More precisely, the authors in [5] studied the following problem

$$\begin{cases} -\Delta(a(k-1,\Delta u(k-1))) + |u(k)|^{p(k)-2}u(k) = f(k), \ k \in \mathbb{Z} \\ \lim\limits_{|k|\to\infty} u(k) = 0. \end{cases} \tag{2}$$

Received January 03, 2015.

They proved an existence result of a weak homoclinic solution of (2).
In this paper, we also prove an existence result of (1) and for that, we define other new spaces and new associated norms compared to that of [5]. Some of the norms defined may be equivalent in order to prove the main result of this paper. Note also that in our study, we show some competition phenomena between $\alpha(.)$ and $\delta(.)$ and between $p(.)$ and $q(.)$. Such competition phenomena are also necessary for the proof of the existence of weak homoclinic solutions of (1).
The paper is organized as follows : Section 2 is devoted to the mathematical preliminary. In Section 3, we study problem (1), where δ is a positive constant and where we prove the existence of weak homoclinic solutions of (1). In the last section, we study problem (1) for a more general δ.

2. Auxiliary results

For the data f, α and a, we assume the following.

$$(H_1)\begin{cases} a(k,.):\mathbb{R}\to\mathbb{R},\ \forall k\in\mathbb{Z} \text{ and there exists a mapping} \\ A:\mathbb{Z}\times\mathbb{R}\to\mathbb{R} \text{ which satisfies} \\ a(k,\xi)=\frac{\partial}{\partial\xi}A(k,\xi),\ \forall k\in\mathbb{Z} \text{ and } A(k,0)=0,\ \forall k\in\mathbb{Z}. \end{cases}$$

(H_2) $|\xi|^{p(k)}\leq a(k,\xi)\xi\leq p(k)A(k,\xi),\ \forall k\in\mathbb{Z}$ and $\xi\in\mathbb{R}$.

(H_3) $\exists\ C_1>0;\ |a(k,\xi)|\leq C_1(j(k)+|\xi|^{p(k)-1}),\ \forall k\in\mathbb{Z}$ and $\xi\in\mathbb{R}$ with $j\in l^{p'(.)}$ such that $\dfrac{1}{p(k)}+\dfrac{1}{p'(k)}=1$.

(H_4) $f:\mathbb{Z}\times\mathbb{R}\to\mathbb{R}$, such that for every $k\in\mathbb{Z}$ and $t\in\mathbb{R}$, $\lim\limits_{t\to 0}=\dfrac{|f(k,t)|}{|t|^{p(k)-1}}=0$.

(H_5) $\alpha:\mathbb{Z}\to\mathbb{R}^+$ such that $\alpha(k)\geq\alpha_0>0$, for all $k\in\mathbb{Z}$.

(H_6) $p:\mathbb{Z}\to(1,+\infty)$ with $1<p^-\leq p^+<+\infty$, where $p^+=\sup\limits_{k\in\mathbb{Z}}p(k)$ and $p^-=\inf\limits_{k\in\mathbb{Z}}p(k)$.

(H_7) $\alpha_0p^->\delta p^+$.

(H_8) $(a(k,\xi)-a(k,\eta)).(\xi-\eta)>0,\ \forall k\in\mathbb{Z}$ and $\xi,\eta\in\mathbb{R}$ such that $\xi\neq\eta$.

In order to present the main results of our paper, we introduce for each $p(.):\mathbb{Z}\to(1,+\infty)$, the spaces

$$l^{p(.)}:=\{u:\mathbb{Z}\to\mathbb{R};\rho_{p(.)}(u):=\sum_{k\in\mathbb{Z}}|u(k)|^{p(k)}<\infty\},$$

$$l^{p(.)}_{\alpha(.)}:=\{u:\mathbb{Z}\to\mathbb{R};\rho_{\alpha(.),p(.)}(u):=\sum_{k\in\mathbb{Z}}\alpha(k)|u(k)|^{p(k)}<\infty\},$$

$$l^{\infty}:=\{u:\mathbb{Z}\to\mathbb{R};\sup_{k\in\mathbb{Z}}|u(k)|<\infty\}$$

and

$$W^{1,p(.)}_{\alpha(.)}:=\{u:\mathbb{Z}\to\mathbb{R};\rho_{1,\alpha(.),p(.)}(u):=\sum_{k\in\mathbb{Z}}\alpha(k)|u(k)|^{p(k)}+\sum_{k\in\mathbb{Z}}|\Delta u(k)|^{p(k)}<\infty\}.$$

On $l^{p(.)}$, $l^{p(.)}_{\alpha(.)}$ and $W^{1,p(.)}_{\alpha(.)}$, we introduce the Luxemburg norms

$$\|u\|_{p(.)} := \inf\left\{\lambda > 0; \sum_{k\in\mathbb{Z}}\left|\frac{u(k)}{\lambda}\right|^{p(k)} \leq 1\right\},$$

$$\|u\|_{\alpha(.),p(.)} := \inf\left\{\lambda > 0; \sum_{k\in\mathbb{Z}}\alpha(k)\left|\frac{u(k)}{\lambda}\right|^{p(k)} \leq 1\right\}$$

and

$$\|u\|_{1,\alpha(.),p(.)} := \|u\|_{\alpha(.),p(.)} + \|\Delta u\|_{p(.)}.$$

As in [5], we can prove the following results.

Lemma 2.1. *Under the assumption* (H_6)*, we have :*

(a) $\rho_{\alpha(.),p(.)}(u+v) \leq 2^{p+}(\rho_{\alpha(.),p(.)}(u) + \rho_{\alpha(.),p(.)}(v)); \quad \forall u, v \in l^{p(.)}_{\alpha(.)}$.

(b) For $u \in l^{p(.)}_{\alpha(.)}$*, if* $\lambda > 1$ *we have*

$$\rho_{\alpha(.),p(.)}(u) \leq \lambda\rho_{\alpha(.),p(.)}(u) \leq \lambda^{p^-}\rho_{\alpha(.),p(.)}(u) \leq \rho_{\alpha(.),p(.)}(\lambda u) \leq \lambda^{p^+}\rho_{\alpha(.),p(.)}(u)$$

and if $0 < \lambda < 1$,

$$\lambda^{p^+}\rho_{\alpha(.),p(.)}(u) \leq \rho_{\alpha(.),p(.)}(\lambda u) \leq \lambda^{p^-}\rho_{\alpha(.),p(.)}(u) \leq \lambda\rho_{\alpha(.),p(.)}(u) \leq \rho_{\alpha(.),p(.)}(u).$$

(c) For every fixed $u \in l^{p(.)}_{\alpha(.)} \setminus \{0\}$, $\rho_{\alpha(.),p(.)}(\lambda u)$ *is a continuous convex even function in* λ*, and it increases strictly when* $\lambda \in [0,\infty)$.

Proposition 2.2. *Let* $u \in l^{p(.)}_{\alpha(.)} \setminus \{0\}$*, then* $\|u\|_{\alpha(.),p(.)} = \gamma \Leftrightarrow \rho_{\alpha(.),p(.)}\left(\frac{u}{\gamma}\right) = 1$.

Proposition 2.3. *If* $u \in l^{p(.)}_{\alpha(.)}$ *and* $p^+ < +\infty$*, then the following properties hold :*

1) $\|u\|_{\alpha(.),p(.)} < 1(= 1; > 1) \Leftrightarrow \rho_{\alpha(.),p(.)}(u) < 1(= 1; > 1)$;
2) $\|u\|_{\alpha(.),p(.)} > 1 \Rightarrow \|u\|^{p^-}_{\alpha(.),p(.)} \leq \rho_{\alpha(.),p(.)}(u) \leq \|u\|^{p+}_{\alpha(.),p(.)}$;
3) $\|u\|_{\alpha(.),p(.)} < 1 \Rightarrow \|u\|^{p^+}_{\alpha(.),p(.)} \leq \rho_{\alpha(.),p(.)}(u) \leq \|u\|^{p-}_{\alpha(.),p(.)}$;
4) $\|u\|_{\alpha(.),p(.)} \to 0 \Leftrightarrow \rho_{\alpha(.),p(.)}(u) \to 0$.

Proposition 2.4. *Let* $u \in W^{1,p(.)}_{\alpha(.)} \setminus \{0\}$*, then* $\|u\|_{1,\alpha(.),p(.)} = a \Leftrightarrow \rho_{1,\alpha(.),p(.)}\left(\frac{u}{a}\right) = 1$.

Proposition 2.5. *If* $u \in W^{1,p(.)}_{\alpha(.)}$ *and* $p^+ < +\infty$*, then the following properties hold :*

1) $\|u\|_{1,\alpha(.),p(.)} < 1(= 1; > 1) \Leftrightarrow \rho_{1,\alpha(.),p(.)}(u) < 1(= 1; > 1)$;
2) $\|u\|_{1,\alpha(.),p(.)} > 1 \Rightarrow \|u\|^{p^-}_{1,\alpha(.),p(.)} \leq \rho_{1,\alpha(.),p(.)}(u) \leq \|u\|^{p+}_{1,\alpha(.),p(.)}$;
3) $\|u\|_{1,\alpha(.),p(.)} < 1 \Rightarrow \|u\|^{p^+}_{1,\alpha(.),p(.)} \leq \rho_{1,\alpha(.),p(.)}(u) \leq \|u\|^{p-}_{1,\alpha(.),p(.)}$;
4) $\|u\|_{1,\alpha(.),p(.)} \to 0 \Leftrightarrow \rho_{1,\alpha(.),p(.)}(u) \to 0$.

We also have the following lemma (see [5]).

Lemma 2.6 (Hölder type inequality)**.** *Let* $u \in l^{p(.)}$ *and* $v \in l^{q(.)}$ *such that* $\frac{1}{p(k)} + \frac{1}{q(k)} = 1, \quad \forall\, k \in \mathbb{Z}$ *then,*

$$\sum_{k\in\mathbb{Z}}|uv| \leq \left(\frac{1}{p^-} + \frac{1}{q^-}\right)\|u\|_{p(.)}\|v\|_{q(.)}.$$

3. Existence of homoclinic solutions

In this section, we study the existence of weak homoclinic solutions of (1) where δ is a positive constant.

Definition 3.1. A weak homoclinic solution of (1) is a function $u \in W^{1,p(.)}_{\alpha(.)}$ such that

$$\begin{aligned}\sum_{k\in\mathbb{Z}} a(k-1,\Delta u(k-1))\Delta v(k-1) \quad &+ \quad \sum_{k\in\mathbb{Z}} \alpha(k)|u(k)|^{p(k)-2}u(k)v(k) \\ &= \quad \delta\sum_{k\in\mathbb{Z}} f(k,u(k))v(k),\end{aligned} \tag{3}$$

for any $v \in W^{1,p(.)}_{\alpha(.)}$.

The main result of this section is the following.

Theorem 3.1. *Assume that* (H_1)*–*(H_8) *hold true. Then, there exists at least one weak homoclinic solution of* (1).

Remark 3.1. The competition phenomena is relative to the condition $\alpha_0 p^- > \delta p^+$ on the data. It means that the parameter $\alpha(.)$ should be bigger than the parameter δ in order to get homoclinic solutions of (1).

We denote $F(k,t) = \displaystyle\int_0^t f(k,\tau)d\tau, \ \ \forall k \in \mathbb{Z}, \ t \in \mathbb{R}$.
The energy functional corresponding to problem (1) is defined by $J : W^{1,p(.)}_{\alpha(.)} \to \mathbb{R}$ such that

$$J(u) = \sum_{k\in\mathbb{Z}} A(k-1,\Delta u(k-1)) + \sum_{k\in\mathbb{Z}} \frac{\alpha(k)}{p(k)}|u(k)|^{p(k)} - \delta\sum_{k\in\mathbb{Z}} F(k,u(k)). \tag{4}$$

We first present some basic properties of J.

Proposition 3.2. *The functional* J *is well-defined on* $W^{1,p(.)}_{\alpha(.)}$ *and is of class* $C^1(W^{1,p(.)}_{\alpha(.)},\mathbb{R})$ *with the derivative given by*

$$\begin{aligned}\langle J'(u),v\rangle = \sum_{k\in\mathbb{Z}} a(k-1,\Delta u(k-1))\Delta v(k-1) \quad &+ \quad \sum_{k\in\mathbb{Z}} \alpha(k)|u(k)|^{p(k)-2}u(k)v(k) \\ &- \quad \delta\sum_{k\in\mathbb{Z}} f(k,u(k))v(k),\end{aligned} \tag{5}$$

for all $u, v \in W^{1,p(.)}_{\alpha(.)}$.

Proof. Let $J(u) = I(u) + L(u) - \Lambda(u)$ with

$$I(u)=\sum_{k\in\mathbb{Z}}A(k-1,\Delta u(k-1)),\ L(u)=\sum_{k\in\mathbb{Z}}\frac{\alpha(k)}{p(k)}|u(k)|^{p(k)} \text{ and } \Lambda(u)=\sum_{k\in\mathbb{Z}}F(k,u(k)).$$

One has

$$\begin{aligned}
|I(u)| &= \left|\sum_{k\in\mathbb{Z}}A(k-1,\Delta u(k-1))\right| \\
&\leq \sum_{k\in\mathbb{Z}}|A(k-1,\Delta u(k-1))| \\
&\leq \sum_{k\in\mathbb{Z}}C_1\left(j(k-1)+\frac{1}{p(k-1)}|\Delta u(k-1)|^{p(k-1)-1}\right)|\Delta u(k-1)| \\
&\leq \sum_{k\in\mathbb{Z}}C_1 j(k-1)|\Delta u(k-1)|+\sum_{k\in\mathbb{Z}}\frac{C_1}{p(k-1)}|\Delta u(k-1)|^{p(k-1)} \\
&< \infty.
\end{aligned}$$

$$|L(u)|=\left|\sum_{k\in\mathbb{Z}}\frac{\alpha(k)}{p(k)}|u(k)|^{p(k)}\right|\leq\frac{1}{p^-}\sum_{k\in\mathbb{Z}}\alpha(k)|u(k)|^{p(k)}<\infty.$$

By (H_4), there exists $\nu>0$ such that $|f(k,t)|\leq|t|^{p(k)-1}$ for all $k\in\mathbb{Z}$, $|t|\leq\nu$. By integrating, we deduce that $|F(k,t)|\leq\dfrac{|t|^{p(k)}}{p(k)}$ for all $k\in\mathbb{Z}$, $|t|\leq\nu$.

For all $u\in W^{1,p(.)}_{\alpha(.)}$, there exists $h\in\mathbb{N}$ such that $|u(k)|\leq\nu$ for all $k\in\mathbb{Z}$, $|k|>h$, so

$$\begin{aligned}
|\Lambda(u)| &= \left|\sum_{k\in\mathbb{Z}}F(k,u(k))\right| \\
&\leq \sum_{k\in\mathbb{Z}}|F(k,u(k))| \\
&\leq \sum_{|k|\leq h}|F(k,u(k))|+\sum_{|k|>h}\frac{|u(k)|^{p(k)}}{p(k)} \\
&\leq \sum_{|k|\leq h}|F(k,u(k))|+\frac{1}{p^-\alpha_0}\sum_{k\in\mathbb{Z}}\alpha(k)|u(k)|^{p(k)}<\infty.
\end{aligned}$$

Therefore, J is well-defined.

Clearly I, L and Λ are in $C^1(W^{1,p(.)}_{\alpha(.)},\mathbb{R})$. In what follows, we prove (5).

Let us choose $u,v\in W^{1,p(.)}_{\alpha(.)}$. We have

$$\langle I'(u),v\rangle=\lim_{\eta\to0^+}\frac{I(u+\eta v)-I(u)}{\eta},\quad \langle L'(u),v\rangle=\lim_{\eta\to0^+}\frac{L(u+\eta v)-L(u)}{\eta}$$

and

$$\langle\Lambda'(u),v\rangle=\lim_{\eta\to0^+}\frac{\Lambda(u+\eta v)-\Lambda(u)}{\eta}.$$

Since

$$\begin{aligned}
\lim_{\eta\to 0^+} & \frac{I(u+\eta v)-I(u)}{\eta} \\
&= \lim_{\eta\to 0^+}\sum_{k\in\mathbb{Z}} \frac{A(k-1,\Delta u(k-1)+\eta\Delta v(k-1))-A(k-1,\Delta u(k-1))}{\eta} \\
&= \sum_{k\in\mathbb{Z}} \lim_{\delta\to 0^+} \frac{A(k-1,\Delta u(k-1)+\eta\Delta v(k-1))-A(k-1,\Delta u(k-1))}{\eta} \\
&= \sum_{k\in\mathbb{Z}} a(k-1,\Delta u(k-1))\Delta v(k-1),
\end{aligned}$$

$$\begin{aligned}
\lim_{\eta\to 0^+}\frac{L(u+\eta v)-L(u)}{\eta} &= \lim_{\eta\to 0^+}\sum_{k\in\mathbb{Z}} \frac{\alpha(k)(|u(k)+\eta v(k)|^{p(k)}-|u(k)|^{p(k)})}{p(k)\eta} \\
&= \sum_{k\in\mathbb{Z}} \lim_{\eta\to 0^+} \frac{\alpha(k)(|u(k)+\eta v(k)|^{p(k)}-|u(k)|^{p(k)})}{p(k)\eta} \\
&= \sum_{k\in\mathbb{Z}} \alpha(k)|u(k)|^{p(k)-2}u(k)v(k)
\end{aligned}$$

and

$$\begin{aligned}
\lim_{\eta\to 0^+}\frac{\Lambda(u+\eta v)-\Lambda(u)}{\eta} &= \lim_{\delta\to 0^+}\sum_{k\in\mathbb{Z}} \frac{F(k,u(k)+\eta v(k))-F(k,u(k))}{\eta} \\
&= \sum_{k\in\mathbb{Z}} \lim_{\eta\to 0^+} \frac{F(k,u(k)+\eta v(k))-F(k,u(k))}{\eta} \\
&= \sum_{k\in\mathbb{Z}} f(k,u(k))v(k),
\end{aligned}$$

we obtain (5). □

Lemma 3.3. *The functional I is weakly lower semi-continuous.*

Proof. From (H_1) and (H_8), I is convex with respect to the second variable. Thus, it is enough to show that I is lower semi-continuous (see Corollary III.8 in [3]). For this, we fix $u \in W^{1,p(.)}_{\alpha(.)}$ and $\epsilon > 0$. Since I is convex, we deduce that for any $v \in W^{1,p(.)}_{\alpha(.)}$,

$$\begin{aligned}
I(v) &\geq I(u) + \langle I'(u), v-u\rangle \\
&\geq I(u) + \sum_{k\in\mathbb{Z}} a(k-1,\Delta u(k-1))\left(\Delta v(k-1)-\Delta u(k-1)\right) \\
&\geq I(u) - C(\frac{1}{p^-}+\frac{1}{p'^-})\|g\|_{p'(.)}\|\Delta(u-v)\|_{p(.)} \\
&\geq I(u) - K\Big(\|u-v\|_{\alpha(.),p(.)}+\|\Delta(u-v)\|_{p(.)}\Big) \\
&\geq I(u) - K\|u-v\|_{1,\alpha(.),p(.)} \;\geq\; I(u)-\epsilon,
\end{aligned}$$

for all $v \in W^{1,p(.)}_{\alpha(.)}$ with $\|u-v\|_{1,\alpha(.),p(.)} < \xi = \dfrac{\epsilon}{K}$. Here $g(k) = j(k) + |\Delta u(k-1)|^{p(k-1)-1}$.

Hence, we conclude that I is weakly lower semi-continuous. □

Proposition 3.4. *The functional J is bounded from below, coercive and weakly lower semi-continuous.*

Proof. By Lemma 3.3, J is weakly lower semi-continuous. We will only prove the coerciveness of the energy functional since the boundedness from below of J is a consequence of its coerciveness. To prove the coerciveness of J, we may assume that $\|u\|_{1,\alpha(.),p(.)} > 1$.

$$\begin{aligned}
J(u) &= \sum_{k\in\mathbb{Z}} A(k-1,\Delta u(k-1)) + \sum_{k\in\mathbb{Z}} \frac{\alpha(k)}{p(k)}|u(k)|^{p(k)} - \delta\sum_{k\in\mathbb{Z}} F(k,u(k)) \\
&\geq \sum_{k\in\mathbb{Z}} \frac{1}{p(k)}|\Delta u(k-1)|^{p(k-1)} + \sum_{k\in\mathbb{Z}} \frac{\alpha(k)}{p(k)}|u(k)|^{p(k)} - \delta\sum_{k\in\mathbb{Z}} |F(k,u(k))| \\
&\geq \frac{1}{p^+}\sum_{k\in\mathbb{Z}} |\Delta u(k-1)|^{p(k-1)} + \frac{1}{p^+}\sum_{k\in\mathbb{Z}} \alpha(k)|u(k)|^{p(k)} - \delta\sum_{|k|>h} \frac{1}{p(k)}|u(k)|^{p(k)} - M(\delta) \\
&\geq \frac{1}{p^+}(\rho_{p(.)}(\Delta u) + \rho_{\alpha(.),p(.)}(u)) - \frac{\delta}{p^-\alpha_0}\sum_{k\in\mathbb{Z}} \alpha(k)|u(k)|^{p(k)} - M(\delta) \\
&\geq \frac{1}{p^+}\rho_{1,\alpha(.),p(.)}(u) - \frac{\delta}{p^-\alpha_0}\rho_{1,\alpha(.)p(.)}(u) - M(\delta) \\
&\geq \left(\frac{1}{p^+} - \frac{\delta}{\alpha_0 p^-}\right)\rho_{1,\alpha(.),p(.)}(u) - M(\delta) \\
&\geq \left(\frac{1}{p^+} - \frac{\delta}{\alpha_0 p^-}\right)\|u\|_{1,\alpha(.),p(.)}^{p^-} - M(\delta),
\end{aligned}$$

where $M(\delta)$ is a positive constant depending on δ.
By (H_7), J is coercive.

Therefore, as $\|u\|_{1,\alpha(.),p(.)} \to +\infty$ then $J(u) \to +\infty$ and so, there exists $c \in \mathbb{R}$ such that $J(u) \geq c$.

If $\|u\|_{1,\alpha(.),p(.)} \leq 1$, then

$$\begin{aligned}
J(u) &\geq \frac{1}{p^+}\rho_{1,\alpha(.),p(.)}(u) - \frac{\delta}{p^-\alpha_0}\rho_{\alpha(.),p(.)}(u) - M(\delta) \\
&\geq -\frac{\delta}{p^-\alpha_0}\rho_{\alpha(.),p(.)}(u) - M(\delta) \\
&\geq -C > -\infty.
\end{aligned}$$

Thus, J is bounded below.

As I is weakly lower semi-continuous, then J is weakly lower semi-continuous. □

We can now give the proof of Theorem 3.1.

Proof of Theorem 3.1. By Proposition 3.2, J has a minimizer which is a weak solution of (1).
In order to complete the proof of Theorem 3.1, we will show that every weak solution u is homoclinic, i.e $u(k) \to 0$ as $|k| \to +\infty$.

Let u be a weak solution to problem (1), then, as $u \in W^{1,p(.)}_{\alpha(.)}$, we get

$$\sum_{k\in\mathbb{Z}} \alpha_0 |u(k)|^{p(k)} \leq \sum_{k\in\mathbb{Z}} \alpha(k)|u(k)|^{p(k)} < \infty.$$

Let $S_1 = \{k \in \mathbb{Z}, |u(k)| < 1\}$ and $S_2 = \{k \in \mathbb{Z}, |u(k)| \geq 1\}$.

S_2 is a finite set, then

$$\begin{aligned}\sum_{k\in\mathbb{Z}} |u(k)|^{p(k)} &= \sum_{k\in S_1} |u(k)|^{p(k)} + \sum_{k\in S_2} |u(k)|^{p(k)} \\ &\leq \sum_{k\in S_1} |u(k)|^{p(k)} + R < \infty,\end{aligned}$$

where R is a positive constant.

So,

$$\sum_{k\in S_1} |u(k)|^{p^+} + R \leq \sum_{k\in S_1} |u(k)|^{p(k)} + R.$$

Therefore, as S_2 is a finite set, we get

$$\sum_{k\in\mathbb{Z}} |u(k)|^{p^+} < \infty.$$

Thus, $\lim\limits_{|k|\to+\infty} |u(k)| = 0.$ □

4. Extension

In this section, we show that the existence result obtained for (1) can be extended to more general anisotropic homoclinic boundary value problems of the form.

$$\begin{cases} -\Delta(a(k-1, \Delta u(k-1))) + \alpha(k)|u(k)|^{p(k)-2}u(k) = \delta(k)f(k, u(k)), \ k \in \mathbb{Z} \\ \lim\limits_{|k|\to\infty} u(k) = 0. \end{cases} \tag{6}$$

For the data f, δ and α, we assume the following.

(H_9) $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$, such that there exists $b : \mathbb{Z} \to \mathbb{R}^+$, with $b \in l^{q'(.)r'(.)}$ and $|f(k,t)| \leq b(k) + c|t|^{q(k)-1}$.

(H_{10}) $\delta : \mathbb{Z} \to \mathbb{R}$, $\delta \in l^{r(.)}$ with $\delta^* = \inf\limits_{k\in\mathbb{Z}} |\delta(k)| > 0$ and $\alpha : \mathbb{Z} \to \mathbb{R}$, $\alpha \in l^\infty$ with $\alpha^- > 0$.

(H_{11}) $r^- > 1$ and $+\infty > p^+ \geq p^- > q^+ \geq q^- > 1$.

We also need the following space.

$$W^{1,p(.)} := \{u : \mathbb{Z} \to \mathbb{R}; u \in l^{p(.)}, \Delta u \in l^{p(.)}\}.$$

On $W^{1,p(.)}$ and l^∞, we introduce the following norms.

$$\begin{aligned}\|u\|_{1,p(.)} &:= \inf\left\{\lambda > 0; \sum_{k\in\mathbb{Z}} \left|\frac{u(k)}{\lambda}\right|^{p(k)} + \sum_{k\in\mathbb{Z}} \left|\frac{\Delta u(k)}{\lambda}\right|^{p(k)} \leq 1\right\} \\ &= \|u\|_{1,p(.)} \\ &= \|u\|_{p(.)} + \|\Delta u\|_{p(.)},\end{aligned}$$

$$\|u\|_\infty := \sup_{k\in\mathbb{Z}} |u(k)|,$$

and

$$\begin{aligned} \|u\|_{1,\delta(.),p(.)} &:= \inf\left\{\lambda>0; \sum_{k\in\mathbb{Z}}|\delta(k)|\left|\frac{u(k)}{\lambda}\right|^{p(k)} + \sum_{k\in\mathbb{Z}}\left|\frac{\Delta u(k)}{\lambda}\right|^{p(k)} \le 1\right\} \\ &= \|u\|_{\delta(.),p(.)} + \|\Delta u\|_{p(.)}. \end{aligned}$$

Lemma 4.1. *For all* $r(.) < +\infty$, $l^{r(.)} \subset l^{\infty}$.

Proof.

$$\sup_{k\in\mathbb{Z}}|u(k)| \le \left(\rho_{r(.)}(u)\right)^{s} \quad \text{with} \quad \begin{cases} s = \dfrac{1}{r^-} & \text{if } \rho_{r(.)}(u) \ge 1, \\ s = \dfrac{1}{r^+} & \text{if } \rho_{r(.)}(u) \le 1. \end{cases}$$

So, we have $\delta \in l^{r(.)} \Rightarrow \delta \in l^{\infty}$. □

In this section, we need the following results.

Lemma 4.2. *(see* [13]*) Assume that* (H_{11}) *is fulfilled. Then,* $l^{q(.)} \subset l^{p(.)}$.

Lemma 4.3. $\|.\|_{1,\delta(.),p(.)}$ *and* $\|.\|_{1,p(.)}$ *are equivalent norms in* $W^{1,p(.)}$.

Proof. As $\delta \in l^{r(.)}$, by Lemma 4.1, we have

$$\begin{aligned} A &= \sum_{k\in\mathbb{Z}}|\delta(k)|\left|\frac{u(k)}{\lambda}\right|^{p(k)} + \sum_{k\in\mathbb{Z}}\left|\frac{\Delta u(k)}{\lambda}\right|^{p(k)} \\ &\le \max(1,\|\delta\|_\infty)\left(\sum_{k\in\mathbb{Z}}\left|\frac{u(k)}{\lambda}\right|^{p(k)} + \sum_{k\in\mathbb{Z}}\left|\frac{\Delta u(k)}{\lambda}\right|^{p(k)}\right) = \max(1,\|\delta\|_\infty)B, \end{aligned}$$

where $B = \sum_{k\in\mathbb{Z}}\left|\frac{u(k)}{\lambda}\right|^{p(k)} + \sum_{k\in\mathbb{Z}}\left|\frac{\Delta u(k)}{\lambda}\right|^{p(k)}$;

elsewhere, we have

$$\begin{aligned} A &= \sum_{k\in\mathbb{Z}}|\delta(k)|\left|\frac{u(k)}{\lambda}\right|^{p(k)} + \sum_{k\in\mathbb{Z}}\left|\frac{\Delta u(k)}{\lambda}\right|^{p(k)} \\ &\ge \min(1,\delta^*)\left(\sum_{k\in\mathbb{Z}}\left|\frac{u(k)}{\lambda}\right|^{p(k)} + \sum_{k\in\mathbb{Z}}\left|\frac{\Delta u(k)}{\lambda}\right|^{p(k)}\right) = \min(1,\delta^*)B. \end{aligned}$$

So, $\forall u \in W^{1,p(.)}$, $\min(1,\delta^*)\|u\|_{1,p(.)} \le \|u\|_{1,\delta(.),p(.)} \le \max(1,\|\delta\|_\infty)\|u\|_{1,p(.)}$. □

Definition 4.1. A weak homoclinic solution of (6) is a function $u \in W^{1,p(.)}$ such that

$$\begin{aligned} \sum_{k\in\mathbb{Z}} a(k-1,\Delta u(k-1))\Delta v(k-1) &+ \sum_{k\in\mathbb{Z}}\alpha(k)|u(k)|^{p(k)-2}u(k)v(k) \\ &= \sum_{k\in\mathbb{Z}}\delta(k)f(k,u(k))v(k), \end{aligned} \tag{7}$$

for any $v \in W^{1,p(.)}$.

The main result of this section is the following.

Theorem 4.4. *Assume that* (H_1)–(H_3) *and* (H_8)–(H_{11}) *hold true. Then, there exists at least one weak homoclinic solution of* (6).

The energy functional corresponding to problem (1) is defined by $J : W^{1,p(.)} \to \mathbb{R}$ such that

$$J(u) = \sum_{k\in\mathbb{Z}} A(k-1, \Delta u(k-1)) + \sum_{k\in\mathbb{Z}} \frac{\alpha(k)}{p(k)}|u(k)|^{p(k)} - \sum_{k\in\mathbb{Z}} \delta(k)F(k, u(k)), \tag{8}$$

where $F(k,t) = \displaystyle\int_0^t f(k,\tau)d\tau$, $\forall k \in \mathbb{Z}$ and $t \in \mathbb{R}$.

We first present some basic properties of J.

Proposition 4.5. *The functional* J *is well-defined on* $W^{1,p(.)}$ *and is of class* $C^1(W^{1,p(.)}, \mathbb{R})$ *with the derivative given by*

$$\begin{aligned}\langle J'(u), v\rangle = \sum_{k\in\mathbb{Z}} a(k-1, \Delta u(k-1))\Delta v(k-1) \quad &+ \quad \sum_{k\in\mathbb{Z}} \alpha(k)|u(k)|^{p(k)-2}u(k)v(k) \\ &- \quad \sum_{k\in\mathbb{Z}} \delta(k)f(k, u(k))v(k),\end{aligned} \tag{9}$$

for all $u, v \in W^{1,p(.)}$.

Proof. Let $J(u) = I(u) + L(u) - \Lambda(u)$ with $I(u) = \displaystyle\sum_{k\in\mathbb{Z}} A(k-1, \Delta u(k-1))$,

$L(u) = \displaystyle\sum_{k\in\mathbb{Z}} \frac{\alpha(k)}{p(k)}|u(k)|^{p(k)}$ and $\Lambda(u) = \displaystyle\sum_{k\in\mathbb{Z}} \delta(k)F(k, u(k)$.

As in Section 3, we have

$$|I(u)| < \infty \text{ and } |L(u)| = \left|\sum_{k\in\mathbb{Z}} \frac{\alpha(k)}{p(k)}|u(k)|^{p(k)}\right| \le \frac{\alpha^+}{p^-}\sum_{k\in\mathbb{Z}} |u(k)|^{p(k)} < \infty.$$

By (H_8), we have $|f(k, u(k))| \le b(k) + c|u(k)|^{q(k)-1}$ and by Young inequality, we have for all $u \in W^{1,p(.)}$,

$$\begin{aligned}|F(k, u(k))| &\le |b(k)||u(k)| + \frac{c}{q(k)}|u(k)|^{q(k)} \\ &\le \frac{1}{q'^-}|b(k)|^{q'(k)} + \frac{(1+c)}{q^-}|u(k)|^{q(k)}.\end{aligned}$$

So,

$$\begin{aligned}|\Lambda(u)| = \left|\sum_{k\in\mathbb{Z}} \delta(k)F(k, u(k))\right| &\le \sum_{k\in\mathbb{Z}} |\delta(k)||F(k, u(k))| \\ &\le \frac{1}{q'^-}\sum_{k\in\mathbb{Z}} |\delta(k)||b(k)|^{q'(k)} + \frac{(1+c)}{q^-}\sum_{k\in\mathbb{Z}} |\delta(k)||u(k)|^{q(k)}\end{aligned}$$

$$
\begin{aligned}
&\leq \quad \frac{1}{q'^{-}r^{-}}\sum_{k\in\mathbb{Z}}|\delta(k)|^{r(k)}+\frac{1}{q'^{-}r'^{-}}\sum_{k\in\mathbb{Z}}|b(k)|^{q'(k)r'(k)}\\
&+ \quad \frac{(1+c)\|\delta\|_{\infty}}{q^{-}}\sum_{k\in\mathbb{Z}}|u(k)|^{q(k)}<\infty.
\end{aligned}
$$

Then, J is well-defined.

Clearly I, L and Λ are in $C^1(W^{1,p(.)},\mathbb{R})$ and we have

$$\langle I'(u),v\rangle=\lim_{\eta\to 0^+}\frac{I(u+\eta v)-I(u)}{\eta}=\sum_{k\in\mathbb{Z}}a(k-1,\Delta u(k-1))\Delta v(k-1),$$

$$\langle L'(u),v\rangle=\lim_{\eta\to 0^+}\frac{L(u+\eta v)-L(u)}{\eta}=\sum_{k\in\mathbb{Z}}\alpha(k)|u(k)|^{p(k)-2}u(k)v(k)$$

and

$$\langle \Lambda'(u),v\rangle=\lim_{\eta\to 0^+}\frac{\Lambda(u+\eta v)-\Lambda(u)}{\eta}=\sum_{k\in\mathbb{Z}}\delta(k)f(k,u(k))v(k).$$

□

As in Lemma 3.3, we can prove the following lemma.

Lemma 4.6. *The functional I is weakly lower semi-continuous.*

Proposition 4.7. *The functional J is bounded from below, coercive and weakly lower semi-continuous.*

Proof. By Lemma 4.6, J is weakly lower semi-continuous. We will only prove the coerciveness of the energy functional since the boundedness from below of J is a consequence of its coerciveness. We assume that $\|u\|_{1,p(.)}>1$. Then

$$
\begin{aligned}
J(u) \quad &= \quad \sum_{k\in\mathbb{Z}}A(k-1,\Delta u(k-1))+\sum_{k\in\mathbb{Z}}\frac{\alpha(k)}{p(k)}|u(k)|^{p(k)}-\sum_{k\in\mathbb{Z}}\delta(k)F(k,u(k))\\
&\geq \quad \sum_{k\in\mathbb{Z}}\frac{1}{p(k)}|\Delta u(k-1)|^{p(k-1)}+\sum_{k\in\mathbb{Z}}\frac{\alpha(k)}{p(k)}|u(k)|^{p(k)}-\sum_{k\in\mathbb{Z}}|\delta(k)|\,|F(k,u(k))|\\
&\geq \quad \frac{1}{p^+}\sum_{k\in\mathbb{Z}}|\Delta u(k-1)|^{p(k-1)}+\frac{\alpha^-}{p^+}\sum_{k\in\mathbb{Z}}|u(k)|^{p(k)}\\
&- \quad \sum_{k\in\mathbb{Z}}|\delta(k)|\left(\frac{1}{q'^-}|b(k)|^{q'(k)}+\frac{(1+c)}{q^-}|u(k)|^{q(k)}\right)\\
&\geq \quad \min(\frac{1}{p^+},\frac{\alpha^-}{p^+})(\rho_{p(.)}(\Delta u)+\rho_{p(.)}(u))-\left(\frac{1}{r^+q'^-}\sum_{k\in\mathbb{Z}}|\delta(k)|^{r(k)}\right)\\
&- \quad \left(\frac{1}{r'^+q'^-}\sum_{k\in\mathbb{Z}}|b(k)|^{q'(k)r'(k)}\right)-\frac{(1+c)}{q^-}\sum_{k\in\mathbb{Z}}|\delta(k)||u(k)|^{q(k)}
\end{aligned}
$$

$$
\begin{aligned}
&\geq \min\left(\frac{1}{p^+}, \frac{\alpha^-}{p^+}\right)\rho_{1,p(.)}(u) - M - \frac{(1+c)}{q^-}\rho_{|\delta(.)|,q(.)}(u) \\
&\geq C_1\|u\|_{1,p(.)}^{p^-} - C_2\|u\|_{|\delta(.)|,q(.)}^{q^+} - M \\
&\geq C_1\|u\|_{1,p(.)}^{p^-} - C_3\|u\|_{1,p(.)}^{q^+} - M,
\end{aligned}
$$

where C_1, C_2, C_3 and M are positive constants.

As $\|u\|_{1,p(.)} \to +\infty$, then $J(u) \to +\infty$. So, there exists $c \in \mathbb{R}$ such that $J(u) \geq c$.

If $\|u\|_{1,p(.)} \leq 1$, then

$$
\begin{aligned}
J(u) &\geq C_1\rho_{1,p(.)}(u) - M - C_2\rho_{|\delta(.)|,q(.)}(u) \\
&\geq C_1\|u\|_{1,p(.)}^{p^+} - C_2\|u\|_{\alpha(.),q(.)}^{q^-} - M \\
&\geq C_1\|u\|_{1,p(.)}^{p^+} - C_4\|u\|_{1,p(.)}^{q^-} - C_2 \\
&\geq -(C_4 + C_2) = -C_0,
\end{aligned}
$$

where C_4 is a positive constant and thus J is bounded below. □

We can now give the proof of Theorem 4.4.

Proof of Theorem 4.4. By Proposition 4.5, J has a minimizer which is a weak solution of (1.1) and as in Section 3, we prove that the solution satisfies $\lim_{|k|\to+\infty} u(k) = 0$. □

Remark 4.1. In this section,the competition phenomena is relative to the exponent. Indeed, $p(.)$ must be bigger than $q(.)$ in order to get weak homoclinic solutions to problem (6).

Acknowledgements

This work was done within the framework of the visit of the authors at the Abdus Salam International Centre for Theoretical Physics (ICTP), Trieste, Italy. The authors thank the mathematics section of ICTP for their hospitality and for financial support and all facilities.

References

[1] B. K. Bonzi, S. Ouaro, F D. Zongo, Entropy solutions for nonlinear elliptic anisotropic homogeneous Neumann problem, *Int. J. Differ. Equ* (2013) Article ID 476781, 14 p.

[2] M-M. Boureanu, V. Rădulescu, Anisotropic Neumann problems in Sobolev spaces with variable exponent, *Nonlinear Analysis: Theory, Methods & Applications* **75**(12) (2012), 4471–4482.

[3] H. Brézis, *Analyse Fonctionnelle: Théorie et Applications*, Paris, Masson (1983).

[4] A. Guiro, I. Nyanquini, S. Ouaro, On the solvability of discrete nonlinear Neumann problems involving the p(x)-Laplacian, *Adv. Differ. Equ.* **32** (2011), 1–14.

[5] A. Guiro, B. Koné, S. Ouaro, Weak Homoclinic solutions of anisotropic difference equations with variable exponents, *Adv. Differ. Equ.* **154** (2012), 1–13.

[6] B. Koné, S. Ouaro, S. Traoré, Weak solutions for anisotropic nonlinear elliptic equations with variable exponent, *Electron. J. Diff. Equ.* **144** (2009), 1–11.

[7] B. Koné, S. Ouaro, Weak solutions for anisotropic discrete boundary value problems, *J. Difference Equ. Appl.* **17**(10) (2011), 1537–1547.

[8] B. Koné, S. Ouaro, On the solvability of discrete nonlinear two point boundary value problems, *Int. J. Math. Math. Math. Sci.* **2012**, Art. ID. 927607, 16 p.

[9] J. Leray, J. L. Lions, Quelques résultats de Visik sur les problèmes elliptiques nonlinéaires par les méthodes de Minty et Browder, *Bull. Soc. Math. France.* **93** (1965), 97–107.
[10] M. Mihăilescu, P. Pucci, V. Rădulescu, Nonhomogeneous boundary value problems in anisotropic Sobolev spaces, *C. R. Acad. Sci. Paris, Ser. I* **345** (2007), 561–566.
[11] M. Mihăilescu, P. Pucci, V. Rădulescu, Eigenvalue problems for anisotropic quasilinear elliptic equations with variable exponent, *J. Math. Anal. Appl.* **340** (2008), 687–698.
[12] M. Mihăilescu, V. Rădulescu, S. Tersian, Eigenvalue problems for anisotropic discrete boundary value problems, *J. Difference Equ. Appl.* **15**(6) (2009), 557–567.
[13] M. Mihăilescu, V. Rădulescu, S. Tersian, Homoclinic solutions of difference equations with variable exponent, *Top. Meth. Non. Ana* **38** (2011), 277–289.
[14] S. Ouaro, Well-posedness results for anisotropic nonlinear elliptic equations with variable exponent and L^1–data, *Cubo J.* **12**(1) (2010), 133–148.
[15] X. Fan, Anisotropic variable exponent Sobolev spaces and $\overrightarrow{p}(.)$-Laplacian equations, *Complex variables and Elliptic Equations* **55** (2010), 1–20.

(Aboudramane Guiro) LAboratoire de Mathématiques et Informatique (LAMI), UFR Sciences et Technologies, Université Polytechnique de Bobo Dioulasso
01 BP 1091 Bobo-Dioulasso 01, Burkina Faso
E-mail address: `abouguiro@yahoo.fr, abouguiro@gmail.com`

(Blaise Koné) LAboratoire de Mathématiques et Informatique (LAMI), Institut Burkinabè des Arts et Metiers, Université Ouaga 1 Pr Joseph KI-ZERBO
03 BP 7021 Ouaga 03, Ouagadougou, Burkina Faso
E-mail address: `leizon71@yahoo.fr`

(Stanislas Ouaro) LAboratoire de Mathématiques et Informatique (LAMI), UFR Sciences Exactes et Appliquées, Université Ouaga 1 Pr Joseph KI-ZERBO
03 BP 7021 Ouaga 03, Ouagadougou, Burkina Faso
E-mail address: `souaro@univ-ouaga.bf, ouaro@yahoo.fr`