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# New error estimation of Chebyshev functional and application to the onepoint and twopoint weighted integral formula

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Abstract. Recently there have been proven many results about error bounds for Chebyshev functional. The aim of our paper is to extend those results and give some new error estimation of the Chebyshev functional and applications to the onepoint and twopoint weighted integral formulas.

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## 1. Introduction

The weighted one-point integral formula of Matić, Pečarić and Ujević [1] is introduced from the general $m-$point integral identity in [2] and states:

$$\int_a^b w(t)f(t)dt \quad = \quad \sum_{j=1}^{n} A_{w,j}(x)f^{(j-1)}(x) + (-1)^n \int_a^b W_{n,w}(t,x)f^{(n)}(t)dt, \qquad (1)$$

where $f : [a,b] \to \mathbf{R}$ is such that $f^{(n-1)}$ is absolutely continuous function, $w : [a,b] \to [0,\infty)$ is weight function, $x \in [a,b]$,

$$A_{w,j}(x) = \frac{(-1)^{j-1}}{(j-1)!}\int_a^b (x-s)^{j-1}w(s)ds, \quad \text{for } j = 1,\dots,n \qquad (2)$$

and

$$W_{n,w}(t,x) = \begin{cases} w_{1n}(t) = \frac{1}{(n-1)!}\int_a^t (t-s)^{n-1}w(s)ds & \text{for } t \in [a,x], \\ w_{2n}(t) = \frac{1}{(n-1)!}\int_b^t (t-s)^{n-1}w(s)ds & \text{for } t \in (x,b]. \end{cases} \qquad (3)$$

If we put $w_{k0}(t) = w(t)$ for $k = 1,2$, then it is easy to check that sequences $\{w_{kn}\}_{n\in\mathbf{N}}$ are $w-$harmonic sequences of functions, i.e.

$$w'_{kn}(t) = w_{k,n-1}(t), \quad n \geq 1.$$

In [2] is also given the following $L_p-$inequality:
If $f^{(n)} \in \mathrm{L_p}$ for some $1 \leq p \leq \infty$, then we have

$$\Big|\int_a^b w(t)f(t)dt - \sum_{j=1}^{n} A_{j,w}(x)f^{(j-1)}(x)\Big| \leq C_1(n,p,x,w)\cdot\|f^{(n)}\|_p,$$

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for $\frac{1}{p}+\frac{1}{q}=1$, where

$$C_1(n,p,x,w)=\frac{1}{(n-1)!}\left[\int_a^x\left|\int_a^t(t-s)^{n-1}w(s)ds\right|^q dt+\int_x^b\left|\int_b^t(t-s)^{n-1}w(s)ds\right|^q dt\right]^{\frac{1}{q}}, \tag{4}$$

for $1<p\le\infty$, and

$$C_1(n,1,x,w)=\frac{1}{(n-1)!}\max\{\sup_{t\in[a,x]}\left|\int_a^t(t-s)^{n-1}w(s)ds\right|,\sup_{t\in[x,b]}\left|\int_b^t(t-s)^{n-1}w(s)ds\right|\}. \tag{5}$$

The inequality is the best possible for $p=1$ and sharp for $1<p\le\infty$.
In [3] authors have obtained the general two-point integral identity. Let $w:[a,b]\to\mathbf{R}$ be some integrable function and $x\in[a,\frac{a+b}{2}]$. Consider a subdivision

$$\sigma:=\{x_0=a,x_1=x,x_2=a+b-x<x_3=b\}$$

of $[a,b]$. Let $\{Q_{k,x}\}_{k\in\mathbf{N}}$ be sequence of polynomials such that $\deg Q_{k,x}\le k-1$, $Q'_{k,x}(t)=Q_{k-1,x}(t)$, $k\in\mathbf{N}$ and $Q_{0,x}\equiv 0$. Define functions $w_{jk}(t)$ on $[x_{j-1},x_j]$, for $j=1,2,3$ and $k\in\mathbf{N}$:

$$\begin{aligned} w_{1k}^{(2)}(t) &= \frac{1}{(k-1)!}\int_a^t(t-s)^{k-1}w(s)\mathrm{d}s \\ w_{2k}^{(2)}(t) &= \frac{1}{(k-1)!}\int_x^t(t-s)^{k-1}w(s)\mathrm{d}s+Q_{k,x}(t) \\ w_{3k}^{(2)}(t) &= -\frac{1}{(k-1)!}\int_t^b(t-s)^{k-1}w(s)\mathrm{d}s. \end{aligned} \tag{6}$$

Obviously, $\{w_{jk}^{(2)}\}_{k\in\mathbf{N}}$ are sequences of $w-$harmonic functions on $[x_{j-1},x_j]$, for every $j=1,2,3$. Let us define coefficients $A_k^{(2)}(x)$ and $B_k^{(2)}(x)$ by following:

$$A_k^2(x)=(-1)^{k-1}\left[\frac{1}{(k-1)!}\int_a^x(x-s)^{k-1}w(s)ds-Q_{k,x}(x)\right], \tag{7}$$

and

$$B_k^2(x)=(-1)^{k-1}\left[\frac{1}{(k-1)!}\int_x^b(a+b-x-s)^{k-1}w(s)ds+Q_{k,x}(a+b-x)\right]. \tag{8}$$

Let $f:[a,b]\to\mathbf{R}$ be such that $f^{(n-1)}$ exists on $[a,b]$ for some $n\in\mathbf{N}$. We introduce the following notation:

$$\begin{aligned} T_{n,w}^{(2)}(x) &= 0, \quad \text{for } n=1, \\ T_{n,w}^{(2)}(x) &:= \sum_{k=2}^n\left[A_k^{(2)}(x)f^{(k-1)}(x)+B_k^{(2)}(x)f^{(k-1)}(a+b-x)\right], \quad \text{for } n\ge 2. \end{aligned}$$

**Theorem 1.1.** *Let $f:[a,b]\to\mathbf{R}$ be such that $f^{(n)}$ is piecewise continuous on $[a,b]$, for some $n\in\mathbf{N}$. Then*

$$\int_a^b w(t)f(t)dt=A_1^{(2)}(x)f(x)+B_1^{(2)}(x)f(a+b-x)+T_{n,w}^{(2)}(x)+(-1)^n\int_a^b W_{n,w}^{(2)}(t,x)f^{(n)}(t)dt, \tag{9}$$

*where*

$$W_{n,w}(t,x) = \begin{cases} w_{1n}^{(2)}(t) & \text{for } t \in [a,x], \\ w_{2n}^{(2)}(t) & \text{for } t \in (x, a+b-x], \\ w_{3n}^{(2)}(t) & \text{for } t \in (a+b-x, b]. \end{cases}$$

Assume $(p,q)$ is a pair of conjugate exponents, that is $1 \le p, q \le \infty, \frac{1}{p} + \frac{1}{q} = 1$. Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n)} \in L_p[a,b]$. Then we have

$$\left| \int_a^b w(t)f(t)dt - A_1^{(2)}(x)f(x) - B_1^{(2)}(x)f(a+b-x) - T_{n,w}^{(2)}(x) \right| \le C_2(n,q,x,w) \cdot \|f^{(n)}\|_p, \tag{10}$$

where for $1 \le q < \infty$

$$C_2(n,q,x,w) = \frac{1}{(n-1)!} \left[ \int_a^x \left| w_{1n}^{(2)}(t) \right|^q dt + \int_x^{a+b-x} \left| w_{2n}^{(2)}(t) \right|^q dt + \int_{a+b-x}^b \left| w_{3n}^{(2)}(t) \right|^q dt \right]^{\frac{1}{q}}, \tag{11}$$

and for $q = \infty$

$$C_2(n,\infty,x,w) = \frac{1}{(n-1)!} \max\left\{ \sup_{t\in[a,x]} \left| w_{1n}^{(2)}(t) \right|, \sup_{t\in[x,a+b-x]} \left| w_{2n}^{(2)}(t) \right|, \sup_{t\in[a+b-x,b]} \left| w_{3n}^{(2)}(t) \right| \right\}. \tag{12}$$

The inequality is the best possible for $p = 1$, and sharp for $1 < p \le \infty$.

For two Lebesque integrable function s $f, g : [a,b] \to \mathbf{R}$ let us consider the Čhebyšev functional:

$$T(f,g) := \frac{1}{b-a} \int_a^b f(t)g(t)dt - \frac{1}{b-a} \int_a^b f(t)dt \cdot \frac{1}{b-a} \int_a^b g(t)dt. \tag{13}$$

P. Cerone and S.S. Dragomir have obtained in [4] the following bounds for Čebyšev functional $T(\varphi, \varphi)$:

**Lemma 1.2.** *If $\varphi : [a,b] \to \mathbf{R}$ is an absolutely continuous function with*

$$(\cdot - a)(b - \cdot)(\varphi')^2 \in L[a,b],$$

*then we have the inequality*

$$T(\varphi,\varphi) \le \frac{1}{2(b-a)} \int_a^b (x-a)(b-x) \left[\varphi'(x)\right]^2 dx. \tag{14}$$

*The constant $\frac{1}{2}$ is the best possible.*

The following inequality of Grüss type has been also delivered in [4]:

**Theorem 1.3.** *Assume that $g : [a,b] \to \mathbf{R}$ is monotonic nondecreasing on $[a,b]$ and $f : [a,b] \to \mathbf{R}$ is absolutely continuous with $f' \in L_\infty[a,b]$. Then we have the inequality*

$$|T(f,g)| \le \frac{1}{2(b-a)} ||f'||_\infty \int_a^b (x-a)(b-x)dg(x). \tag{15}$$

*The constant $\frac{1}{2}$ is the best possible.*

In [5] authors have proved the following inequality of Grüss type:

**Lemma 1.4.** *Let $a \le x$ and let $g, h : [a, x] \to \mathbf{R}$ be two integrable functions. If*

$$\alpha \le g(t) \le A, \quad \forall t \in [a, x],$$

*for some constants $\alpha$ and $A$, then*

$$|T(g,h)| \le \frac{1}{2}(A-\alpha)\sqrt{T(h,h)}. \tag{16}$$

In this paper we shall give new bounds for general one-point (1) and twopoint formula (9) by using upper results for Chebyshev functional.

## 2. Main result

Let us apply identity (13) for $f \leftrightarrow (-1)^n W_{n,w}(\cdot, x)$ and $g \leftrightarrow f^{(n)}$ :

$$\begin{aligned} T((-1)^n W_{n,w}(\cdot,x), f^{(n)}) &= \frac{1}{b-a}\int_a^b (-1)^n W_{n,w}(t,x) f^{(n)}(t)dt \qquad (17)\\ &- \frac{1}{b-a}\int_a^b (-1)^n W_{n,w}(t,x)dt \cdot \frac{1}{b-a}\int_a^b f^{(n)}(t)dt. \end{aligned}$$

**Theorem 2.1.** *Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n)}$ is absolutely continuous function, $w : [a,b] \to [0,\infty)$ is weight function and $x \in [a,b]$. Then the following identity holds:*

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(t)w(t)dt - \frac{1}{b-a}\sum_{j=1}^n A_{w,j}(x) f^{(j-1)}(x) - \frac{A_{w,n}(x)}{(b-a)^2}\left(f^{(n-1)}(b) - f^{(n-1)}(a)\right) \\ &= T((-1)^n W_{n,w}(\cdot,x), f^{(n)}). \qquad (18) \end{aligned}$$

*Proof.* Divide relation (1) by $b-a$ and add to both sides of new identity term

$$-\frac{1}{b-a}\int_a^b (-1)^n W_{n,w}(t,x)dt \cdot \frac{1}{b-a}\int_a^b f^{(n)}(t)dt = -\frac{A_{w,n}(x)}{(b-a)^2}\left(f^{(n-1)}(b) - f^{(n-1)}(a)\right).$$

Now the right side of the identity appears to be Chebyshev functional $T((-1)^n W_{n,w}(\cdot,x), f^{(n)})$ so the theorem is proved. □

If we apply identity (13) for $f \leftrightarrow (-1)^n W^{(2)}_{n,w}(\cdot, x)$ and $g \leftrightarrow f^{(n)}$, then we get

$$\begin{aligned} T((-1)^n W^{(2)}_{n,w}(\cdot,x), f^{(n)}) &= \frac{1}{b-a}\int_a^b (-1)^n W^{(2)}_{n,w}(t,x) f^{(n)}(t)dt \qquad (19)\\ &- \frac{1}{b-a}\int_a^b (-1)^n W^{(2)}_{n,w}(t,x)dt \cdot \frac{1}{b-a}\int_a^b f^{(n)}(t)dt. \end{aligned}$$

**Theorem 2.2.** *Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n)}$ is absolutely continuous function, $w : [a,b] \to [0,\infty)$ is weight function and $x \in \left[a, \frac{a+b}{2}\right]$. Then the following identity holds:*

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(t)w(t)dt - \frac{1}{b-a}\left[A^{(2)}_1(x)f(x) + B^{(2)}_1(x)f(a+b-x) + T^{(2)}_{n,w}(x)\right] \\ &-\frac{A^{(2)}_n(x) + B^{(2)}_n(x)}{(b-a)^2}\left(f^{(n-1)}(b) - f^{(n-1)}(a)\right) = T((-1)^n W^{(2)}_{n,w}(\cdot,x), f^{(n)}). \qquad (20) \end{aligned}$$

*Proof.* Divide relation (9) by $b-a$ and add to both sides of new identity term

$$-\frac{1}{b-a}\int_a^b(-1)^n W^{(2)}_{n,w}(t,x)dt\cdot\frac{1}{b-a}\int_a^b f^{(n)}(t)dt=-\frac{A^{(2)}_n(x)+B^{(2)}_n(x)}{(b-a)^2}\Big(f^{(n-1)}(b)-f^{(n-1)}(a)\Big).$$

Now the right side of the identity appears to be Chebyshev functional $T((-1)^n W^{(2)}_{n,w}(\cdot,x),f^{(n)})$ so the theorem is proved. □

Now we establish error bounds for $T((-1)^n W_{n,w}(\cdot,x),f^{(n)})$ and $T((-1)^n W^{(2)}_{n,w}(\cdot,x),f^{(n)})$.

**Theorem 2.3.** *Let $f:[a,b]\to\mathbf{R}$ be such that $f^{(n)}$ is absolutely continuous function with*

$$(\cdot-a)(b-\cdot)\left(f^{(n+1)}\right)^2\in L[a,b],$$

*$w:[a,b]\to[0,\infty)$ is weight function and $x\in[a,b]$. Then we have*

$$\begin{aligned}\left|T((-1)^n W_{n,w}(\cdot,x),f^{(n)})\right| &\le \frac{1}{b-a}\left[\left(C_1\left(n,2,x,w\right)\right)^2-\frac{\left(A_{n+1,w}(x)\right)^2}{b-a}\right]^{1/2}\\ &\quad\cdot\left[\frac{1}{2}\int_a^b\left(t-a\right)\left(b-t\right)f^{(n+1)}(t)^2dt\right]^{1/2}. \qquad (21)\end{aligned}$$

*Proof.* The proof follows from Cauchy-Schwartz's inequality

$$T\left(\varphi,\psi\right)^2\le T\left(\varphi,\varphi\right)\cdot T\left(\psi,\psi\right)$$

applied to $T((-1)^n W_{n,w}(\cdot,x),f^{(n)})$. So we have

$$\left|T((-1)^n W_{n,w}(\cdot,x),f^{(n)})\right|\le\sqrt{T((-1)^n W_{n,w}(\cdot,x),(-1)^n W_{n,w}(\cdot,x))}\cdot\sqrt{T(f^{(n)},f^{(n)})}.$$

We compute

$$\begin{aligned}&T((-1)^n W_{n,w}(\cdot,x),(-1)^n W_{n,w}(\cdot,x))\\ &\quad=\frac{1}{b-a}\int_a^b(-1)^{2n}W_{n,w}\left(t,x\right)^2dt-\frac{1}{(b-a)^2}\left[\int_a^b(-1)^n W_{n,w}\left(t,x\right)dt\right]^2\\ &\quad=\frac{\left[C_1\left(n,2,x,w\right)\right]^2}{b-a}-\frac{\left[A_{n+1,w}(x)\right]^2}{(b-a)^2}.\end{aligned}$$

On the other side, according to the Lemma 1.2. we have

$$T\left(f^{(n)},f^{(n)}\right)\le\frac{1}{2(b-a)}\int_a^b\left(t-a\right)\left(b-t\right)f^{(n+1)}(t)^2dt,$$

which finishes the proof. □

**Theorem 2.4.** *Let $f:[a,b]\to\mathbf{R}$ be such that $f^{(n)}$ is absolutely continuous function with*

$$(\cdot-a)(b-\cdot)\left(f^{(n+1)}\right)^2\in L[a,b],$$

*$w : [a, b] \to [0, \infty)$ is weight function and $x \in \left[a, \frac{a+b}{2}\right]$. Then we have*

$$\left|T((-1)^n W_{n,w}^{(2)}(\cdot, x), f^{(n)})\right| \leq \frac{1}{b-a}\left[\left(C_2\left(n, 2, x, w\right)\right)^2 - \frac{\left(A_{n+1}^{(2)}(x) + B_{n+1}^{(2)}(x)\right)^2}{b-a}\right]^{1/2}$$
$$\cdot \left[\frac{1}{2}\int_a^b \left(t-a\right)\left(b-t\right) f^{(n+1)}(t)^2 dt\right]^{1/2}. \tag{22}$$

*Proof.* The proof follows from Cauchy-Schwartz's inequality

$$T\left(\varphi, \psi\right)^2 \leq T\left(\varphi, \varphi\right) \cdot T\left(\psi, \psi\right)$$

applied to $T((-1)^n W_{n,w}^{(2)}(\cdot, x), f^{(n)})$. So we have

$$\left|T((-1)^n W_{n,w}^{(2)}(\cdot, x), f^{(n)})\right| \leq \sqrt{T((-1)^n W_{n,w}^{(2)}(\cdot, x), (-1)^n W_{n,w}^{(2)}(\cdot, x))} \cdot \sqrt{T(f^{(n)}, f^{(n)})}.$$

We compute

$$\begin{aligned} &T((-1)^n W_{n,w}^{(2)}(\cdot, x), (-1)^n W_{n,w}^{(2)}(\cdot, x)) \\ &= \frac{1}{b-a}\int_a^b (-1)^{2n} W_{n,w}^{(2)}\left(t, x\right)^2 dt - \frac{1}{(b-a)^2}\left[\int_a^b (-1)^n W_{n,w}^{(2)}\left(t, x\right) dt\right]^2 \\ &= \frac{\left[C_2\left(n, 2, x, w\right)\right]^2}{b-a} - \frac{\left[A_{n+1}^{(2)}(x) + B_{n+1}^{(2)}\right]^2}{(b-a)^2}. \end{aligned}$$

On the other side, according to the Lemma 1.2. we have

$$T\left(f^{(n)}, f^{(n)}\right) \leq \frac{1}{2(b-a)}\int_a^b \left(t-a\right)\left(b-t\right) f^{(n+1)}(t)^2 dt,$$

which finishes the proof. □

## 3. Applications for special weight functions

In this section we shall apply results from the Theorem 2.1 and Theorem 2.3 to some special weight functions such as $w(t) = 1, t \in [a, b]$; $w(t) = \frac{1}{\sqrt{1-t^2}}, t \in \langle -1, 1\rangle$; $w(t) = \sqrt{1-t^2}, t \in [-1, 1]$; $w(t) = \sqrt{\frac{1-t}{1-t}}$, $t \in \langle -1, 1\rangle$; $w(t) = \sqrt{t}, t \in [0, 1]$ and $w(t) = \frac{1}{\sqrt{t}}, t \in \langle 0, 1]$.

**Example 3.1.** For the weight $w(t) = 1$, $t \in [a, b]$ we have (see [2],4.1):

$$W_{n,w}(t, x) = \begin{cases} w_{1n}(t) = \frac{(t-a)^n}{n!} & \text{for } t \in [a, x], \\ w_{2n}(t) = \frac{(t-b)^n}{n!} & \text{for } t \in (x, b], \end{cases}$$

$$A_{w,j}(x) = \frac{(b-x)^j - (a-x)^j}{j!}$$

and

$$C_1(n, 2, x, 1) = \frac{1}{n!}\left[\frac{(x-a)^{2n+1} + (b-x)^{2n+1}}{2n+1}\right]^{\frac{1}{2}},$$

so identity (18) becomes

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(t)dt &- \frac{1}{b-a}\sum_{j=1}^{n}\frac{(b-x)^j-(a-x)^j}{j!}f^{(j-1)}(x)\\ &- \frac{(b-x)^n-(a-x)^n}{n!(b-a)^2}\Big(f^{(n-1)}(b)-f^{(n-1)}(a)\Big)\\ &= T((-1)^n W_{n,1}(\cdot,x),f^{(n)}).\end{aligned}$$

If the assumptions of Theorem 2.3 hold, then we have

$$\begin{aligned}&\left|T((-1)^n W_{n,1}(\cdot,x),f^{(n)})\right|\\ &\le \frac{1}{(b-a)n!}\left[\frac{(x-a)^{2n+1}+(b-x)^{2n+1}}{2n+1}-\frac{(b-x)^{n+1}-(a-x)^{n+1}}{(b-a)(n+1)^2}\right]^{1/2}\\ &\cdot\left[\frac{1}{2}\int_a^b (t-a)(b-t)\,f^{(n+1)}(t)^2dt\right]^{1/2}.\end{aligned}$$

In particular, for $n=1$ we have

$$\frac{1}{b-a}\int_a^b f(t)dt - f(x) - \frac{f(b)-f(a)}{b-a} = T(-W_{1,1}(\cdot,x),f')$$

and

$$\begin{aligned}|T(-W_{1,1}(\cdot,x),f')| &\le \frac{1}{b-a}\left[\frac{(x-a)^3+(b-x)^3}{3}-\frac{a+b-2x}{4}\right]^{1/2}\\ &\quad\cdot\left[\frac{1}{2}\int_a^b (t-a)(b-t)\,f''(t)^2dt\right]^{1/2},\end{aligned}$$

which for $x=\frac{a+b}{2}$ becomes

$$\left|T(-W_{1,1}(\cdot,\frac{a+b}{2}),f')\right| \le \frac{\sqrt{b-a}}{2\sqrt{3}}\cdot\left[\frac{1}{2}\int_a^b (t-a)(b-t)\,f''(t)^2dt\right]^{1/2}.$$

**Example 3.2.** For the weight $w(t)=\frac{1}{\sqrt{1-t^2}}$, $t\in\langle -1,1\rangle$ we have (see [2],4.2.1):

$$A_{w,j}(x)=\begin{cases}\frac{(-1-x)^{j-1}}{(j-1)!}F(1-j,\frac{1}{2},1;\frac{2}{x+1}), & x\ne -1,\\ \frac{2^{j-1}}{(j-1)!}B(\frac{1}{2},j-\frac{1}{2}), & x=-1\end{cases}$$

and

$$W_{n,w}(t,x)=\begin{cases}\frac{2^{-\frac{1}{2}}(t+1)^{n-\frac{1}{2}}}{(n-1)!}B(\frac{1}{2},n)F(\frac{1}{2},\frac{1}{2},\frac{1}{2}+n;\frac{t+1}{2}) & \text{for } t\in[-1,x],\\ (-1)^n\frac{2^{-\frac{1}{2}}(1-t)^{n-\frac{1}{2}}}{(n-1)!}B(\frac{1}{2},n)F(\frac{1}{2},\frac{1}{2},\frac{1}{2}+n;\frac{1-t}{2}) & \text{for } t\in(x,1],\end{cases}$$

where

$$B(u,v)=\int_0^1 s^{u-1}(1-s)^{v-1}ds$$

is the Beta function, and

$$F(\alpha,\beta,\gamma;z)=\frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}dt,$$

for $\gamma>\beta>0$ and $z<1$ is the hypergeometric function. We also use notation of the hypergeometric function when integral $\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}dt$ converges. Specially, when $\alpha<0$, then any $z\in\mathbf{R}$ is allowed.

Also $C_1(n,2,x,w)=\frac{1}{(n-1)!}\left[\int_{-1}^x\left|\int_{-1}^t\frac{(t-s)^{n-1}}{\sqrt{1-s^2}}ds\right|^2dt+\int_x^1\left|\int_1^t\frac{(t-s)^{n-1}}{\sqrt{1-s^2}}ds\right|^2dt\right]^{\frac{1}{2}}$.

As a special for $n=1$, $x=0$, we have $C_1(1,2,0,w)=\sqrt{(2\pi-4)}$, $A_{w,1}(x)=\pi$, and

$$W_{1,w}(t,0)=\begin{cases}\sin^{-1}t+\dfrac{\pi}{2}, & t\in[-1,0]\\ \sin^{-1}t-\dfrac{\pi}{2}, & t\in(0,1]\end{cases}$$

which gives $T\left(-W_{1,w}\left(.,0\right),f'\right)=\frac{1}{2}\int_{-1}^1\frac{f(t)}{\sqrt{1-t^2}}dt+\frac{\pi}{4}\left[f(-1)-2f(0)-f(1)\right],$

where $\left|T\left(-W_{1,w}\left(.,0\right),f'\right)\right|\le\frac{\sqrt{\pi-2}}{2}\left[\int_{-1}^1\left(1-t^2\right)\left(f''(t)\right)^2\right]^{\frac{1}{2}}$.

**Example 3.3.** For the weight $w(t)=\frac{1}{\sqrt{1-t^2}}$, $t\in\langle-1,1\rangle$ we have (see [2],4.2.1):

$$A_{w,j}(x)=\begin{cases}\frac{(-1-x)^{j-1}}{(j-1)!}F(1-j,\frac{1}{2},1;\frac{2}{x+1}), & x\neq-1,\\ \frac{2^{j-1}}{(j-1)!}B(\frac{1}{2},j-\frac{1}{2}), & x=-1\end{cases}$$

and

$$W_{n,w}(t,x)=\begin{cases}\frac{2^{-\frac{1}{2}}(t+1)^{n-\frac{1}{2}}}{(n-1)!}B(\frac{1}{2},n)F(\frac{1}{2},\frac{1}{2},\frac{1}{2}+n;\frac{t+1}{2}) & \text{for } t\in[-1,x],\\ (-1)^n\frac{2^{-\frac{1}{2}}(1-t)^{n-\frac{1}{2}}}{(n-1)!}B(\frac{1}{2},n)F(\frac{1}{2},\frac{1}{2},\frac{1}{2}+n;\frac{1-t}{2}) & \text{for } t\in(x,1],\end{cases}$$

where

$$B(u,v)=\int_0^1 s^{u-1}(1-s)^{v-1}ds$$

is the Beta function, and

$$F(\alpha,\beta,\gamma;z)=\frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}dt,$$

for $\gamma>\beta>0$ and $z<1$ is the hypergeometric function. We also use notation of the hypergeometric function when integral $\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}dt$ converges. Specially, when $\alpha<0$, then any $z\in\mathbf{R}$ is allowed.

Also $C_1(n,2,x,w)=\frac{1}{(n-1)!}\left[\int_{-1}^x\left|\int_{-1}^t\frac{(t-s)^{n-1}}{\sqrt{1-s^2}}ds\right|^2dt+\int_x^1\left|\int_1^t\frac{(t-s)^{n-1}}{\sqrt{1-s^2}}ds\right|^2dt\right]^{\frac{1}{2}}$.

As a special for $n=1$, $x=0$, we have $C_1(1,2,0,w)=\sqrt{(2\pi-4)}$, $A_{w,1}(x)=\pi$, and

$$W_{1,w}(t,0)=\begin{cases}\sin^{-1}t+\dfrac{\pi}{2}, & t\in[-1,0]\\ \sin^{-1}t-\dfrac{\pi}{2}, & t\in(0,1]\end{cases}$$

which gives $T\left(-W_{1,w}\left(.,0\right),f'\right)=\frac{1}{2}\int_{-1}^{1}\frac{f(t)}{\sqrt{1-t^2}}dt+\frac{\pi}{4}\left[f(-1)-2f(0)-f(1)\right],$

where $\left|T\left(-W_{1,w}\left(.,0\right),f'\right)\right|\leq\frac{\sqrt{\pi-2}}{2}\left[\int_{-1}^{1}\left(1-t^2\right)\left(f''(t)\right)^2\right]^{\frac{1}{2}}.$

**Example 3.4.** Here $w(t)=\sqrt{\frac{1-t}{1-t}}$, $t\in\langle-1,1\rangle$ we have (see [2],4.2.3):

$$A_{w,j}(x)=\begin{cases}\frac{2(-1-x)^{j-1}}{(j-1)!}B(\frac{3}{2},\frac{1}{2})F(1-j,\frac{1}{2},2;\frac{2}{x+1}), & x\neq-1,\\ \frac{2^j}{(j-1)!}B(\frac{3}{2},j-\frac{1}{2}), & x=-1\end{cases}$$

and

$$W_{n,w}(t,x)=\begin{cases}\frac{2^{\frac{1}{2}}(t+1)^{n-\frac{1}{2}}}{(n-1)!}B(\frac{1}{2},n)F(-\frac{1}{2},\frac{1}{2},\frac{1}{2}+n;\frac{t+1}{2}) & \text{for } t\in[-1,x],\\ (-1)^n\frac{2^{-\frac{1}{2}}(1-t)^{n+\frac{1}{2}}}{(n-1)!}B(\frac{3}{2},n)F(\frac{1}{2},\frac{3}{2},\frac{3}{2}+n;\frac{1-t}{2}) & \text{for } t\in(x,1],\end{cases}$$

for this example we get

$$C_1(n,2,x,w)=\frac{1}{(n-1)!}\left[\int_{-1}^{x}\left|\int_{-1}^{t}(t-s)^{n-1}\sqrt{\frac{1-s}{1+s}}ds\right|^2dt+\int_{x}^{1}\left|\int_{1}^{t}(t-s)^{n-1}\sqrt{\frac{1-s}{1+s}}ds\right|^2dt\right]^{\frac{1}{2}}$$

Then for $n=1$, $x=0$, we have $C_1(1,2,0,w)=\sqrt{\frac{6\pi-8}{3}}$,$A_{w,1}(x)=\pi$, $A_{w,2}(x)=-\frac{\pi}{2}$,

$$W_{1,w}(t,0)=\begin{cases}\dfrac{\pi}{2}+\sqrt{1-t^2}+\sin^{-1}t, & t\in[-1,0]\\ \dfrac{\pi}{2}+\sqrt{1-t^2}+\sin^{-1}t, & t\in(0,1]\end{cases}$$

which gives $T\left(-W_{1,w}\left(.,0\right),f'\right)=\frac{1}{2}\int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}f(t)dt+\frac{\pi}{4}\left[f(-1)-2f(0)-f(1)\right],$

where $\left|T\left(-W_{1,w}\left(.,0\right),f'\right)\right|\leq\frac{\sqrt{48\pi-64-3\pi^2}}{8\sqrt{3}}\left[\int_{-1}^{1}\left(1-t^2\right)\left(f''(t)\right)^2\right]^{\frac{1}{2}}.$

**Example 3.5.** Here $w(t)=\sqrt{t}$, $t\in[0,1]$ we have (see [2],4.2.4):

$$A_{w,j}(x)=\begin{cases}\frac{(-x)^j}{(j-1)!}B(1,\frac{3}{2})F(1-j,\frac{3}{2},\frac{5}{2},\frac{1}{x}), & x\neq0,\\ \frac{1}{(j-1)!}B(1,j+\frac{1}{2}), & x=0\end{cases}$$

and

$$W_{n,w}(t,x)=\begin{cases}\frac{t^{n+\frac{1}{2}}}{(n-1)!}B(n,\frac{3}{2}) & \text{for } t\in[0,x],\\ \frac{(t-1)^n}{n!}F(-\frac{1}{2},1,n+1;1-t) & \text{for } t\in(x,1],\end{cases}$$

which gives $T\left(-W_{1,w}\left(.,0\right),f'\right)=\int_{0}^{1}\sqrt{t}f(t)dt-\frac{2}{3}f(1),$

where $\left|T\left(-W_{1,w}\left(.,0\right),f'\right)\right|\leq\frac{1}{10}\left[\int_{0}^{1}\left(t-t^2\right)\left(f''(t)\right)^2\right]^{\frac{1}{2}}.$

**Example 3.6.** Here $w(t)=\frac{1}{\sqrt{t}}$, $t\in\langle0,1]$ we have (see [2],4.2.5):

$$A_{w,j}(x)=\begin{cases}\frac{2(-x)^j}{(j-1)!}F(1-j,\frac{1}{2},\frac{3}{2};\frac{1}{x}), & x\neq0,\\ \frac{2}{(2j-1)(j-1)!}, & x=0\end{cases}$$

and

$$W_{n,w}(t,x)=\begin{cases}\frac{t^{n-\frac{1}{2}}}{(n-1)!}B(n,\frac{1}{2}) & \text{for } t\in\langle 0,x],\\ \frac{(t-1)^n}{n!}F(\frac{1}{2},1,n+1;1-t) & \text{for } t\in\langle x,1],\end{cases}$$

which gives $T\left(-W_{1,w}\left(.,0\right),f'\right)=\int_0^1\frac{f(t)}{\sqrt{t}}dt-2f(1)$,
where $\left|T\left(-W_{1,w}\left(.,0\right),f'\right)\right|\leq\frac{1}{3}\left[\int_0^1\left(t-t^2\right)\left(f''(t)\right)^2\right]^{\frac{1}{2}}$.

## 4. Applications to the two points formula

In this section we shall apply results from the Theorem 2.2 and Theorem 2.4 for the two point formula to a special weight function $w(t)=1, t\in[a,(a+b)/2]$ for $n=1$ and for $x=a, x=\frac{a+3b}{4}$ and $x=\frac{a+b}{2}$ and will find bounds in these cases:

**Example 4.1.** For $x=a$, we have $A_1^{(2)}(a)=B_1^{(2)}(a)=\frac{b-a}{2}$, $A_2^{(2)}(a)=B_2^{(2)}(a)=\frac{-(b-a)^2}{8}$, where as $C_2(1,2,a,1)=\sqrt{\frac{(b-a)^3}{8}}$, which gives
$T\left(-W_{1,1}^{(2)}(.,a),f'\right)=\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{2}\left(f(a)+f(b)\right)-\frac{1}{b-a}\left(f(b)-f(a)\right)$,
where $T\left(-W_{1,1}^{(2)}(.,a),f'\right)\leqslant\frac{\sqrt{b-a}}{4\sqrt{2}}\left(\int_a^b(t-a)(b-t)(f''(t))^2dt\right)^{1/2}$.

**Example 4.2.** For $x=\frac{3a+b}{4}$, we have $A_1^{(2)}(\frac{3a+b}{4})=B_1^{(2)}(\frac{3a+b}{4})=\frac{b-a}{2}$, $A_2^{(2)}(\frac{3a+b}{4})=B_2^{(2)}(\frac{3a+b}{4})=0$, where as $C_2(1,2,\frac{3a+b}{4},1)=\sqrt{\frac{5(b-a)^3}{192}}$, which gives
$T\left(-W_{1,1}^{(2)}(.,\frac{3a+b}{4}),f'\right)=\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{2}\left(f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right)-\frac{1}{b-a}\left(f(b)-f(a)\right)$,
where
$T\left(-W_{1,1}^{(2)}(.,\frac{3a+b}{4}),f'\right)\leqslant\frac{\sqrt{5(b-a)}}{8\sqrt{6}}\left(\int_a^b(t-a)(b-t)(f''(t))^2dt\right)^{1/2}$.

**Example 4.3.** For $x=\frac{a+b}{2}$, we have $A_1^{(2)}(\frac{a+b}{2})=B_1^{(2)}(\frac{a+b}{2})=\frac{b-a}{2}$, $A_2^{(2)}(\frac{a+b}{2})=B_2^{(2)}(\frac{a+b}{2})=\frac{(b-a)^2}{8}$, where as $C_2(1,2,\frac{a+b}{2},1)=\sqrt{\frac{(b-a)^3}{12}}$, which gives
$T\left(-W_{1,1}^{(2)}(.,\frac{a+b}{2}),f'\right)=\frac{1}{b-a}\int_a^b f(t)dt-f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\left(f(b)-f(a)\right)$,
where
$T\left(-W_{1,1}^{(2)}(.,\frac{a+b}{2}),f'\right)\leqslant\frac{\sqrt{(b-a)}}{4\sqrt{6}}\left(\int_a^b(t-a)(b-t)(f''(t))^2dt\right)^{1/2}$.

## 5. Conclusion

If we make a comparison of the results obtained in this section, it can be concluded that the optimal error bound is achieved for $x=\frac{a+b}{2}$, i.e. for the generalization of the midpoint quadrature formula, and the worst error bound is obtained for $x=a$, i.e. generalized trapezoidal formula.

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