Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 43(2), 2016, Pages 128–140
ISSN: 1223-6934

Estimations of the difference between two weighted integral means and application of the Steffensen's inequality

Sanja Kovač, Josip Pečarić, and Anamarija Perušić

Abstract. We present an estimate of the difference between two weighted integral means related to the general one-point integral formula of Matić, Pečarić and Ujević.

Key words and phrases. weighted integral formula, one-point formula, weight functions, L^p inequalities, Steffensen's inequality.

1. Introduction

Let us recall Steffensen's inequality which was introduced and proved in [9]:

Theorem 1.1. *Suppose that f is decreasing and g is integrable on $[a,b]$ with $0 \leq g \leq 1$ and $\lambda = \int_0^1 g(t)dt$. Then we have*

$$\int_{b-\lambda}^{b} f(t)dt \leq \int_a^b f(t)g(t)dt \leq \int_a^{a+\lambda} f(t)dt. \tag{1}$$

The inequalities are reversed for f increasing.

In [4] and [5] the following one-point integral formula is introduced from the general $m-$point integral identity:

$$\int_a^b w(t)f(t)dt = \sum_{j=1}^{n} A_{w,j}(x)f^{(j-1)}(x) + (-1)^n \int_a^b W_{n,w}(t,x)f^{(n)}(t)dt, \tag{2}$$

where $f : [a,b] \to \mathbf{R}$ is such that $f^{(n-1)}$ is absolutely continuous function, $w : [a,b] \to [0,\infty)$ is weight function, $x \in [a,b]$

$$A_{w,j}(x) = \frac{(-1)^{j-1}}{(j-1)!} \int_a^b (x-s)^{j-1} w(s)ds, \quad \text{for } j = 1,\dots,n \tag{3}$$

and

$$W_{n,w}(t,x) = \begin{cases} w_{1n}(t) = \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} w(s)ds & \text{for } t \in [a,x], \\ w_{2n}(t) = \frac{1}{(n-1)!} \int_b^t (t-s)^{n-1} w(s)ds & \text{for } t \in (x,b]. \end{cases} \tag{4}$$

Let us define function $W_{n,w}(t,x)$ outside of the interval $[a,b]$:

$$W_{n,w}(t,x) = 0, \text{ for } t < a$$
$$W_{n,w}(t,x) = W_{n,w}(b,x), \text{ for } t > b.$$

Received March 5, 2015. Accepted December 21, 2015.
This work has been supported in part by Croatian Science Foundation under the project 5435.

Let us define for $n \geq 2$:

$$T_{w,n}^{[a,b]}(x) := \frac{1}{\int_a^b w(t)dt} \sum_{k=2}^{n} A_{w,k}(x) f^{(k-1)}(x)$$

and $T_{w,1}^{[a,b]}(x) = 0$.

In [1] identity (2) is obtained as the extension of the weighted Montgomery identity via Taylor's formula. Also, the difference between two integral means, each having its own weight, w and u defined on two different intervals $[a,b]$ and $[c,d]$ such that $[a,b] \cap [c,d] \neq \emptyset$ is obtained:

$$\begin{aligned} \frac{1}{\int_a^b w(t)dt} & \int_a^b w(t)f(t)dt - \frac{1}{\int_c^d u(t)dt} \int_c^d u(t)f(t)dt - T_{w,n}^{[a,b]}(x) + T_{u,n}^{[c,d]}(x) \\ &= (-1)^n \int_{\min\{a,c\}}^{\max\{b,d\}} K_n(t,x) f^{(n)}(t)dt, \end{aligned} \tag{5}$$

where

$$K_n(t,x) \quad = \quad \frac{W_{n,w}(t,x)}{\int_a^b w(t)dt} - \frac{W_{n,u}(t,x)}{\int_c^d u(t)dt}. \tag{6}$$

Assume (p,q) is a pair of conjugate exponents, $1 \leq p, q \leq \infty$. The following inequality is also obtained in [1]: If $f^{(n)} \in L_p[a,d]$, then we have

$$\left| \int_a^b w(t)f(t)dt - \int_c^d u(t)f(t)dt - T_{w,n}^{[a,b]}(x) + T_{u,n}^{[c,d]}(x) \right| \leq ||K_n(\cdot,x)||_q \cdot ||f^{(n)}||_p.$$

The inequality is sharp for $1 < p \leq \infty$ and the best possible for $p = 1$.

In this paper we deal with $n-$convex functions. The following definitions and theorem can be found in [8].

Definition 1.1. Let f be a real-valued function defined on the segment $[a,b]$. A $k-$th order divided difference of f at distinct points $x_0, \ldots, x_k \in [a,b]$ is defined recursively by

$$\begin{aligned} f[x_i] &= f(x_i), \quad i = 0, \ldots, k \\ f[x_0, \ldots, x_k] &= \frac{f[x_1, \ldots, x_k] - f[x_0, \ldots, x_{k-1}]}{x_k - x_0}. \end{aligned}$$

The value $f[x_0, \ldots, x_k]$ is independent of the order of the points $x_0, \ldots, x_k$. The definition may be extended to include the case that some (or all) of the points coincide by assuming that $x_0 \leq \ldots \leq x_k$ and letting

$$f[\underbrace{x, \ldots, x}_{j \text{ times}}] = \frac{f^{(j)}(x)}{j!}, \tag{7}$$

provided that $f^{(j)}(x)$ exists.

Definition 1.2. A function $f : [a,b] \to \mathbb{R}$ is said to be $n-$convex $(n \geq 0)$ if for all choices of $n+1$ distinct points in $[a,b]$

$$f[x_0, \ldots, x_n] \geq 0.$$

Theorem 1.2. *If $f^{(n)}$ exists, then f is $n-$convex if and only if $f^{(n)} \geq 0$.*

The aim of this paper is to give the extension of the Steffensen's inequality (1) by using identity (5).

2. Generalization of the Steffensen's inequality

We shall start with general inequality for $n-$convex functions.

Theorem 2.1. *Let $f : [a,b] \cup [c,d] \to \mathbb{R}$ be a $n-$convex function, $x \in [a,b] \cap [c,d]$ and let $w : [a,b] \to \mathbb{R}$ and $u : [c,d] \to \mathbb{R}$ be integrable functions (weights). If $(-1)^n K_n(t,x) \geq 0$ for every $t \in [a,b] \cup [c,d]$, then*

$$\frac{1}{\int_a^b w(t)dt} \int_a^b w(t)f(t)dt - \frac{1}{\int_c^d u(t)dt} \int_c^d u(t)f(t)dt \geq T_{w,n}^{[a,b]}(x) - T_{u,n}^{[c,d]}(x). \tag{8}$$

If $(-1)^n K_n(t,x) \leq 0$, for every $t \in [a,b] \cup [c,d]$ inequality (8) is reversed.

Proof. Without loss of generality we may assume that $f^{(n)}$ is continuous (see [8]). The result follows from the identity (5) and Theorem 1.2. □

Theorem 2.2. *Let $f : [a, \max\{b, a+\lambda\}] \to \mathbb{R}$ be a $n-$convex function for $n \geq 1$, $0 \leq \lambda$ and let $w : [a,b] \to [0,\infty\rangle$ be integrable on $[a,b]$. If $x \in [a,b] \cap [a,a+\lambda]$ and $(-1)^n K_n(t,x) \geq 0$ for every $t \in [a, \max\{b, a+\lambda\}]$, then we have*

$$\frac{1}{\int_a^b w(t)dt} \int_a^b w(t)f(t)dt - T_{w,n}^{[a,b]}(x) \geq \frac{1}{\lambda} \int_a^{a+\lambda} f(t)dt - T_{1,n}^{[a,a+\lambda]}(x). \tag{9}$$

Proof. We put $c = a$ and $d = a + \lambda$, so we have two possibilities: $[c,d] \subseteq [a,b]$ and $[a,b] \subseteq [c,d]$.

a) Case $[a, a+\lambda] \subseteq [a,b]$
For $u \equiv 1$ on $[a, a+\lambda]$ we have

$$(-1)^n K_n(t,x) = \begin{cases} (-1)^n \left[\frac{1}{(n-1)! \int_a^b w(t)dt} \int_a^t (t-s)^{n-1} w(s)ds - \frac{(t-a)^n}{n!\lambda} \right], & a \leq t \leq x \\ (-1)^n \left[\frac{1}{(n-1)! \int_a^b w(t)dt} \int_b^t (t-s)^{n-1} w(s)ds - \frac{(t-a-\lambda)^n}{n!\lambda} \right], & x < t \leq a+\lambda \\ (-1)^n \frac{1}{(n-1)! \int_a^b w(t)dt} \int_b^t (t-s)^{n-1} w(s)ds, & a+\lambda < t \leq b. \end{cases}$$

Apply Theorem 2.1 to finish the proof.

b) Case $[a,b] \subseteq [a, a+\lambda]$
For $u \equiv 1$ on $[a, a+\lambda]$ we have

$$(-1)^n K_n(t,x) = \begin{cases} (-1)^n \left[\frac{1}{(n-1)! \int_a^b w(t)dt} \int_a^t (t-s)^{n-1} w(s)ds - \frac{(t-a)^n}{n!\lambda} \right], & a \leq t \leq x \\ (-1)^n \left[\frac{1}{(n-1)! \int_a^b w(t)dt} \int_b^t (t-s)^{n-1} w(s)ds - \frac{(t-a-\lambda)^n}{n!\lambda} \right], & x < t \leq b \\ -\frac{(a+\lambda-t)^n}{n!\lambda}, & b < t \leq a+\lambda. \end{cases}$$

Apply Theorem 2.1 to finish the proof.

□

Let us introduce the following classes of functions for $n > 1$:

$$M_n[a,b] := \left\{ w : [a,b] \to [0,1] : \left(\int_a^b w(t)dt \right)^n \leq n \int_a^b (t-a)^{n-1} w(t)dt \right\}$$

and

$$M_n'[a,b] := \left\{ w : [a,b] \to [0,1] : \left(\int_a^b w(t)dt \right)^n \geq n \int_a^b (t-a)^{n-1} w(t)dt \right\}.$$

Let us denote $W := \int_a^b w(t)dt$.

Corollary 2.3. *Let $w : [a,b] \to [0,1]$ be integrable function on $[a,b]$ and $n > 1$.*

a) *If $\lambda = \int_a^b w(t)dt$ and $f : [a,b] \to \mathbb{R}$ is a $1-$convex function then we have*

$$\int_a^b w(t)f(t)dt \geq \int_a^{a+\lambda} f(t)dt. \tag{10}$$

b) *If $w \in M_n[a,b]$,*

$$\lambda := \left[n \cdot \int_a^b (t-a)^{n-1} w(t)dt \right]^{\frac{1}{n}}$$

and $f : [a, \max\{b, a+\lambda\}] \to \mathbb{R}$ is a $n-$convex function then we have

$$\frac{1}{\int_a^b w(t)dt} \int_a^b w(t)f(t)dt - T_{w,n}^{[a,b]}(a) \geq \frac{1}{\lambda} \int_a^{a+\lambda} f(t)dt - T_{1,n}^{[a,a+\lambda]}(a). \tag{11}$$

c) *If $w \in M_n'[a,b]$,*

$$\lambda := \left[\frac{n}{\int_a^b w(t)dt} \cdot \int_a^b (t-a)^{n-1} w(t)dt \right]^{\frac{1}{n-1}}$$

and $f : [a, \max\{b, a+\lambda\}] \to \mathbb{R}$ is a $n-$convex function then we have

$$\frac{1}{\int_a^b w(t)dt} \int_a^b w(t)f(t)dt - T_{w,n}^{[a,b]}(a) \geq \frac{1}{\lambda} \int_a^{a+\lambda} f(t)dt - T_{1,n}^{[a,a+\lambda]}(a). \tag{12}$$

Proof. a) Since $\lambda = \int_a^b w(t)dt \leq b-a$, we have for $x = a$

$$-K_1(t,a) = \begin{cases} \frac{1}{\int_a^b w(t)dt} \int_t^b w(s)ds - \frac{t-a}{\lambda} + 1, & a < t \leq a+\lambda \\ \frac{1}{\int_a^b w(t)dt} \int_t^b w(s)ds, & a+\lambda < t \leq b. \end{cases}$$

It is easy to check that $-K_1(t,a) \geq 0$, so the assertion follows from Theorem 2.2.

b) In this case we have

$$\lambda = \left[n \cdot \int_a^b (t-a)^{n-1} w(t)dt \right]^{\frac{1}{n}} \leq \left[n \int_a^b (t-a)^{n-1} \right]^{\frac{1}{n}} = \left[n \frac{(b-a)^n}{n} \right]^{\frac{1}{n}} = b-a,$$

so for $x = a$ we have

$$(-1)^n K_n(t,a) = \begin{cases} 0, & t = a \\ \frac{1}{(n-1)! \int_a^b w(t)dt} \int_t^b (s-t)^{n-1} w(s)ds - \frac{(a+\lambda-t)^n}{n!\lambda}, & a < t \leq a+\lambda \\ \frac{1}{(n-1)! \int_a^b w(t)dt} \int_t^b (s-t)^{n-1} w(s)ds, & a+\lambda < t \leq b. \end{cases}$$

Obviously, $(-1)^n K_n(t,a) \geq 0$, for $t \in [a+\lambda, b]$.
In order to prove that $(-1)^n K_n(t,a) \geq 0$, for $t \in \langle a+\lambda, b]$, we shall prove that

$$\frac{1}{(n-1)!\int_a^b w(t)dt}\int_t^b (s-t)^{n-1}w(s)ds \geq \frac{(a+\lambda-t)^n}{n!\lambda}.$$

From the definition of the set M_n it is obvious that $\lambda \geq W$. We compute

$$\begin{aligned}
\frac{1}{(n-1)!W}\int_t^b (s-t)^{n-1}w(s)ds &= \frac{1}{(n-1)!W}\int_t^b \left(\frac{s-t}{s-a}\right)^{n-1}(s-a)^{n-1}w(s)ds \\
&= \frac{1}{(n-1)!W}\int_t^b (n-1)\frac{(s-t)^{n-2}(t-a)}{(s-a)^n}\cdot\left(\int_s^b (z-a)^{n-1}w(z)dz\right)ds \\
&\geq \frac{1}{(n-1)!W}\int_t^{a+\lambda} (n-1)\frac{(s-t)^{n-2}(t-a)}{(s-a)^n}\cdot\left(\int_s^b (z-a)^{n-1}w(z)dz\right)ds \\
&= \frac{1}{(n-1)!W}\int_t^{a+\lambda} (n-1)\frac{(s-t)^{n-2}(t-a)}{(s-a)^n} \\
&\cdot \left(\int_a^b (z-a)^{n-1}w(z)dz - \int_a^s (z-a)^{n-1}w(z)dz\right)ds \\
&\geq \frac{1}{(n-1)!W}\int_t^{a+\lambda} (n-1)\frac{(s-t)^{n-2}(t-a)}{(s-a)^n} \\
&\cdot \left(\int_a^b (z-a)^{n-1}w(z)dz - \int_a^s (z-a)^{n-1}dz\right)ds \\
&= \frac{1}{(n-1)!W}\int_t^{a+\lambda} (n-1)\frac{(s-t)^{n-2}(t-a)}{(s-a)^n} \\
&\cdot \left(\int_a^b (z-a)^{n-1}w(z)dz - \frac{(s-a)^n}{n}\right)ds \\
&\left| u = \frac{1}{(n-1)!W}\left(\int_a^b (z-a)^{n-1}w(z)dz - \frac{(s-a)^n}{n}\right),\right. \\
&\left. dv = (n-1)\frac{(s-t)^{n-2}(t-a)}{(s-a)^n}ds\right| \\
&= \frac{1}{(n-1)!W}\left(\int_a^b (z-a)^{n-1}w(z)dz - \frac{\lambda^n}{n}\right) \\
&\cdot \frac{(a+\lambda-t)^{n-1}}{\lambda^{n-1}} + \frac{(a+\lambda-t)^n}{n!W} \geq \frac{(a+\lambda-t)^n}{n!\lambda}.
\end{aligned}$$

The assertion follows from the Theorem 2.2.

c) In this case we have $\lambda^{n-1}\cdot W = \int_a^b (t-a)^{n-1}w(t)dt \leq W^n$ (since $w \in M_n'[a,b]$), so $\lambda \leq W \leq b-a$. Therefore for $x = a$ we have

$$(-1)^n K_n(t,a) = \begin{cases} \frac{1}{(n-1)!\int_a^b w(t)dt}\int_t^b (s-t)^{n-1}w(s)ds - \frac{(a+\lambda-t)^n}{n!\lambda}, & a \leq t \leq a+\lambda \\ \frac{1}{(n-1)!\int_a^b w(t)dt}\int_t^b (s-t)^{n-1}w(s)ds, & a+\lambda < t \leq b. \end{cases}$$

Obviously, $(-1)^n K_n(t,a) \geq 0$, for $t \in [a+\lambda, b]$.
In order to prove that $(-1)^n K_n(t,a) \geq 0$, for $t \in \langle a+\lambda, b]$, we shall prove that

$$\frac{1}{(n-1)! \int_a^b w(t)dt} \int_t^b (s-t)^{n-1} w(s)ds \geq \frac{(a+\lambda-t)^n}{n!\lambda}.$$

We compute

$$\frac{1}{(n-1)!W} \int_t^b (s-t)^{n-1} w(s)ds = \frac{1}{(n-1)!W} \int_t^b \left(\frac{s-t}{s-a}\right)^{n-1} (s-a)^{n-1} w(s)ds$$

$$= \frac{1}{(n-1)!W} \int_t^b (n-1) \frac{(s-t)^{n-2}(t-a)}{(s-a)^n} \cdot \left(\int_s^b (z-a)^{n-1} w(z)dz\right) ds$$

$$\geq \frac{1}{(n-1)!W} \int_t^{a+\lambda} (n-1) \frac{(s-t)^{n-2}(t-a)}{(s-a)^n} \cdot \left(\int_s^b (z-a)^{n-1} w(z)dz\right) ds$$

$$= \frac{1}{(n-1)!W} \int_t^{a+\lambda} (n-1) \frac{(s-t)^{n-2}(t-a)}{(s-a)^n}$$

$$\cdot \left(\int_a^b (z-a)^{n-1} w(z)dz - \int_a^s (z-a)^{n-1} w(z)dz\right) ds$$

$$\geq \frac{1}{(n-1)!W} \int_t^{a+\lambda} (n-1) \frac{(s-t)^{n-2}(t-a)}{(s-a)^n}$$

$$\cdot \left(\int_a^b (z-a)^{n-1} w(z)dz - \int_a^s (z-a)^{n-1} dz\right) ds$$

$$= \frac{1}{(n-1)!W} \int_t^{a+\lambda} (n-1) \frac{(s-t)^{n-2}(t-a)}{(s-a)^n}$$

$$\cdot \left(\int_a^b (z-a)^{n-1} w(z)dz - \frac{(s-a)^n}{n}\right) ds$$

$$\left| u = \frac{1}{(n-1)!W} \left(\int_a^b (z-a)^{n-1} w(z)dz - \frac{(s-a)^n}{n}\right), \right.$$

$$\left. dv = (n-1) \frac{(s-t)^{n-2}(t-a)}{(s-a)^n} ds \right|$$

$$= \frac{1}{(n-1)!} \left[\left(\frac{1}{W} \int_a^b (z-a)^{n-1} w(z)dz - \frac{\lambda^n}{nW}\right)\right.$$

$$\left. \cdot \quad \frac{(a+\lambda-t)^{n-1}}{\lambda^{n-1}} + \frac{(a+\lambda-t)^n}{nW}\right]$$

$$= \frac{(a+\lambda-t)^{n-1}}{n!} \left[\frac{n}{\lambda^{n-1}W} \int_a^b (z-a)^{n-1} w(z)dz - \frac{\lambda}{W} + \frac{a+\lambda-t}{W}\right]$$

$$= \frac{(a+\lambda-t)^{n-1}}{n!} \left(1 - \frac{t-a}{W}\right) \geq \frac{(a+\lambda-t)^{n-1}}{n!} \left(1 - \frac{t-a}{\lambda}\right) = \frac{(a+\lambda-t)^n}{n!\lambda}.$$

The assertion follows from the Theorem 2.2. □

Remark 2.1. Inequality (10) is the right-hand side of Steffensen inequality.

Theorem 2.4. *Let $f : [\min\{a, b-\lambda\}, b] \to \mathbb{R}$ be a $n-$convex function for $n \geq 1$, $0 \leq \lambda$ and let $w : [a, b] \to [0, \infty\rangle$ be integrable on $[a, b]$. If $x \in [a, b] \cap [b-\lambda, b]$ and $(-1)^n K_n(t, x) \leq 0$ for every $t \in [\min\{a, b-\lambda\}, b]$, then we have*

$$\frac{1}{\lambda}\int_{b-\lambda}^{b} f(t)dt - T_{1,n}^{[b-\lambda,b]}(x) \geq \frac{1}{\int_a^b w(t)dt}\int_a^b w(t)f(t)dt - T_{w,n}^{[a,b]}(x). \tag{13}$$

Proof. We put $c = b-\lambda$ and $d = b$, so we have two possibilities: $[c, d] \subseteq [a, b]$ and $[a, b] \subseteq [c, d]$.

a) Case $[b-\lambda, b] \subseteq [a, b]$

For $u \equiv 1$ on $[b-\lambda, b]$ we have

$$(-1)^n K_n(t,x) = \begin{cases} (-1)^n \frac{1}{(n-1)!\int_a^b w(t)dt}\int_a^t (t-s)^{n-1}w(s)ds, a \leq t \leq b-\lambda \\ (-1)^n \left[\frac{1}{(n-1)!\int_a^b w(t)dt}\int_a^t (t-s)^{n-1}w(s)ds - \frac{(t-b+\lambda)^n}{n!\lambda}\right], b-\lambda < t \leq x \\ (-1)^n \left[\frac{1}{(n-1)!\int_a^b w(t)dt}\int_b^t (t-s)^{n-1}w(s)ds - \frac{(t-b)^n}{n!\cdot\lambda}\right], x < t \leq b. \end{cases}$$

Apply Theorem 2.1 to finish the proof.

b) Case $[a, b] \subseteq [b-\lambda, b]$

For $u \equiv 1$ on $[b-\lambda, b]$ we have

$$(-1)^n K_n(t,x) = \begin{cases} -\frac{(b-\lambda-t)^n}{n!\lambda}, b-\lambda \leq t \leq a \\ (-1)^n \left[\frac{1}{(n-1)!\int_a^b w(t)dt}\int_a^t (t-s)^{n-1}w(s)ds - \frac{(t-b+\lambda)^n}{n!\lambda}\right], a < t \leq x \\ (-1)^n \left[\frac{1}{(n-1)!\int_a^b w(t)dt}\int_a^t (t-s)^{n-1}w(s)ds - \frac{(t-\lambda)^n}{n!\lambda}\right], x < t \leq b. \end{cases}$$

Apply Theorem 2.1 to finish the proof.

□

Corollary 2.5. *Let $w : [a, b] \to [0, 1]$ be integrable function on $[a, b]$. If $f : [a, b] \to \mathbb{R}$ is a $1-$convex function then we have*

$$\int_{b-\lambda}^{b} f(t)dt \geq \int_a^b w(t)f(t)dt. \tag{14}$$

Remark 2.2. Inequality (14) is the left -hand side of reversed Steffensen inequality (1).

3. $n-$exponential convexity of Steffensen's inequality via one-point integral formula

In this section we shall generate means from the differences of weighted integrals, and from Steffensen's inequality via one-point integral formula.

Let $x \in [a, b] \cap [c, d]$ and let $w : [a, b] \to \mathbb{R}$ and $u : [c, d] \to \mathbb{R}$ be integrable functions (weights). Let us define functional $A : C[a, \max\{b, d\}] \to \mathbb{R}$ with

$$Af := \int_a^b w(t)f(t)dt - T_{n,w}^{[a,b]}(x) - \int_c^d u(t)f(t)dt + T_{n,w}^{[c,d]}(x). \tag{15}$$

It is easy to check that A is linear functional.

Corollary 3.1. *Let* $f : [a,b] \cup [c,d] \to \mathbb{R}$ *be a* $n-$*convex function,* $x \in [a,b] \cap [c,d]$ *and let* $w : [a,b] \to \mathbb{R}$ *and* $u : [c,d] \to \mathbb{R}$ *be integrable functions (weights). If* $(-1)^n K_n(t,x) \geq 0$ *for every* $t \in [a,b] \cup [c,d]$*, then* $Af \geq 0$.

Proof. The proof follows immediatelly from the Theorem 2.1. □

Theorem 3.2. *Assume that* $w : [a,b] \to \mathbb{R}$ *and* $u : [c,d] \to \mathbb{R}$ *are weights such that* $(-1)^n K_n(t,x) \geq 0$ *for every* $t \in [a,b] \cup [c,d]$*. Then for every* $f \in C^n[a, \max\{b,d\}]$ *there exists* $\xi \in [a, \max\{b,d\}]$ *such that*

$$Af = f^{(n)}(\xi) A(P_n), \tag{16}$$

where $P_n(t) = \frac{(t-a)^n}{n!}$.

Proof. For given function $f \in C^n[a, \max\{b,d\}]$ let us define

$$m := \min\{f^{(n)}(t) : a \leq t \leq \max\{b,d\}\}$$

and

$$M := \max\{f^{(n)}(t) : a \leq t \leq \max\{b,d\}\}.$$

Now, let us define functions $h_1, h_2 : [a, \max\{b,d\}] \to \mathbb{R}$ with

$$h_1(t) = M \cdot P_n(t) - f(t), \quad h_2(x) = f(t) - m \cdot P_n(t).$$

Note that $h_1^{(n)}(t) = M - f^{(n)}(t) \geq 0$ for $t \in [a, \max\{b,d\}]$, so we conclude from the Corollary 3.1 that $Ah_1 \geq 0$ and consequently $Af \leq M \cdot A(P_n)$. On the other hand, from $h_2^{(n)}(t) = f^{(n)}(t) - m \geq 0$ we conclude $m \cdot A(P_n) \leq Af$. Now from

$$m \cdot A(P_n) \leq Af \leq MA(P_n)$$

and continuity of $f^{(n)}$ we conclude that there exists $\xi \in [a, \max\{b,d\}]$ such that (16) is valid. □

Corollary 3.3. *Assume that* $w : [a,b] \to \mathbb{R}$ *and* $u : [c,d] \to \mathbb{R}$ *are weights such that* $(-1)^n K_n(t,x) \geq 0$ *for every* $t \in [a,b] \cup [c,d]$ *and* $A(P_n) \neq 0$*. Then for every* $f, g \in C^n[a, \max\{b,d\}]$ *there exists* $\xi \in [a, \max\{b,d\}]$ *such that*

$$\frac{Af}{Ag} = \frac{f^{(n)}(\xi)}{g^{(n)}(\xi)} \tag{17}$$

provided that neither of the denominator equals zero.

Proof. Let us define function $h(t) = f(t)A(g) - g(t)A(f)$. Since $h^{(n)}(t) = f^{(n)}(t)Ag - g^{(n)}(t)Af$ is continuous, then according to the Theorem 3.2, there exits $\xi \in [a, \max\{b,d\}]$ such that $Ah = h^{(n)}(\xi)A(P_n)$. It is obvious that $Ah = 0$, so we conclude that $f^{(n)}(\xi)Ag - g^{(n)}(\xi)Af = 0$ which is equivalent to (17). □

Remark 3.1. If $\frac{f^{(n)}}{g^{(n)}}$ has inverse function, then from (17) we have

$$\xi = \left(\frac{f^{(n)}}{g^{(n)}}\right)^{-1} \left(\frac{Af}{Ag}\right), \tag{18}$$

so ξ is a mean.

Using above results, we now make a list of linear functional that will give us particular examples of Cauchy means. Motivated by inequalities (9),(13),(10) and (14) we define functionals $A_1(f), A_2(f), A_3(f)$ and $A_4(f)$ by

$$A_1(f) = \frac{1}{\int_a^b w(t)dt}\int_a^b w(t)f(t)dt - T_{w,n}^{[a,b]}(x) - \frac{1}{\lambda}\int_a^{a+\lambda} f(t)dt + T_{1,n}^{[a,a+\lambda]}(x) \tag{19}$$

$$A_2(f) = \frac{1}{\lambda}\int_{b-\lambda}^b f(t)dt - T_{1,n}^{[b-\lambda,b]}(x) - \frac{1}{\int_a^b w(t)dt}\int_a^b w(t)f(t)dt + T_{w,n}^{[a,b]}(x) \tag{20}$$

$$A_3(f) = \frac{1}{\int_a^b w(t)dt}\int_a^b w(t)f(t)dt - \frac{1}{\lambda}\int_a^{a+\lambda} f(t)dt \tag{21}$$

$$A_4(f) = \frac{1}{\lambda}\int_{b-\lambda}^b f(t)dt - \frac{1}{\int_a^b w(t)dt}\int_a^b w(t)f(t)dt. \tag{22}$$

Also, we define $I_1 = [a, \max\{b, a+\lambda\}]$, $I_2 = [\max\{b-\lambda, a\}, b]$ and $I_3 = I_4 = [a, b]$.

Now we will use above defined functionals to construct exponentially convex functions. We start this part of the section with some definitions and facts about exponentially convex functions which are used in our results (see[7]).

Definition 3.1. A function $\psi : I \to \mathbb{R}$ is n-exponentially convex in the Jensen sense on I if

$$\sum_{i,j=1}^{n} \xi_i \xi_j \, \psi\left(\frac{x_i + x_j}{2}\right) \geq 0,$$

hold for all choices $\xi_1, \dots, \xi_n \in \mathbb{R}$ and all choices $x_1, \dots, x_n \in I$.

A function $\psi : I \to \mathbb{R}$ is n-exponentially convex if it is n-exponentially convex in the Jensen sense and continuous on I.

Remark 3.2. It is clear from the definition that 1-exponentially convex functions in the Jensen sense are in fact nonnegative functions. Also, n-exponentially convex function in the Jensen sense are k-exponentially convex in the Jensen sense for every $k \in \mathbb{N},\ k \leq n$.

Definition 3.2. A function $\psi : I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is n-exponentially convex in the Jensen sense for all $n \in \mathbb{N}$.

A function $\psi : I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

Remark 3.3. A positive function is log-convex in the Jensen sense if and only if it is 2-exponentially convex in the Jensen sense.

A positive function is log-convex if and only if it is 2-exponentially convex. (see [3])

Proposition 3.4. *If f is a convex function on I and if $x_1 \leq y_1,\ x_2 \leq y_2,\ x_1 \neq x_2,\ y_1 \neq y_2$, then the following inequality is valid*

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(y_2) - f(y_1)}{y_2 - y_1}.$$

If the function f is concave, the inequality is reversed.

Theorem 3.5. *Let $\Upsilon = \{f_s : s \in J\}$, where J an interval in $\mathbb{R}$, be a family of functions defined on an interval $I_i,\ i = 1, 2, 3, 4$, in $\mathbb{R}$, such that the function $s \mapsto f_s[z_0, \ldots, z_l]$ is n-exponentially convex in the Jensen sense on J for every $(l+1)$ mutually different points $z_0, \ldots, z_l \in I_i$. Let $A_i,\ i = 1, 2, 3, 4$, be linear functionals defined with (19)-(22). Then $s \mapsto A_i(f_s)$ is an n-exponentially convex function in the Jensen sense on J.*
If in the addition the function $s \mapsto A_i(f_s)$ is continuous on J, then it is n-exponentially convex on J.

Proof. For $\xi_j \in \mathbb{R},\ j = 1, \ldots, n$ and $s_j \in J,\ j = 1, \ldots, n$, we define the function

$$g(z) = \sum_{j,k=1}^{n} \xi_j \xi_k f_{\frac{s_j+s_k}{2}}(z).$$

Since the function $s \mapsto f_s[z_0, \ldots, z_l]$ is n-exponentially convex in the Jensen sense, we have

$$g[z_0, \ldots, z_l] = \sum_{j,k=1}^{n} \xi_j \xi_k f_{\frac{s_j+s_k}{2}}[z_0, \ldots, z_l] \geq 0,$$

so we conclude that g is a l-convex function on J, and thus $A_i(g) \geq 0,\ i = 1, 2, 3, 4$, hence

$$\sum_{j,k=1}^{n} \xi_j \xi_k A_i\left(f_{\frac{s_j+s_k}{2}}\right) \geq 0.$$

We conclude that the function $s \mapsto A_i(f_s)$ is n-exponentially convex on J in the Jensen sense.

If the function $s \mapsto A_i(f_s)$ is also continuous on J, then $s \mapsto A_i(f_s)$ is n-exponentially convex by definition. □

The following corollaries are an immediate consequences of the above theorem:

Corollary 3.6. *Let $\Upsilon = \{f_s : s \in J\}$, where J an interval in $\mathbb{R}$, be a family of functions defined on an interval $I_i,\ i = 1, 2, 3, 4$, in $\mathbb{R}$, such that the function $s \mapsto f_s[z_0, \ldots, z_l]$ is exponentially convex in the Jensen sense on J for every $(l+1)$ mutually different points $z_0, \ldots, z_l \in I_i$. Let $A_i(f),\ i = 1, 2, 3, 4$, be linear functionals defined with (19)-(22). Then $s \mapsto A_i(f_s)$ is an exponentially convex function in the Jensen sense on J. If the function $s \mapsto A_i(f_s)$ is continuous on J, then it is exponentially convex on J.*

Corollary 3.7. *Let $\Upsilon = \{f_s : s \in J\}$, where J an interval in $\mathbb{R}$, be a family of functions defined on an interval $I_i,\ i = 1, 2, 3, 4$, in $\mathbb{R}$, such that the function $s \mapsto f_s[z_0, \ldots, z_l]$ is 2-exponentially convex in the Jensen sense on J for every $(l+1)$ mutually different points $z_0, \ldots, z_l \in I_i$. Let $A_i(f),\ i = 1, 2, 3, 4$, be linear functional defined as in (19)-(22). Then the following statements hold:*

(i) *If the function $s \mapsto A_i(f_s)$ is continuous on J, then it is 2-exponentially convex function on J. If $s \mapsto A_i(f_s)$ is additionally strictly positive, then it is also log-convex on J. Furthermore, the following inequality holds true:*

$$[A_i(f_s)]^{t-r} \leq [A_i(f_r)]^{t-s} [A_i(f_t)]^{s-r} \tag{23}$$

for every choice $r, s, t \in J$, such that $r < s < t$.

(ii) *If the function $s \mapsto A_i(f_s)$ is strictly positive and differentiable on J, then for every $s, q, u, v \in J$, such that $s \le u$ and $q \le v$, we have*

$$\mu_{s,q}(A_i, \Upsilon) \le \mu_{u,v}(A_i, \Upsilon), \tag{24}$$

where

$$\mu_{s,q}(A_i, \Upsilon) = \begin{cases} \left(\frac{A_i(f_s)}{A_i(f_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(\frac{\frac{d}{ds}A_i(f_s)}{A_i(f_s)}\right), & s = q, \end{cases} \tag{25}$$

for $f_s, f_q \in \Upsilon$.

Proof. (i) This is an immediate consequence of Theorem 3.5 and Remark 3.3.

(ii) Since the function $s \mapsto A_i(f_s)$, $i = 1, 2, 3, 4$ is positive and continuous, according to (i) the function $s \mapsto A_i(f_s)$ is log-convex on J, and thus the function $s \mapsto \log A_i(f_s)$ is convex on J. So, we get

$$\frac{\log A_i(f_s) - \log A_i(f_q)}{s-q} \le \frac{\log A_i(f_u) - \log A_i(f_v)}{u-v}, \tag{26}$$

for $s \le u, q \le v, s \neq q, u \neq v$, and there form conclude that

$$\mu_{s,q}(A_i, \Upsilon) \le \mu_{u,v}(A_i, \Upsilon).$$

Cases $s = q$ and $u = v$ follows from (26) as limit cases.

□

Remark 3.4. Note that the results from above theorem and corollaries still hold when two of the points $z_0, \ldots, z_l \in I_i$ coincide, say $z_1 = z_0$, for a family of differentiable functions f_s such that the function $s \mapsto f_s[z_0, \ldots, z_l]$ is n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense), and furthermore, they still hold when all $(l+1)$ points coincide for a family of l differentiable functions with the same property. The proofs are obtained by (7) and suitable characterization of convexity.

4. Applications to Stolarsky type means

In this section, we present few families of functions which fulfill the conditions of Theorem 3.5, Corollary 3.6, Corollary 3.7 and Remark 3.4. This enable us to establish a lots of families of functions which are exponentially convex.

Example 4.1. Consider a family of functions

$$\Omega_1 = \{f_s : \mathbb{R} \to \mathbb{R} : s \in \mathbb{R}\}$$

defined by

$$f_s(t) = \begin{cases} \frac{e^{st}}{s^n}, & s \neq 0, \\ \frac{t^n}{n!}, & s = 0. \end{cases}$$

We have

$$f_s^{(n)}(t) = \begin{cases} e^{st} > 0, & s \neq 0 \\ 1 > 0, & s = 0. \end{cases},$$

so f_s is n-convex on $\mathbb{R}$ for every $s \in \mathbb{R}$ and $s \mapsto f_s^{(n)}(t)$ is exponentially convex by definition. Using analogous arguing as in the proof of Theorem 3.5 we also have that $s \mapsto f_s[z_0, \ldots, z_m]$ is exponentially convex (and so exponentially convex in the Jensen sense). Using Corollary 3.6 we conclude that $s \mapsto A_i(f_s), i = 1, 2, 3, 4$, are exponentially convex in the Jensen sense. It is easy to verify that this mapping is continuous (although mapping $s \mapsto f_s$ is not continuous for $s = 0$), so it is exponentially convex.

For this family of functions, $\mu_{s,q}(A_i, \Omega_1),\ i = 1, 2, 3, 4$, from (25) we have

$$\mu_{s,q}(A_i, \Omega_1) = \begin{cases} \left(\frac{A_i(f_s)}{A_i(f_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(\frac{A_i(id \cdot f_s)}{A_i(f_s)} - \frac{n}{s}\right), & s = q \neq 0, \\ \exp\left(\frac{1}{n+1}\frac{A_i(id \cdot f_0)}{A_i(f_0)}\right), & s = q = 0. \end{cases}$$

Also, by (24) it is monotonous function in parameters s and q.

We observe here that $\left(\frac{\frac{d^n f_s}{dt^n}}{\frac{d^n f_q}{dt^n}}\right)^{\frac{1}{s-q}}(\log t) = t$ so using Corollary 3.3 it follows that

$$M_{s,q}(A_i, \Omega_1) = \log \mu_{s,q}(A_i, \Omega_1),\ i = 1, 2, 3, 4$$

satisfies:

$$\begin{aligned} &a \leq M_{s,q}(A_1, \Omega_1) \leq \max\{b, a + \lambda\}, \\ &\min\{a, b - \lambda\} \leq M_{s,q}(A_2, \Omega_1) \leq b, \\ &a \leq M_{s,q}(A_3, \Omega_1) \leq b, \\ &a \leq M_{s,q}(A_4, \Omega_1) \leq b. \end{aligned}$$

So, $M_{s,q}(A_i, \Omega_1)$ is monotonic mean.

Example 4.2. Consider a family of functions

$$\Omega_2 = \{f_s : (0, \infty) \to \mathbb{R} : s \in \mathbb{R}\}$$

defined by

$$f_s(t) = \begin{cases} \frac{t^s}{s(s-1)\cdots(s-n+1)}, & s \notin \{0, 1, \ldots, n-1\}, \\ \frac{t^j \ln t}{(-1)^{n-1-j} j!(n-1-j)!}, & s = j \in \{0, 1, \ldots, n-1\}. \end{cases}$$

Here, $f_s^{(n)}(t) = t^{s-n} = e^{(s-n)\ln t} > 0$ which shows that f_s is n-convex for $t > 0$ and $s \mapsto f_s^{(n)}(t)$ is exponentially convex by definition. Analogue as in Example 4.1 we get that the mappings $s \mapsto A_i(f_s), i = 1, 2, 3, 4$ are exponentially convex. Now (25) equals to:

$$\mu_{s,q}(A_i, \Omega_2) = \begin{cases} \left(\frac{A_i(f_s)}{A_i(f_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left((-1)^{n-1}(n-1)!\frac{A_i(f_0 f_s)}{A_i(f_s)} + \sum_{k=0}^{n-1} \frac{1}{k-s}\right), & s = q \notin \{0, 1, \ldots, n-1\}, \\ \exp\left((-1)^{n-1}(n-1)!\frac{A_i(f_0 f_s)}{2A_i(f_s)} + \sum_{\substack{k=0 \\ k \neq s}}^{n-1} \frac{1}{k-s}\right), & s = q \in \{0, 1, \ldots, n-1\}. \end{cases}$$

Again, using Corollary 3.3 we conclude that

$$a \leq \left(\frac{A_1(f_s)}{A_1(f_q)}\right)^{\frac{1}{s-q}} \leq \max\{b, a+\lambda\},$$

$$\min\{a, b-\lambda\} \leq \left(\frac{A_2(f_s)}{A_2(f_q)}\right)^{\frac{1}{s-q}} \leq b,$$

$$a \leq \left(\frac{A_i(f_s)}{A_i(f_q)}\right)^{\frac{1}{s-q}} \leq b, \ i = 3, 4$$

which shows that $\mu_{s,q}(A_i, \Omega_2), i = 1, 2, 3, 4$, is mean.

References

[1] A. Aglić Aljinović, J. Pečarić, On some Ostrowski type inequalities via Montgomery and Taylor's formula,*Tamkang Journal of Mathematics* **36**(3) (2005), 199-218.
[2] A. Aglić Aljinović, J. Pečarić, I. Perić, Estimates of the difference between two weighted integral means via weighted Montgomery identity, *Math.Inequal.Appl.* **7** (3) (2004), 315-336.
[3] J. Jakšetić, J. Pečarić, Exponential convexity method, *Journal of Convex Analysis* **20**(1) (2013), 181-197.
[4] S. Kovač, J. Pečarić, Weighted version of general integral formula of Euler type, *Mathematical Inequalities & Applications* **13** (3) (2010), 579-599.
[5] M. Matić, J. Pečarić, N. Ujević, Generalizations of weighted version of Ostrowski's inequality and some related results, *J. Inequal. Appl.* **5**(6) (2000), 639-666.
[6] G.V. Milovanović, J.E. Pečarić, The Steffensen inequality for convex function of order n, *Univ.Beograd.Publ.Elektrotehn.Fak.Ser.Mat.Fiz.* No. 634-677 (1979), 97-100.
[7] J. Pečarić, J. Perić, Improvements of the Giaccardi and the Petrović inequality and related results, *An. Univ.Craiova Ser.Mat.Inform.* **39** (1) (2012), 65-75.
[8] J. E. Pečarić, F. Proschan, Y. L. Tong, *Convex functions, partial orderings, and statistical applications*, Mathematics in science and engineering 187, Academic Press, 1992.
[9] J.F. Steffensen, On certain inequalities between mean values and their application to actuarial problems, *Skand. Aktuarietids* (1918), 82-97.

(Sanja Kovač) Faculty of Geotechnical Engineering, Varaždin, University of Zagreb, Hallerova aleja 7, 42000 Varaždin, Croatia
E-mail address: skovac@gfv.hr

(Josip Pečarić) Faculty of Textile Technology, University of Zagreb, Prilaz baruna Filipovića 28a, 10000 Zagreb, Croatia
E-mail address: pecaric@hazu.hr

(Anamarija Perušić) Faculty of Civil Engineering, University of Rijeka, Radmile Matejčić 3, 51000 Rijeka, Croatia
E-mail address: anamarija.perusic@gradri.hr