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The Gauss-Airy functions and their properties

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Abstract. In this paper, in connection with the generating function of three-variable Hermite polynomials, we introduce the Gauss-Airy function

$$\mathrm{GAi}(x;y,z)=\frac{1}{\pi}\int_0^\infty e^{-yt^2}\cos(xt+\frac{zt^3}{3})dt,\quad y\geq 0,\ x,z\in\mathbb{R}.$$

Some properties of this function such as behaviors of zeros, orthogonal relations, corresponding inequalities and their integral transforms are investigated.

Key words and phrases. Gauss-Airy function, Airy function, Inequality, Orthogonality, Integral transforms.

1. Introduction

We consider the three-variable Hermite polynomials

$$ {}_3H_n(x,y,z)=n!\sum_{j=0}^{[\frac{n}{3}]}\sum_{k=0}^{[\frac{n-3j}{2}]}\frac{z^j}{j!}\frac{y^k}{k!(n-2k)!}x^{n-3j-2k}, \tag{1}$$

as the coefficient set of the generating function $e^{xt+yt^2+zt^3}$

$$e^{xt+yt^2+zt^3}=\sum_{n=0}^{\infty}{}_3H_n(x,y,z)\frac{t^n}{n!}. \tag{2}$$

For the first time, these polynomials [8] and their generalization [9] was introduced by Dattoli et al. and later Torre [10] used them to describe the behaviors of the Hermite-Gaussian wavefunctions in optics. These are wavefunctions appeared as Gaussian apodized Airy polynomials [3, 7] in elegant and standard forms. Also, similar families to the generating function (2) have been stated in literature, see [11, 12]. Now in this paper, it is our motivation to introduce the Gauss-Airy function derived from operational relation

$$ {}_3H_n(x,y,z)=e^{y\frac{d^2}{dx^2}+z\frac{d^3}{dx^3}}x^n. \tag{3}$$

For this purpose, in view of the operational calculus of the Mellin transform [4–6], we apply the following integral representation

$$e^{ys^2+zs^3}=\int_{-\infty}^{\infty}e^{s\xi}\mathrm{GAi}(\xi,y,z)\,d\xi, \tag{4}$$

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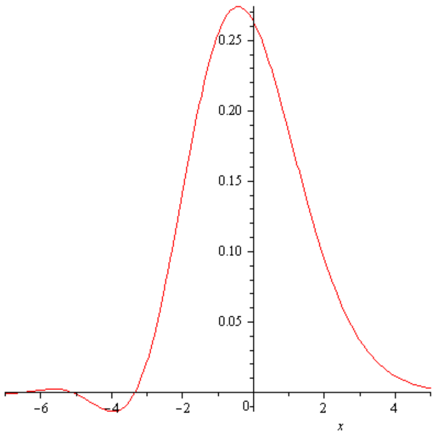

FIGURE 1. The function $\mathrm{GAi}(x)$.

where the function $\mathrm{GAi}(\xi, y, z)$ is given by

$$\mathrm{GAi}(\xi, y, z) = \frac{1}{\pi}\int_0^\infty e^{-yr^2}\cos(r\xi - zr^3)dr. \tag{5}$$

If we set $s = \frac{d}{dx}$ in (4) and incorporate it with the relation (3), we see that the three-variable Hermite polynomials is presented by the following integral representation with respect to the Gauss-Airy function as a convolution product of two distributions

$$_3H_n(x, y, z) = \int_{-\infty}^{\infty} \xi^n \mathrm{GAi}(\xi - x, y, z)\, d\xi. \tag{6}$$

The main object of this paper is devoted to the properties of Gauss-Airy function GAi, which are summarized as the follows. In Section 2, we state the Weierstrass infinite product for the entire function $\mathrm{GAi}(x)$ with respect to its zeros ga_n. We define the Gauss-Airy function zeta and state a formula for obtaining its values. In Section 3, we show that the Gauss-Airy function is logarithmically concave and state some inequality for this function using this property. In Section 4, we get a relationship between the Gauss-Airy function and Airy function, and present orthogonal relations for the Gauss-Airy function with resect to its zeros. In last section, we try to find some identities for the Gauss-Airy function in view of the Mellin, Laplace and Fourier transforms of this function.

2. Zeros of the Gauss-Airy function and the Gauss-Airy zeta function

In this section, for describing the behavior of function $\mathrm{GAi}(x, y, z)$, we confine ourself to the function $\mathrm{GAi}(x) = \mathrm{GAi}(x, 1, -\frac{1}{3})$. For the other parameters y and z, the same analysis can be applied. As we see in Figure 1, $\mathrm{GAi}(x)$ is an oscillatory function for negative x and tends to zero algebraically. We denote the zeros by $ga_n, n = 1, 2, \cdots$, and show the first six roots of this function, see Table 1. Also, we

ga_1	-3.338107410459767038489
ga_2	-5.087949444130970616637
ga_3	-6.52055982809555105913
ga_4	-7.78670809007175899878
ga_5	-8.944133587120853123138
ga_6	-10.022650853340980380158

TABLE 1. The first six real roots of GAi(x).

can write the Weierstrass infinite product for this function as

$$\mathrm{GAi}(x) = \mathrm{GAi}(0)e^{-kx}\prod_{n=1}^{\infty}\left(1+\frac{x}{|ga_n|}\right)e^{-\frac{x}{|ga_n|}}, \quad k = |\frac{\mathrm{GAi}'(0)}{\mathrm{GAi}(0)}| = 0.1763219672. \tag{7}$$

Now, using the above infinite product and zeros ga_n, we define the Gauss-Airy zeta function and evaluate some values for it. Let us start with the following lemma.

Lemma 2.1. *The Gauss-Airy function* GAi(x) *satisfy the following ordinary differential equation*

$$y'' - 2y' - xy = 0, \quad x \in \mathbb{R}. \tag{8}$$

Proof. By using the Laplace integral

$$y(x) = \int_{\Gamma} e^{xr}v(r)dr, \tag{9}$$

and substituting into the ordinary differential equation (8), we get a first order differential equation for $v(r)$ as follows

$$v'(r) + (r^2 - 2r)v(r) = 0. \tag{10}$$

After the deformation and normalization of integral (9), we rewrite y as follows

$$y = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{xr+r^2-\frac{r^3}{3}}dr, \tag{11}$$

which implies that the solution of differential equation (8) is the Gauss-Airy function. □

Remark 2.1. By the same procedure to Lemma 2.1, we can show the Gauss-Airy function GAi(x, a, b) satisfy the following differential equation

$$-by'' - 2ay' - xy = 0, \quad x, b \in \mathbb{R},\ a > 0. \tag{12}$$

Theorem 2.2. *The value of the Gauss-Airy zeta function*

$$\zeta_{ga}(s) = \sum_{n=1}^{\infty}\frac{1}{|ga_n|^s}, \tag{13}$$

is given by

$$\zeta_{ga}(s) = \frac{d^s}{dx^s}\left(\ln(\mathrm{GAi}(x))\right)_{x=0}, \tag{14}$$

and in special case, we get $\zeta_{ga}(2) = k^2 + 2k$ *and* $\zeta_{ga}(3) = \frac{1}{2} - k(k+1)(k+2)$.

Proof. If we take the logarithmic derivative of the Weierstrass infinite product, we obtain

$$\frac{d}{dx}\ln(\mathrm{GAi}(x)) = -k + \sum_{n=1}^{\infty}\left(\frac{1}{x+|ga_n|} - \frac{1}{|ga_n|}\right). \tag{15}$$

Applying the higher order derivatives on the above relation and setting $x = 0$, we get the identity (14). The special values $\zeta_{ga}(2)$ and $\zeta_{ga}(3)$ are given by (14) and differential equation (8) simultaneously. □

3. Some inequalities for the Gauss-Airy function

In this section, we get some inequalities for the Gauss-Airy function. We start with the logarithmic concavity concept and state the following lemma.

Lemma 3.1. *The Gauss-Airy function* $\mathrm{GAi}(x)$ *is logarithmically concave over* (ga_1, ∞)*, where* ga_1 *is the first negative root on real axis.*

Proof. It suffices to show that $\frac{d^2\ln(\mathrm{GAi}(x))}{dx^2} \leq 0$ over the interval (ga_1, ∞). For this purpose, by using the Weierstrass infinite product (7) we see that

$$\frac{d^2\ln(\mathrm{GAi}(x))}{dx^2} = \sum_{n=1}^{\infty} - \frac{1}{(1+\frac{x}{ga_n})^2}, \tag{16}$$

is negative except at the zeros of $\mathrm{GAi}(x)$. □

Remark 3.1. Since the Gauss-Airy function $\mathrm{GAi}(x)$ is logarithmically concave over (ga_1, ∞), then $f(x) = \frac{1}{\mathrm{GAi}(x)}$ is logarithmically convex over and satisfy the following inequality

$$f(ux + (1-u)y) \leq f(x)^u f(y)^{1-u}, \quad 0 \leq u \leq 1,\ x, y > ga_1. \tag{17}$$

Theorem 3.2. *The following inequalities hold for the Gauss-Airy function* $\mathrm{GAi}(x)$

(1) $x\mathrm{GAi}^2(x) \leq \mathrm{GAi}'^2(x), \quad x \in (ga_1, \infty)$.
(2) $\mathrm{GAi}(x)\mathrm{GAi}(y) \leq \mathrm{GAi}^2(\frac{x+y}{2}), \quad x \in (ga_1, \infty)$.
(3) $\mathrm{GAi}^{\alpha}(x)\mathrm{GAi}^{1-\alpha}(0) \leq \mathrm{GAi}(\alpha x), \quad x \in (ga_1, \infty),\ 0 \leq \alpha \leq 1$.
(4) $\mathrm{GAi}(x) \leq |\mathrm{GAi}(z)|, \quad x \in (ga_1, \infty),\ z = x + iy$.
(5) $|\mathrm{GAi}(z)|\mathrm{GAi}(0) \leq \mathrm{GAi}(x)|\mathrm{GAi}(iy)|, \quad x \geq 0,\ z = x + iy$.

Proof. To prove these inequalities, we apply the approaches of [14] which generalize the results of the Airy function. For the inequality (1), we use Lemma 3.1 for the logarithmic concavity of $\mathrm{GAi}(x)$, which implies that

$$\frac{d^2\ln(\mathrm{GAi}(x))}{dx^2} = \frac{x\mathrm{GAi}^2(x) - \mathrm{GAi}'^2(x)}{\mathrm{GAi}^2(x)} \leq 0, \tag{18}$$

and proof is completed. The inequalities (2) and (3) are deduced from Remark 3.1 by setting $u = \frac{1}{2}$ and $u = \alpha, y = 0$ in (17), respectively. To prove (4), using the infinite product representation for $x \in (ga_1, \infty)$, we have

$$\begin{aligned}\frac{\mathrm{GAi}(x)}{|\mathrm{GAi}(x+iy)|} &= \frac{\mathrm{GAi}(0)e^{-kx}\prod_{n=1}^{\infty}\left(1+\frac{x}{|ga_n|}\right)e^{-\frac{x}{|ga_n|}}}{|\mathrm{GAi}(0)e^{-k(x+iy)}\prod_{n=1}^{\infty}\left(1+\frac{x+iy}{|ga_n|}\right)e^{-\frac{x+iy}{|ga_n|}}|} \\ &= \prod_{n=1}^{\infty}\frac{1+\frac{x}{ga_n}}{\sqrt{(1+\frac{x}{ga_n})^2+\frac{y^2}{ga_n^2}}}, \end{aligned} \tag{19}$$

which implies that each factor in the infinite product is less than or equal to unit, and proof is completed. Moreover, since for the each factor in the infinite product we have

$$\frac{1+\frac{x}{ga_n}}{\sqrt{(1+\frac{x}{ga_n})^2+\frac{y^2}{ga_n^2}}} = \frac{1}{\sqrt{1+\frac{y^2}{ga_n^2(1+\frac{x}{ga_n})^2}}} \geq \frac{1}{\sqrt{1+\frac{y^2}{ga_n^2}}}, \tag{20}$$

and

$$|\mathrm{GAi}(iy)| = \mathrm{GAi}(0)\prod_{n=1}^{\infty}\frac{1}{\sqrt{1+\frac{y^2}{ga_n^2}}}. \tag{21}$$

Therefore, we can write the following inequality

$$\frac{\mathrm{GAi}(x)}{|\mathrm{GAi}(z)|} \geq \frac{\mathrm{GAi}(0)}{|\mathrm{GAi}(iy)|}, \tag{22}$$

or equivalently

$$|\mathrm{GAi}(z)|\mathrm{GAi}(0) \leq \mathrm{GAi}(x)|\mathrm{GAi}(iy)|. \tag{23}$$

□

4. Orthogonality of the Gauss-Airy functions

In this section, we intend to introduce an orthonormal basis on the interval $(0, \infty)$ in terms of the Gauss-Airy function and its zeros. Let us start with the following lemma.

Lemma 4.1. *The following relation holds between the Gauss-Airy function and Airy function* $\mathrm{Ai}(x)$

$$\mathrm{Ai}(1+x) = e^{-(x+\frac{2}{3})}\mathrm{GAi}(x), \tag{24}$$

where the Airy function is given by [1, 2]

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^{\infty}\cos(xr+\frac{r^3}{3})dr. \tag{25}$$

Proof. If we use the differential equation of Gauss-Airy function

$$y'' - 2y' - xy = 0, \quad x \in \mathbb{R}, \tag{26}$$

and set $z(x) = Ce^{-x}y(x)$, then the above differential equation changes to the following Airy type differential equation

$$z'' - (1+x)z = 0, \quad x \in \mathbb{R}, \tag{27}$$

with the solution $z(x) = \mathrm{Ai}(1+x)$. After matching the Gauss-Airy function and Airy function at zero point, we get $C = e^{-\frac{2}{3}}$ and the relation (24) is obtained. □

Lemma 4.2. *By the same procedure to Lemma 4.1 and applying differential equation* (12), *we generalize the relation* (24) *as follows*

$$a^{-\frac{1}{3}}\mathrm{Ai}(\frac{b^2}{a^{\frac{4}{3}}}+a^{-\frac{1}{3}}x)=e^{-(\frac{b}{a}x+\gamma)}\mathrm{GAi}(x;b,a), \tag{28}$$

where

$$\gamma=\ln\left(\frac{a^{-\frac{1}{3}}\mathrm{Ai}(\frac{b^2}{a^{\frac{4}{3}}})}{\mathrm{GAi}(0;b,a)}\right). \tag{29}$$

Remark 4.1. The relation (24) implies that the difference between the zeros of Airy function a_n and the zeros of Gauss-Airy function ga_n is unity, i.e., $a_n-ga_n=1, n=1,2,\cdots$.

Theorem 4.3. *The functions* $\{\frac{\mathrm{GAi}(x+ga_n)}{\mathrm{GAi}'(ga_n)}, n\in\mathbb{N}\}$ *form an orthonormal basis on the interval* $(0,\infty)$ *with respect to the weight function* $w(x)=e^{-2x}$ *and zeros* ga_n.

Proof. We recall the following orthonormal basis on the interval $(0,\infty)$ in terms of the Airy function and different zeros a_n and $a_{n'}$ [15, 16]

$$\begin{aligned}\int_0^\infty \mathrm{Ai}(x+a_n)\mathrm{Ai}(x+a_{n'})dx &= 0, \quad n\neq n',\ n,n'\in\mathbb{N}, &(30)\\ \int_0^\infty \mathrm{Ai}^2(x+a_n)dx &= \mathrm{Ai}'^2(a_n). &(31)\end{aligned}$$

At this point, if we substitute the relation (24) into the above integrals, we get an orthonormal basis for the Gauss-Airy function in terms of the zeros ga_n and the weight function $w(x)=e^{-2x}$

$$\begin{aligned}\int_0^\infty e^{-2x}\mathrm{GAi}(x+ga_n)\mathrm{GAi}(x+ga_{n'})dx &= 0, \quad n\neq n',\ n,n'\in\mathbb{N}, &(32)\\ \int_0^\infty e^{-2x}\mathrm{GAi}^2(x+ga_n)dx &= \mathrm{GAi}'^2(ga_n). &(33)\end{aligned}$$

$\square$

Also, in connection with the orthogonality of Airy functions on interval $(-\infty,\infty)$ [16]

$$\int_{-\infty}^\infty \mathrm{Ai}(\frac{\xi+a}{\alpha})\mathrm{Ai}(\frac{\xi+b}{\beta})d\xi=\begin{cases}\delta(b-a), & \alpha=\beta,\\ \frac{|\alpha\beta|}{|\beta^3-\alpha^3|^{\frac{1}{3}}}\mathrm{Ai}\left(\frac{b-a}{(\beta^3-\alpha^3)^{\frac{1}{3}}}\right), & \beta\neq\alpha,\end{cases} \tag{34}$$

we can extend the orthogonality of the Gauss-Airy functions as follows.

Theorem 4.4. *The functions* $\mathrm{GAi}(x)$ *are orthogonal on the interval* $(-\infty,\infty)$ *with respect to the weight function of exponential form as follows*

$$\int_{-\infty}^\infty e^{-\frac{\xi(\alpha+\beta)}{\alpha\beta}}\mathrm{GAi}(\frac{\xi+a-\alpha}{\alpha})\mathrm{GAi}(\frac{\xi+b-\beta}{\beta})d\xi=\begin{cases}\delta(b-a), & \alpha=\beta,\\ I, & \beta\neq\alpha,\end{cases} \tag{35}$$

where

$$I=\frac{|\alpha\beta|}{|\beta^3-\alpha^3|^{\frac{1}{3}}}e^{\frac{a}{\alpha}+\frac{b}{\beta}-\frac{1}{3}-\frac{b-a}{(\beta^3-\alpha^3)^{\frac{1}{3}}}}\mathrm{GAi}\left(\frac{b-a-(\beta^3-\alpha^3)^{\frac{1}{3}}}{(\beta^3-\alpha^3)^{\frac{1}{3}}}\right). \tag{36}$$

5. Integral Transforms of the Gauss-Airy function

In this section, we want to obtain the integral transforms type of the Gauss-Airy function. We focus on the Mellin, Laplace and Fourier transforms of this function in view of the Meijer G functions and Airy functions.

Theorem 5.1. *The Mellin transform of the Gauss-Airy function is given by the following relation*

$$\begin{aligned}\mathcal{M}\{\mathrm{GAi}(x);s\} &= \frac{\Gamma(s)\cos(s\frac{\pi}{2})}{2\sqrt{\pi}}3^{-\frac{s}{2}}G^{1,3}_{3,2}\left(\frac{3}{4}\Big|\begin{matrix}\Delta(3,\frac{s+1}{2})\\ \Delta(1,2),\Delta(1,-\frac{1}{2})\end{matrix}\right)\\ &- \frac{\Gamma(s)\sin(s\frac{\pi}{2})}{2\sqrt{\pi}}3^{-\frac{s}{2}}G^{1,3}_{3,2}\left(\frac{3}{4}\Big|\begin{matrix}\Delta(3,\frac{s+1}{2})\\ \Delta(1,1),\Delta(1,0)\end{matrix}\right), \qquad (37)\end{aligned}$$

where Δ *is denoted by* $\Delta(k,a)=\frac{a}{k},\frac{a+1}{k},\cdots,\frac{a+k-1}{k}$.

Proof. If we use the definition of Mellin transform

$$\mathcal{M}\{f(x);s\}=\int_0^\infty x^{s-1}f(x)dx, \qquad (38)$$

for the Gauss-Airy function $\mathrm{GAi}(x)$

$$\mathrm{GAi}(x)=\frac{1}{\pi}\int_0^\infty e^{-r^2}\cos(rx+\frac{1}{3}r^3)dr,$$

and apply the following facts

$$\int_0^\infty x^{s-1}\cos(rx)dx = \frac{\Gamma(s)}{r^s}\cos(s\frac{\pi}{2}), \qquad (39)$$

$$\int_0^\infty x^{s-1}\sin(rx)dx = \frac{\Gamma(s)}{r^s}\sin(s\frac{\pi}{2}), \qquad (40)$$

then, we get the Mellin transform as follows

$$\begin{aligned}\int_0^\infty x^{s-1}\mathrm{GAi}(x)dx &= \frac{\Gamma(s)\cos(s\frac{\pi}{2})}{\pi}\int_0^\infty r^{-s}e^{-r^2}\cos(\frac{1}{3}r^3)dr\\ &- \frac{\Gamma(s)\sin(s\frac{\pi}{2})}{\pi}\int_0^\infty r^{-s}e^{-r^2}\sin(\frac{1}{3}r^3)dr\\ &= \frac{\Gamma(s)\cos(s\frac{\pi}{2})}{2\pi}\int_0^\infty r^{-\frac{s+1}{2}}e^{-r}\cos(\frac{1}{3}r^{\frac{3}{2}})dr\\ &- \frac{\Gamma(s)\sin(s\frac{\pi}{2})}{2\pi}\int_0^\infty r^{-\frac{s+}{2}1}e^{-r}\sin(\frac{1}{3}r^{\frac{3}{2}})dr.\end{aligned}$$

At this point, we evaluate the improper integrals in above equation in terms of the Meijer G functions for $\Re(s)<2$ as follows [13] (page 82)

$$\begin{aligned}\int_0^\infty x^{s-1}\mathrm{GAi}(x)dx &= \frac{\Gamma(s)\cos(s\frac{\pi}{2})}{2\sqrt{\pi}}3^{-\frac{s}{2}}G^{1,3}_{3,2}\left(\frac{3}{4}\Big|\begin{matrix}\Delta(3,\frac{s+1}{2})\\ \Delta(1,2),\Delta(1,-\frac{1}{2})\end{matrix}\right)\\ &- \frac{\Gamma(s)\sin(s\frac{\pi}{2})}{2\sqrt{\pi}}3^{-\frac{s}{2}}G^{1,3}_{3,2}\left(\frac{3}{4}\Big|\begin{matrix}\Delta(3,\frac{s+1}{2})\\ \Delta(1,1),\Delta(1,0)\end{matrix}\right),\end{aligned}$$

which confirms the relation (37). □

In order to obtain the Laplace and Fourier transforms of the Gauss-Airy function, we recall a well-known integral representation for the product of Airy functions as follows [16]

$$\mathrm{Ai}(u)\mathrm{Ai}(v) = \frac{1}{2^{\frac{1}{3}}\pi}\int_{-\infty}^{\infty} \mathrm{Ai}(2^{\frac{2}{3}}(\xi^2 + \frac{u+v}{2}))e^{i(u-v)\xi}d\xi. \tag{41}$$

Theorem 5.2. *The following Laplace transform type of the Gauss-Airy function holds*

$$\int_0^{\infty} e^{-ux}\frac{1}{\sqrt{x}}\mathrm{GAi}(x;\frac{u}{2},\frac{1}{4})dx = 4\sqrt{2}\pi e^{-\gamma}\mathrm{Ai}^2(u), \tag{42}$$

where the constant γ *is given by* (29).

Proof. Setting $u = v$ in the relation (41), we obtain

$$\mathrm{Ai}^2(u) = \frac{1}{2^{\frac{1}{3}}\pi}\int_{-\infty}^{\infty} \mathrm{Ai}(2^{\frac{2}{3}}(\xi^2 + u))d\xi. \tag{43}$$

In this case, after substituting $a = \frac{1}{4}, b = \frac{u}{2}$ into (28) and using the suitable change of variable we easily arrive at (42). □

Theorem 5.3. *The following Fourier transform types hold for the Gauss-Airy function*

$$e^{-yp^2+izp^3} = \int_{-\infty}^{\infty} e^{-ip\xi}\mathrm{GAi}(\xi, y, z)\, d\xi, \tag{44}$$

$$\mathrm{Ai}(u) = \frac{e^{-\gamma}}{2\pi\mathrm{Ai}(0)}\int_{-\infty}^{\infty} e^{-2u\xi^2 - iu\xi}\mathrm{GAi}(\xi^2, \frac{u}{2}, \frac{1}{4})d\xi, \tag{45}$$

where γ *is given by* (29).

Proof. To prove (44), it suffice to substitute $s = -ip$ into (4). For the relation (45), if we set $v = 0$ in the relation (41), we obtain

$$\mathrm{Ai}(u) = \frac{1}{2^{\frac{1}{3}}\pi\mathrm{Ai}(0)}\int_{-\infty}^{\infty} \mathrm{Ai}(2^{\frac{2}{3}}(\xi^2 + u))e^{iu\xi}d\xi. \tag{46}$$

Now, by substituting the relation (28) into the above integral with $a = \frac{1}{4}, b = \frac{u}{2}$, we get the relation (45).

□

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