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Existence and uniqueness of entropy solution for some nonlinear elliptic unilateral problems in Musielak-Orlicz-Sobolev spaces

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Abstract. In this paper, we study the existence and uniqueness of entropy solution for some quasilinear degenerate elliptic unilateral problems of the type

$$\begin{cases} -\operatorname{div} a(x,\nabla u) = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

in the Musielak-Orlicz-Sobolev spaces $W_0^1L_\varphi(\Omega)$, with $f \in L^1(\Omega)$ and by assuming that the conjugate function of the Musielak-Orlicz function $\varphi(x,t)$ satisfies the Δ_2-condition. An example of such equation is given by

$$\begin{cases} -\operatorname{div}\Big(|\nabla u|^{p(x)-2}\log^\sigma(1+|\nabla u|)\ \nabla u\Big) = f & \text{in} \ \ \Omega, \\ u \ = 0 & \text{on} \ \ \partial\Omega, \end{cases} \tag{1}$$

for $1 \leq p(x) < \infty$ and $0 < \sigma < \infty$.

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1. Introduction

Let Ω be a bounded open subset of $I\!R^N$ $(N \geq 2)$, with smooth boundary conditions.

For $2-\frac{1}{N} < p < N$, Boccardo and Gallouët have studied in [11] the elliptic problem of the type

$$\begin{cases} Au = f & \text{in} \quad \Omega, \\ \ u = \ 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $Au = -\operatorname{div} a(x,u,\nabla u)$ is a Leray-Lions operator from $W_0^{1,p}(\Omega)$ into its dual, and f is a bounded Radon measure on Ω. They have proved the existence of solutions $u \in W_0^{1,q}(\Omega)$ for all $1 < q < \bar{q} = \frac{N(p-1)}{N-1}$. Also they proved some regularity results.

Aharouch and Bennouna have treated in [1] the quasilinear elliptic of unilateral problem

$$\begin{cases} -\operatorname{div}\ (a(x,\nabla u)) = f & \text{in} \quad \Omega, \\ \ u = \ 0 \text{ on } \quad \partial\Omega, \end{cases} \tag{2}$$

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where $f \in L^1(\Omega)$. They have proved the existence and uniqueness of entropy solutions in the framework of Orlicz Sobolev spaces $W_0^1 L_M(\Omega)$ without assuming the Δ_2-condition on the $N-$function M of the Orlicz spaces, (see also. [6, 7, 13]).

In [5], Bendahmane and Wittbold have proved existence and uniqueness of a renormalized solution to the nonlinear elliptic equation

$$\begin{cases} -\text{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = f \quad \text{in} \quad \Omega, \\ u = \ 0 \text{ on } \quad \partial\Omega, \end{cases} \tag{3}$$

where the right-hand side $f \in L^1(\Omega)$ and the exponent $p(\cdot) : \overline{\Omega} \mapsto (1, +\infty)$ is continuous, for some related results we refer to [2, 4, 12, 22].

In the recent years, Musielak-Orlicz-Sobolev spaces have attracted the attention of mainly researchers, the impulse for this manly comes from there physical applications, such in electro-rheological fluids, (see [23]). The purpose of this paper is to prove the existence and uniqueness of entropy solutions for some quasilinear unilateral elliptic problem of the form

$$\begin{cases} Au = f \quad \text{in} \quad \Omega, \\ u = \ 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{4}$$

in Musielak-Orlicz-Sobolev spaces, where $f \in L^1(\Omega)$ and $A : D(A) \subset W_0^1 L_\varphi(\Omega) \mapsto W^{-1} L_\psi(\Omega)$ is the Leray-Lions operator defined as:

$$A(u) = -\text{div } a(x, \nabla u),$$

by assuming that the conjugate function of Musielak-Orlicz function $\varphi(x,t)$ satisfies Δ_2-condition, and by using corollary 1 of [9] to construct a complementary system $(W_0^1 L_\varphi(\Omega), W_0^1 E_\varphi(\Omega);\ W^{-1} L_\psi(\Omega),\ W^{-1} E_\psi(\Omega))$.

Note that, the second author has studded in [9] the existence of solution for the problem (4) where f is assumed to be in the dual, and only strict monotonicity is assumed, we refer also to [19] for the elliptic case with large monotonicity, and the interesting works of Gwiazda el al. [16, 17, 18] in the generalized Orlicz Sobolev spaces, also [14] where the author has proved the Poincaré inequality under the Δ_2-condition.

This paper is organized as follows. In the section 2 we recall some definitions and basic properties of Musielak-Orlicz-Sobolev. We introduce in the section 3 the assumptions on $a(x, \xi)$ under which our problem has at least one solution. The section 4 contains some useful lemmas for proving our main results. The section 5 will be devoted to show the existence and uniqueness of entropy solutions for our main problem (4).

2. Preliminaries

In this section, we introduce some definitions and known facts about Musielak-Orlicz-Sobolev spaces. The standard reference is [24].

2.1. Musielak-Orlicz function. Let Ω be an open bounded subset of $I\!R^N$ $(N \geq 2)$ with smooth boundary conditions, and let $\varphi(x,t)$ be a real-valued function defined on $\Omega \times I\!R^+$, and satisfying the following two conditions :

(a): $\varphi(x,\cdot)$ is an N-function, *i.e.* convex, nondecreasing, continuous, $\varphi(x,0)=0$, $\varphi(x,t)>0$ for all $t>0$, and :

$$\lim_{t\to 0}\sup_{x\in\Omega}\frac{\varphi(x,t)}{t}=0 \quad , \quad \lim_{t\to\infty}\inf_{x\in\Omega}\frac{\varphi(x,t)}{t}=\infty,$$

(b): $\varphi(\cdot,t)$ is a measurable function.

A function $\varphi(x,t)$ which satisfies conditions (a) and (b) is called a Musielak-Orlicz function.

For every Musielak-Orlicz function $\varphi(x,t)$, we set $\varphi_x(t)=\varphi(x,t)$ and let $\varphi_x^{-1}(t)$ the reciprocal function with respect to t of $\varphi_x(t)$, i.e.

$$\varphi_x^{-1}(\varphi(x,t))=\varphi(x,\varphi_x^{-1}(t))=t.$$

For any two Musielak-Orlicz functions $\varphi(x,t)$ and $\gamma(x,t)$, we introduce the following ordering:

(c): If there exist two positive constants c and T such that for almost everywhere $x\in\Omega$:

$$\varphi(x,t)\leq\gamma(x,ct) \quad \text{for} \quad t\geq T,$$

we write $\varphi\prec\gamma$, and we say that γ dominate φ globally if $T=0$, and near infinity if $T>0$.

(d): For every positive constant c and almost everywhere $x\in\Omega$, if

$$\lim_{t\to 0}\Big(\sup_{x\in\Omega}\frac{\varphi(x,ct)}{\gamma(x,t)}\Big)=0 \quad \textit{or} \quad \lim_{t\to\infty}\Big(\sup_{x\in\Omega}\frac{\varphi(x,ct)}{\gamma(x,t)}\Big)=0,$$

we write $\varphi\prec\prec\gamma$ at 0 or near ∞ respectively, and we say that φ increases essentially more slowly than γ at 0 or near ∞ respectively.

The Musielak-Orlicz function $\psi(x,t)$ complementary to (or conjugate of) $\varphi(x,t)$, in the sense of Young with respect to the variable t, is given by

$$\psi(x,s)=\sup_{t\geq 0}\{st-\varphi(x,t)\}, \tag{5}$$

and we have

$$st\leq\psi(x,s)+\varphi(x,t) \qquad \forall s,t\in I\!R^+. \tag{6}$$

The Musielak-Orlicz function $\varphi(x,t)$ is said to satisfy the Δ_2-condition if, there exists $k>0$ and a nonnegative function $h(\cdot)\in L^1(\Omega)$, such that

$$\varphi(x,2t)\leq k\varphi(x,t)+h(x) \qquad \text{a.e.} \quad x\in\Omega,$$

for large values of t, or for all values of t.

2.2. Musielak-Orlicz Lebesgue spaces. In this paper, the measurability of a function $u:\Omega\mapsto I\!R$ means the Lebesgue measurability.

We define the functional

$$\varrho_{\varphi,\Omega}(u)=\int_\Omega\varphi(x,|u(x)|)\,dx,$$

where $u:\Omega\mapsto I\!R$ is a measurable function. The set

$$K_\varphi(\Omega)=\{u:\Omega\longmapsto I\!R \quad \text{measurable} \ / \ \varrho_{\varphi,\Omega}(u)<+\infty\}$$

is called the Musielak-Orlicz class (or the generalized Orlicz class). The Musielak-Orlicz spaces (or the generalized Orlicz spaces) $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is, $L_\varphi(\Omega)$ is the smallest linear space containing the set $K_\varphi(\Omega)$; equivalently

$$L_\varphi(\Omega)=\Big\{u:\Omega\longmapsto I\!R \quad \text{measurable} \quad / \ \ \varrho_{\varphi,\Omega}(\frac{|u(x)|}{\lambda})<+\infty, \quad \text{for some } \lambda>0\Big\}.$$

In the space $L_\varphi(\Omega)$, we define the following two norms:

$$||u||_{\varphi,\Omega}=\inf\Big\{\lambda>0 \ \ / \ \ \int_\Omega \varphi(x,\frac{|u(x)|}{\lambda})\,dx\leq 1\Big\},$$

which is called the Luxemburg norm, and the so-called Orlicz norm is given by:

$$|||u|||_{\varphi,\Omega}=\sup_{||v||_{\psi,\Omega}\leq 1}\int_\Omega |u(x)v(x)|\,dx,$$

where $\psi(x,t)$ is the Musielak-Orlicz function complementary (or conjugate) to $\varphi(x,t)$. These two norms are equivalent on the Musielak-Orlicz space $L_\varphi(\Omega)$.

The closure in $L_\varphi(\Omega)$ of the bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_\varphi(\Omega)$. It is a separable space and $(E_\varphi(\Omega))^*=L_\psi(\Omega)$.

We have $E_\varphi(\Omega)=K_\varphi(\Omega)$ if and only if $K_\varphi(\Omega)=L_\varphi(\Omega)$ if and only if $\varphi(x,t)$ has the Δ_2-condition for large values of t, or for all values of t.

2.3. Musielak-Orlicz-Sobolev spaces. We now turn to the Musielak-Orlicz-Sobolev space $W^1L_\varphi(\Omega)$ (resp. $W^1E_\varphi(\Omega)$) is the space of all measurable functions u such that u and its distributional derivatives up to order 1 lie in $L_\varphi(\Omega)$ (resp. $E_\varphi(\Omega)$). Let $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_n)$ with nonnegative integers α_i, $|\alpha|=|\alpha_1|+|\alpha_2|+...+|\alpha_n|$ and $D^\alpha u$ denotes the distributional derivatives.
We define the convex modular and the norm on the Musielak-Orlicz-Sobolev spaces $W^1L_\varphi(\Omega)$ respectively by,

$$\overline{\varrho}_{\varphi,\Omega}(u)=\sum_{|\alpha|\leq 1}\varrho_{\varphi,\Omega}(D^\alpha u) \quad \textit{and} \quad ||u||_{1,\varphi,\Omega}=\inf\Big\{\lambda>0:\overline{\varrho}_{\varphi,\Omega}(\frac{u}{\lambda})\leq 1\Big\},$$

for any $u\in W^1L_\varphi(\Omega)$.
The pair $\langle W^1L_\varphi(\Omega),||u||_{1,\varphi,\Omega}\rangle$ is a Banach space if φ satisfies the following condition

$$\text{there exists a constant} \quad c>0 \quad \text{such that} \quad \inf_{x\in\Omega}\varphi(x,1)\geq c.$$

The spaces $W^1L_\varphi(\Omega)$ and $W^1E_\varphi(\Omega)$ can be identified with subspaces of the product of $n+1$ copies of $L_\varphi(\Omega)$. Denoting this product by $\Pi L_\varphi(\Omega)$, we will use the weak topologies $\sigma(\Pi L_\varphi(\Omega),\Pi E_\psi(\Omega))$ and $\sigma(\Pi E_\psi(\Omega),\Pi L_\varphi(\Omega))$.

The space $W_0^1E_\varphi(\Omega)$ is defined as the (norm) closure of the Schwartz space $\mathrm{D}(\Omega)$ in $W^1E_\varphi(\Omega)$, and the space $W_0^1L_\varphi(\Omega)$ as the $\sigma(\Pi L_\varphi(\Omega),\Pi E_\psi(\Omega))$ closure of $\mathrm{D}(\Omega)$ in $W^1L_\varphi(\Omega)$, (for more details on Musielak-Orlicz-Sobolev spaces we refer to [24]).

2.4. Dual space. Let $W^{-1}L_\psi(\Omega)$ (resp. $W^{-1}E_\psi(\Omega)$) denotes the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_\psi(\Omega)$ (resp. $E_\psi(\Omega)$). It is a Banach space under the usual quotient norm.

If $\psi(x,t)$ has the Δ_2-condition, then the space $\mathrm{D}(\Omega)$ is dense in $W_0^1L_\varphi(\Omega)$ for the topology $\sigma(\Pi L_\varphi(\Omega),\Pi L_\psi(\Omega))$ (see corollary 1 of [9]).

3. Essential assumptions

Let Ω be a bounded open subset of $I\!R^N$ $(N \geq 2)$ with smooth boundary conditions. Let $\varphi(x,t)$ be a Musielak-Orlicz function and $\psi(x,t)$ the Musielak-Orlicz function complementary (or conjugate) to $\varphi(x,t)$. We assume here that $\psi(x,t)$ satisfying the Δ_2-condition near infinity, therefore $L_\psi(\Omega) = E_\psi(\Omega)$.
We assume that there exists an Orlicz function $M(t)$ such that $M(t) \prec \varphi(x,t)$ near infinity, i.e. there exist two constants $c > 0$ and $T \geq 0$ such that

$$M(t) \leq \varphi(x, ct) \qquad \text{a.e. in} \quad \Omega \qquad \text{for} \quad t \geq T. \tag{7}$$

Let $\Psi(\cdot)$ be a measurable function on Ω, such that

$$\Psi^+(\cdot) \in W_0^1 L_\varphi(\Omega) \cap L^\infty(\Omega),$$

and we consider the convex set

$$K_\Psi = \Big\{ v \in W_0^1 L_\varphi(\Omega) \ \textit{such that}\ v \geq \Psi \ \textit{a.e. in}\ \Omega \Big\}.$$

The Leray-Lions operator $A : D(A) \subset W_0^1 L_\varphi(\Omega) \longmapsto W^{-1} L_\psi(\Omega)$ given by

$$A(u) = -\text{div}\ a(x, \nabla u)$$

where $a : \Omega \times I\!R^N \longmapsto I\!R$ is a *Carathéodory* function (measurable with respect to x in Ω for every ξ in $I\!R^N$, and continuous with respect to ξ in $I\!R^N$ for almost every x in Ω) which satisfies the following conditions

$$|a(x,\xi)| \leq \beta\big(K(x) + k_1 \psi_x^{-1}(\varphi(x, k_2|\xi|))\big), \tag{8}$$

$$\big(a(x,\xi) - a(x,\xi^*)\big) \cdot \big(\xi - \xi^*\big) > 0 \qquad \text{for} \quad \xi \neq \xi^*, \tag{9}$$

$$a(x,\xi) \cdot \xi \geq \alpha\ \varphi(x, |\xi|), \tag{10}$$

for a.e. $x \in \Omega$ and all $\xi \in I\!R^N$, where $K(x)$ is a nonnegative function lying in $E_\psi(\Omega)$ and $\alpha, \beta > 0$ and $k_1, k_2 \geq 0$.
We consider the quasilinear unilateral elliptic problem

$$\begin{cases} -\text{div}\, a(x, \nabla u) = f & \text{in } \ \Omega, \\ u = 0 & \text{in } \ \partial\Omega, \end{cases} \tag{11}$$

with $f \in L^1(\Omega)$. We study the existence of entropy solution in the Musielak-Orlicz-Sobolev spaces.

4. Some technical lemmas

Now, we present some lemmas useful in the proof of our main results.

Lemma 4.1. *(see* [20]*, Theorem 13.47) Let* $(u_n)_n$ *be a sequence in* $L^1(\Omega)$ *and* $u \in L^1(\Omega)$ *such that*

(i): $u_n \to u$ *a.e. in* Ω,
(ii): $u_n \geq 0$ *and* $u \geq 0$ *a.e. in* Ω,
(iii): $\displaystyle\int_\Omega u_n\, dx \to \int_\Omega u\, dx$,

then $u_n \to u$ *in* $L^1(\Omega)$.

Lemma 4.2. *Assuming that* (8)$-$(10) *hold, and let* $(u_n)_n$ *be a sequence in* $W_0^1 L_\varphi(\Omega)$ *such that*

(i): *$u_n \rightharpoonup u$ weakly in $W_0^1L_\varphi(\Omega)$ for $\sigma(\Pi L_\varphi(\Omega), \Pi E_\psi(\Omega))$,*
(ii): *$(a(x,\nabla u_n))_n$ is bounded in $(L_\psi(\Omega))^N = (E_\psi(\Omega))^N$,*
(iii): *Let $\Omega_s = \big\{x \in \Omega, \quad |\nabla u| \leq s\big\}$ and χ_s his characteristic function, with*

$$\int_\Omega (a(x,\nabla u_n) - a(x,\nabla u\chi_s))\cdot(\nabla u_n - \nabla u\chi_s)\,dx \longrightarrow 0 \quad as \quad n,s\to\infty, \tag{12}$$

then $\varphi(x,|\nabla u_n|) \longrightarrow \varphi(x,|\nabla u|)$ in $L^1(\Omega)$ for a subsequence.

Proof. Taking $s \geq r > 0$, we have :

$$\begin{aligned}0 &\leq \int_{\Omega_r} (a(x,\nabla u_n) - a(x,\nabla u))\cdot(\nabla u_n - \nabla u)\,dx\\ &\leq \int_{\Omega_s} (a(x,\nabla u_n) - a(x,\nabla u))\cdot(\nabla u_n - \nabla u)\,dx\\ &= \int_{\Omega_s} (a(x,\nabla u_n) - a(x,\nabla u\chi_s))\cdot(\nabla u_n - \nabla u\chi_s)\,dx\\ &\leq \int_{\Omega} (a(x,\nabla u_n) - a(x,\nabla u\chi_s))\cdot(\nabla u_n - \nabla u\chi_s)\,dx.\end{aligned} \tag{13}$$

thanks to (12), we obtain

$$\lim_{n\to\infty} \int_{\Omega_r} (a(x,\nabla u_n) - a(x,\nabla u))\cdot(\nabla u_n - \nabla u)\,dx = 0. \tag{14}$$

Using the same argument as in [15], we claim that,

$$\nabla u_n \longrightarrow \nabla u \qquad \text{a.e. in} \quad \Omega. \tag{15}$$

On the other hand, we have

$$\begin{aligned}\int_\Omega a(x,\nabla u_n)\cdot\nabla u_n\,dx &= \int_\Omega (a(x,\nabla u_n) - a(x,\nabla u\chi_s))\cdot(\nabla u_n - \nabla u\chi_s)\,dx\\ &+ \int_\Omega a(x,\nabla u\chi_s)\cdot(\nabla u_n - \nabla u\chi_s)\,dx + \int_\Omega a(x,\nabla u_n)\cdot\nabla u\chi_s\,dx.\end{aligned} \tag{16}$$

For the second term on the right-hand side of (16), having in mind that $\psi(x,s)$ verify Δ_2−condition, then $L_\psi(\Omega) = E_\psi(\Omega)$, and thanks to (8) we have $a(x,\nabla u\chi_s) \in (E_\psi(\Omega))^N$. Moreover, we have $\nabla u_n \rightharpoonup \nabla u$ weakly in $(L_\varphi(\Omega))^N$ for $\sigma(\Pi L_\varphi(\Omega), \Pi E_\psi(\Omega))$, then

$$\begin{aligned}\lim_{s,n\to\infty} \int_\Omega a(x,\nabla u\chi_s)\cdot(\nabla u_n - \nabla u\chi_s)\,dx &= \lim_{s\to\infty} \int_\Omega a(x,\nabla u\chi_s)\cdot(\nabla u - \nabla u\chi_s)\,dx\\ &= \lim_{s\to\infty} \int_{\Omega/\Omega_s} a(x,0)\cdot\nabla u\,dx = 0.\end{aligned} \tag{17}$$

Concerning the last term on the right-hand side of (16), since $(a(x,\nabla u_n))_n$ is bounded in $(E_\psi(\Omega))^N$ and using (15), we obtain

$$a(x,\nabla u_n) \rightharpoonup a(x,\nabla u) \quad \text{weakly in} \quad (E_\psi(\Omega))^N \quad \text{for} \quad \sigma(\Pi E_\psi(\Omega), \Pi L_\varphi(\Omega)),$$

which implies that

$$\begin{aligned}\lim_{s,n\to\infty} \int_\Omega a(x,\nabla u_n)\cdot\nabla u\chi_s\,dx &= \lim_{s\to\infty} \int_\Omega a(x,\nabla u)\cdot\nabla u\chi_s\,dx\\ &= \int_\Omega a(x,\nabla u)\cdot\nabla u\,dx.\end{aligned} \tag{18}$$

By combining (12) and (16) – (18), we conclude that

$$\int_\Omega a(x,\nabla u_n)\cdot\nabla u_n\,dx \longrightarrow \int_\Omega a(x,\nabla u)\cdot\nabla u\,dx \quad \text{as} \quad n\to\infty. \tag{19}$$

On the other hand, we have $\varphi(x,|\nabla u_n|)\geq 0$ and $\varphi(x,|\nabla u_n|)\to\varphi(x,|\nabla u|)$ a.e. in Ω, by using the Fatou's Lemma we obtain

$$\int_\Omega \varphi(x,|\nabla u|)\,dx \leq \liminf_{n\to\infty}\int_\Omega \varphi(x,|\nabla u_n|)\,dx. \tag{20}$$

Moreover, since $a(x,\nabla u_n)\cdot\nabla u_n-\alpha\varphi(x,|\nabla u_n|)\geq 0$ and

$$a(x,\nabla u_n)\cdot\nabla u_n-\alpha\varphi(x,|\nabla u_n|)\longrightarrow a(x,\nabla u)\cdot\nabla u-\alpha\varphi(x,|\nabla u|) \qquad \text{a.e. in} \quad \Omega,$$

Thanks to Fatou's Lemma, we get

$$\int_\Omega a(x,\nabla u)\cdot\nabla u-\alpha\varphi(x,|\nabla u|)\,dx \leq \liminf_{n\to\infty}\int_\Omega a(x,\nabla u_n)\cdot\nabla u_n-\alpha\varphi(x,|\nabla u_n|)\,dx,$$

using (19), we obtain

$$\int_\Omega \varphi(x,|\nabla u|)\,dx \geq \limsup_{n\to\infty}\int_\Omega \varphi(x,|\nabla u_n|)\,dx. \tag{21}$$

By combining (20) and (21), we deduce

$$\int_\Omega \varphi(x,|\nabla u_n|)\,dx \longrightarrow \int_\Omega \varphi(x,|\nabla u|)\,dx \qquad \text{as} \quad n\to\infty. \tag{22}$$

In view of Lemma 4.1, we conclude that

$$\varphi(x,|\nabla u_n|)\longrightarrow\varphi(x,|\nabla u|) \qquad \text{in} \quad L^1(\Omega), \tag{23}$$

which finishes our proof.

5. Main results

Let $k>0$, we define the truncation function $T_k(\cdot):I\!R\longmapsto I\!R$ by

$$T_k(s)=\begin{cases} s & \text{if} \quad |s|\leq k,\\ k\dfrac{s}{|s|} & \text{if} \quad |s|>k.\end{cases}$$

Definition 5.1. A measurable function u is called an entropy solution of the quasilinear unilateral elliptic problem (11) if

$$\begin{cases} T_k(u)\in K_\Psi \qquad \text{for any} \quad k>\|\Psi^+\|_\infty,\\[2ex] \displaystyle\int_\Omega a(x,\nabla u)\cdot\nabla T_k(u-v)\,dx \leq \int_\Omega fT_k(u-v)\,dx \qquad \forall v\in K_\Psi\cap L^\infty(\Omega).\end{cases} \tag{24}$$

Theorem 5.1. *Assuming that* (7) – (10) *hold, and* $f\in L^1(\Omega)$, *Then, the problem* (11) *has a unique entropy solution.*

5.1. Existence of entropy solution.

Step 1 : Approximate problems. Let $(f_n)_{n\in I\!N} \in W^{-1}E_\psi(\Omega)\cap L^\infty(\Omega)$ be a sequence of smooth functions such that $f_n \to f$ in $L^1(\Omega)$ and $|f_n| \leq |f|$ (for example $f_n = T_n(f)$). We consider the approximate problem

$$(P_n)\begin{cases} u_n \in K_\Psi, \\ \displaystyle\int_\Omega a(x,\nabla u_n)\cdot\nabla(u_n - v)\,dx \leq \int_\Omega f_n(u_n - v)\,dx \quad \text{for any} \quad v \in K_\Psi \cap L^\infty(\Omega). \end{cases} \tag{25}$$

Let $X = K_\Psi$, we define the operator $A : X \longmapsto X^*$ by

$$\langle Au, v\rangle = \int_\Omega a(x,\nabla u)\cdot\nabla v\;dx \qquad \forall v \in K_\Psi.$$

Using (6), we have for any $u, v \in K_\Psi$,

$$\begin{aligned}\Big|\int_\Omega a(x,\nabla u)\cdot\nabla v\,dx\Big| &\leq \int_\Omega \beta\big(K(x) + k_1\psi_x^{-1}(\varphi(x,k_2|\nabla u|))\big)|\nabla v|\,dx \\ &\leq \beta\int_\Omega \psi(x,K(x))\,dx + \beta k_1\int_\Omega \varphi(x,k_2|\nabla u|)\,dx + \beta(1+k_1)\int_\Omega \varphi(x,|\nabla v|)\,dx.\end{aligned} \tag{26}$$

Lemma 5.2. *The operator A acted from $W_0^1L_\varphi(\Omega)$ in to $W^{-1}L_\psi(\Omega) = W^{-1}E_\psi(\Omega)$ is bounded and pseudo-monotone. Moreover,A is coercive in the following sense : there exists $v_0 \in K_\Psi$ such that*

$$\frac{\langle Av, v - v_0\rangle}{||v||_{1,\varphi,\Omega}} \longrightarrow \infty \qquad as \quad ||v||_{1,\varphi,\Omega} \to \infty \qquad for \quad v \in K_\Psi.$$

Proof of Lemma 5.2. In view of (26), the operator A is bounded. For the coercivity, let $\varepsilon > 0$, we have for $v_0 \in K_\Psi$ and any $v \in W_0^1L_\varphi(\Omega)$

$$\begin{aligned}\big|\langle Av, v_0\rangle\big| &\leq \int_\Omega \big|a(x,\nabla v)\big|\,\big|\nabla v_0\big|\,dx \leq \beta\int_\Omega (K(x) + k_1\psi_x^{-1}(\varphi(x,k_2|\nabla v|)))|\nabla v_0|\,dx \\ &\leq \beta\int_\Omega K(x)|\nabla v_0|\,dx + \beta k_1\varepsilon\int_\Omega \psi_x^{-1}(\varphi(x,k_2|\nabla v|))\frac{1}{\varepsilon}|\nabla v_0|\,dx \\ &\leq \beta\int_\Omega \psi(x,K(x))dx + \beta\int_\Omega \varphi(x,|\nabla v_0|)dx + \beta k_1\varepsilon\int_\Omega \varphi(x,k_2|\nabla v|)\,dx \\ &\quad + \beta k_1\varepsilon\int_\Omega \varphi(x,\frac{1}{\varepsilon}|\nabla v_0|)\,dx \\ &\leq c_\varepsilon\int_\Omega \varphi(x,|\nabla v|)dx + \beta(k_1\varepsilon + 1)\int_\Omega \varphi(x,(\frac{1}{\varepsilon}+1)|\nabla v_0|)\,dx + C_1,\end{aligned}$$

with c_ε is a constant depending on ε. By taking ε small enough such that $c_\varepsilon \leq \frac{\alpha}{2}$, we obtain

$$\langle Av, v_0\rangle \leq \frac{\alpha}{2}\int_\Omega \varphi(x,|\nabla v|)dx + \beta(k_1\varepsilon + 1)\int_\Omega \varphi(x,(\frac{1}{\varepsilon}+1)|\nabla v_0|)\,dx + C_1.$$

On the other hand, in view of (10), we have

$$\langle Av, v\rangle = \int_\Omega a(x,\nabla v)\cdot\nabla v\,dx \geq \alpha\int_\Omega \varphi(x,|\nabla v|)\,dx.$$

Therefore

$$\frac{\langle Av, v-v_0\rangle}{||v||_{1,\varphi,\Omega}} = \frac{\langle Av, v\rangle - \langle Av, v_0\rangle}{||v||_{1,\varphi,\Omega}}$$

$$\geq \frac{\alpha\int_\Omega \varphi(x,|\nabla v|)\,dx - \frac{\alpha}{2}\int_\Omega \varphi(x,|\nabla v|)dx - \beta(k_1\varepsilon+1)\int_\Omega \varphi(x,(\frac{1}{\varepsilon}+1)|\nabla v_0|)\,dx + C_1}{\|v\|_{1,\varphi,\Omega}}$$

$$= \frac{\frac{\alpha}{2}\int_\Omega \varphi(x,|\nabla v|)\,dx - \beta(k_1\varepsilon+1)\int_\Omega \varphi(x,(\frac{1}{\varepsilon}+1)|\nabla v_0|)\,dx + C_1}{\|v\|_{1,\varphi,\Omega}} \longrightarrow \infty$$

as $\|v\|_{1,\varphi,\Omega}$ goes to infinity.
It remains to show that A is pseudo-monotone. Let $(u_k)_k$ be a sequence in $W_0^1L_\varphi(\Omega)$ such that

$$\begin{cases} u_k \rightharpoonup u \textit{ in } W_0^1L_\varphi(\Omega) & \text{for} \quad \sigma(\Pi L_\varphi(\Omega), \Pi E_\psi(\Omega)), \\ Au_k \rightharpoonup \chi \textit{ in } W^{-1}E_\psi(\Omega) & \text{for} \quad \sigma(\Pi E_\psi(\Omega), \Pi L_\varphi(\Omega)), \\ \limsup\limits_{k\to\infty}\langle Au_k, u_k\rangle \leq \langle \chi, u\rangle. \end{cases} \tag{27}$$

We will prove that

$$\chi = Au \textit{ and } \langle Au_k, u_k\rangle \to \langle \chi, u\rangle \textit{ as } k\to\infty.$$

Firstly, since $W_0^1L_\varphi(\Omega) \hookrightarrow\hookrightarrow E_\varphi(\Omega)$, then $u_k \to u$ in $E_\varphi(\Omega)$ for a subsequence still denoted $(u_k)_k$.
As $(u_k)_k$ is a bounded sequence in $W_0^1L_\varphi(\Omega)$ and thanks to the growth condition (8), it follows that $(a(x,\nabla u_k))_k$ is bounded in $(E_\psi(\Omega))^N$. Therefore, there exists a function $\xi \in (E_\psi(\Omega))^N$ such that

$$a(x,\nabla u_k) \rightharpoonup \xi \quad \text{in} \quad (E_\psi(\Omega))^N \quad \text{for} \quad \sigma(\Pi E_\psi(\Omega), \Pi L_\varphi(\Omega)) \quad \text{as} \quad k\to\infty. \tag{28}$$

It is clear that, for all $v \in W_0^1L_\varphi(\Omega)$, we have

$$\langle \chi, v\rangle = \lim_{k\to\infty}\langle Au_k, v\rangle = \lim_{k\to\infty}\int_\Omega a(x,\nabla u_k)\cdot\nabla v\,dx = \int_\Omega \xi\cdot\nabla v\,dx. \tag{29}$$

By using (27) and (29), we obtain

$$\limsup_{k\to\infty}\langle Au_k, u_k\rangle = \limsup_{k\to\infty}\int_\Omega a(x,\nabla u_k)\cdot\nabla u_k\,dx \leq \int_\Omega \xi\cdot\nabla u\,dx. \tag{30}$$

On the other hand, thanks to (9), we have

$$\int_\Omega \Big(a(x,\nabla u_k) - a(x,\nabla u)\Big)\cdot(\nabla u_k - \nabla u)\,dx \geq 0, \tag{31}$$

then

$$\int_\Omega a(x,\nabla u_k)\cdot\nabla u_k\,dx \geq \int_\Omega a(x,\nabla u_k)\cdot\nabla u\,dx + \int_\Omega a(x,\nabla u)\cdot(\nabla u_k - \nabla u)\,dx.$$

In view of (28), we have

$$\liminf_{k\to\infty}\int_\Omega a(x,\nabla u_k)\cdot\nabla u_k\,dx \geq \int_\Omega \xi\cdot\nabla u\,dx$$

and (30) yields

$$\lim_{k\to\infty}\int_\Omega a(x,\nabla u_k)\cdot\nabla u_k\,dx=\int_\Omega \xi\cdot\nabla u\,dx. \tag{32}$$

Combining (29) and (32), we find:

$$\langle Au_k,u_k\rangle\to\langle\chi,u\rangle \qquad \text{as} \quad k\to\infty. \tag{33}$$

In view of (32), we have

$$\lim_{k\to\infty}\int_\Omega \Big(a(x,\nabla u_k)-a(x,\nabla u)\Big)\cdot(\nabla u_k-\nabla u)\,dx\to\ 0$$

which implies, thanks to Lemma 4.2, that

$$u_k\to u \quad \text{in} \quad W_0^1L_\varphi(\Omega) \qquad \text{and} \qquad \nabla u_k\to\nabla u \quad \text{a.e. in} \quad \Omega,$$

then

$$a(x,\nabla u_k)\rightharpoonup a(x,\nabla u) \quad \text{in} \quad (E_\psi(\Omega))^N,$$

we deduce that $\chi=Au$, which completes the proof the Lemma 5.2. □

In view of Lemma 5.2, there exists at least one weak solution $u_n\in W_0^1L_\varphi(\Omega)$ of the problem (25), (cf. [10], Lemma 6).

Step 2 : A priori estimates. Taking $v=u_n-\eta T_k(u_n-\Psi^+)\in W_0^1L_\varphi(\Omega),$ for η small enough we have $v\geq\Psi,$ thus v is an admissible test function in (25), and we obtain

$$\int_\Omega a(x,\nabla u_n)\cdot\nabla T_k(u_n-\Psi^+)\,dx\leq\int_\Omega f_nT_k(u_n-\Psi^+)\,dx,$$

Since $\nabla T_k(u_n-\Psi^+)$ is identically zero on the set $\{|u_n-\Psi^+|>k\}$, we can write

$$\int_{\{|u_n-\Psi^+|\leq k\}} a(x,\nabla u_n)\cdot\nabla(u_n-\Psi^+)\ dx\leq\int_\Omega f_nT_k(u_n-\Psi^+)\,dx\leq C_2k,$$

with $C_2=\|f\|_1,$ it follows that

$$\int_{\{|u_n-\Psi^+|\leq k\}} a(x,\nabla u_n)\cdot\nabla u_n\,dx\leq C_2k+\int_{\{|u_n-\Psi^+|\leq k\}} a(x,\nabla u_n)\cdot\nabla\Psi^+\,dx.$$

Let $0<\lambda<\dfrac{\alpha}{\alpha+1}$, it's clear that

$$\int_{\{|u_n-\Psi^+|\leq k\}} a(x,\nabla u_n)\cdot\nabla u_n\,dx\leq C_2k+\lambda\int_{\{|u_n-\Psi^+|\leq k\}} a(x,\nabla u_n)\cdot\frac{\nabla\Psi^+}{\lambda}\,dx. \tag{34}$$

Thanks to (9), we have

$$\int_{\{|u_n-\Psi^+|\leq k\}}\Big(a(x,\nabla u_n)-a(x,\frac{\nabla\Psi^+}{\lambda})\Big)\cdot(\nabla u_n-\frac{\nabla\Psi^+}{\lambda})\,dx\geq 0,$$

then

$$\begin{aligned}\int_{\{|u_n-\Psi^+|\leq k\}} a(x,\nabla u_n)\cdot\frac{\nabla\Psi^+}{\lambda}\,dx \quad &\leq\int_{\{|u_n-\Psi^+|\leq k\}} a(x,\nabla u_n)\cdot\nabla u_n\,dx\\ &-\int_{\{|u_n-\Psi^+|\leq k\}} a(x,\frac{\nabla\Psi^+}{\lambda})\cdot(\nabla u_n-\frac{\nabla\Psi^+}{\lambda})\,dx.\end{aligned}$$

Which yields thanks to (34), that

$$\begin{aligned}\int_{\{|u_n-\Psi^+|\leq k\}} a(x,\nabla u_n)\cdot\nabla u_n\,dx &\leq C_2k+\lambda\int_{\{|u_n-\Psi^+|\leq k\}} a(x,\nabla u_n)\cdot\nabla u_n\,dx\\ &\quad -\lambda\int_{\{|u_n-\Psi^+|\leq k\}} a(x,\frac{\nabla\Psi^+}{\lambda})\cdot(\nabla u_n-\frac{\nabla\Psi^+}{\lambda})\,dx.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}(1-\lambda)\int_{\{|u_n-\Psi^+|\leq k\}} a(x,\nabla u_n)\cdot\nabla u_n\,dx &\leq C_2k+\lambda\int_{\{|u_n-\Psi^+|\leq k\}} a(x,\frac{\nabla\Psi^+}{\lambda})\cdot\frac{\nabla\Psi^+}{\lambda}\,dx\\ &\quad -\lambda\int_{\{|u_n-\Psi^+|\leq k\}} a(x,\frac{\nabla\Psi^+}{\lambda})\cdot\nabla u_n\,dx,\end{aligned} \tag{35}$$

In view of (6), we have

$$\begin{aligned}\Big|\int_{\{|u_n-\Psi^+|\leq k\}} a(x,\frac{\nabla\Psi^+}{\lambda})\cdot\nabla u_n\,dx\Big| &\leq \int_{\{|u_n-\Psi^+|\leq k\}}\psi(x,|a(x,\frac{\nabla\Psi^+}{\lambda})|)\,dx\\ &\quad+\int_{\{|u_n-\Psi^+|\leq k\}}\varphi(x,|\nabla u_n|)\,dx.\end{aligned}$$

Having in mind (10) and (35), we obtain

$$\begin{aligned}&\Big(\alpha(1-\lambda)-\lambda\Big)\int_{\{|u_n-\Psi^+|\leq k\}}\varphi(x,|\nabla u_n|)\,dx\\ &\leq (1-\lambda)\int_{\{|u_n-\Psi^+|\leq k\}} a(x,\nabla u_n)\cdot\nabla u_n\,dx-\lambda\int_{\{|u_n-\Psi^+|\leq k\}}\varphi(x,|\nabla u_n|)\,dx\\ &\leq C_2k+\lambda\int_{\{|u_n-\Psi^+|\leq k\}} a(x,\frac{\nabla\Psi^+}{\lambda})\cdot\frac{\nabla\Psi^+}{\lambda}\,dx+\lambda\int_{\{|u_n-\Psi^+|\leq k\}}\psi(x,|a(x,\frac{\nabla\Psi^+}{\lambda})|)\,dx,\end{aligned}$$

then,

$$\int_{\{|u_n-\Psi^+|\leq k\}}\varphi(x,|\nabla u_n|)\,dx\leq C_3k \qquad \text{for} \quad k\geq 1. \tag{36}$$

On the other hand, since $\{|u_n|\leq k\}\subset\{|u_n-\Psi^+|\leq k+||\Psi^+||_\infty\}$, then

$$\begin{aligned}\int_\Omega\varphi(x,|\nabla T_k(u_n)|)\,dx &= \int_{\{|u_n|\leq k\}}\varphi(x,|\nabla u_n|)\,dx\\ &\leq \int_{\{|u_n-\Psi^+|\leq k+||\Psi^+||_\infty\}}\varphi(x,|\nabla u_n|)\,dx\\ &\leq C_3(k+||\Psi^+||_\infty),\end{aligned}$$

which implies that

$$\int_\Omega\varphi(x,|\nabla T_k(u_n)|)\,dx\leq C_4k \qquad \text{for} \quad k\geq\max(1,\|\Psi^+\|_\infty), \tag{37}$$

with C_4 is a constant that does not depend on n and k.
Thus $(T_k(u_n))_n$ is bounded in $W_0^1L_\varphi(\Omega)$ uniformly in n, then there exists a subsequence still denoted $(T_k(u_n))_{n\in\mathbb{N}}$ and $v_k\in W_0^1L_\varphi(\Omega)$ such that

$$\begin{cases} T_k(u_n)\rightharpoonup v_k & \text{weakly in } W_0^1L_\varphi(\Omega) \text{ for } \sigma(\Pi L_\varphi(\Omega),\Pi E_\psi(\Omega)),\\ T_k(u_n)\to v_k & \text{strongly in } E_\varphi(\Omega) \text{ and a.e in } \Omega.\end{cases} \tag{38}$$

Step 3 : Convergence in measure of u_n. In view of (7), we have

$$M(t) \leq \varphi(x, ct) \quad \text{a.e. in} \quad \Omega \qquad \text{with} \qquad \lim_{t\to 0} \frac{M(t)}{t} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{M(t)}{t} = \infty.$$

In view of ([15], Lemma 5.7), there exists two positive constants C_5 and C_6, and a function $q(\cdot) \in L^1(\Omega)$ such that

$$C_5 \int_\Omega M(|T_k(u_n)|)dx + \int_\Omega q(x)dx \leq \int_\Omega M(C_6|\nabla T_k(u_n)|) + q(x)dx \leq \int_\Omega \varphi(x, |\nabla T_k(u_n)|)dx.$$

So, in virtue of (37), we obtain

$$\int_\Omega M(|T_k(u_n)|)\, dx \leq kC_7 \qquad \text{for} \quad k \geq \max(1, \|\Psi^+\|_\infty). \tag{39}$$

Then, we deduce that,

$$\begin{aligned} M(k)\ \text{meas}(\{|u_n| > k\}) &= \int_{\{|u_n|>k\}} M(|T_k(u_n)|)\, dx \\ &\leq \int_\Omega M(|T_k(u_n)|)\, dx \leq kC_7, \end{aligned}$$

hence,

$$\text{meas}(\{|u_n| > k\}) = \frac{kC_7}{M(k)} \longrightarrow 0 \quad \text{as} \quad k \to +\infty. \tag{40}$$

For all $\delta > 0$, we have

$$\begin{aligned} \text{meas}\{|u_n - u_m| > \delta\} &\leq \text{meas}\{|u_n| > k\} + \text{meas}\{|u_m| > k\} \\ &\quad + \text{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\}. \end{aligned}$$

Let $\varepsilon > 0$, using (40) we may choose $k = k(\varepsilon)$ large enough such that

$$\text{meas}\{|u_n| > k\} \leq \frac{\varepsilon}{3} \qquad \text{and} \qquad \text{meas}\{|u_m| > k\} \leq \frac{\varepsilon}{3}. \tag{41}$$

Moreover, in view of (38) we have $T_k(u_n) \to v_k$ strongly in $E_\varphi(\Omega)$, then, we can assume that $(T_k(u_n))_{n\in I\!N}$ is a Cauchy sequence in measure. Thus, for all $k > 0$ and $\delta, \varepsilon > 0$, there exists $n_0 = n_0(k, \delta, \varepsilon)$ such that

$$\text{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\} \leq \frac{\varepsilon}{3} \qquad \text{for all } m, n \geq n_0(k, \delta, \varepsilon). \tag{42}$$

By combining (41) – (42), we conclude that

$$\forall \delta, \varepsilon > 0 \quad \text{there exists} \quad n_0 = n_0(\delta, \varepsilon) \quad \text{such that} \quad \text{meas}\{|u_n - u_m| > \delta\} \leq \varepsilon$$

for any $n, m \geq n_0(\delta, \varepsilon)$. It follows that $(u_n)_n$ is a Cauchy sequence in measure, then converges almost everywhere, for a subsequence, to some measurable function u. Consequently, we have

$$\begin{cases} T_k(u_n) \rightharpoonup T_k(u) & \text{weakly in} \quad W_0^1 L_\varphi(\Omega) \quad \text{for} \quad \sigma(\Pi L_\varphi(\Omega), \Pi E_\psi(\Omega)), \\ T_k(u_n) \to T_k(u) & \text{strongly in} \quad E_\varphi(\Omega). \end{cases} \tag{43}$$

Step 4 : Strong convergence of truncations. In the sequel, we denote by $\varepsilon_i(n)$, $i = 1, 2, \ldots$ various real-valued functions of real variables that converges to 0 as n tends to infinity.
Let $h > k > 0$, we define

$$M := 4k + h, \qquad z_n := u_n - T_h(u_n) + T_k(u_n) - T_k(u) \qquad \text{and} \qquad \omega_n := T_{2k}(z_n).$$

Taking $v = u_n - \eta\omega_n$, we have $v \geq \Psi$ for η small enough, thus v is an admissible test function in (25), and since $\nabla\omega_n = 0$ on $\{|u_n| \geq M\}$, we obtain

$$\int_{\{|u_n|\leq M\}} a(x, \nabla T_M(u_n)) \cdot \nabla\omega_n \, dx \leq \int_\Omega f_n\omega_n \, dx.$$

We have $\omega_n = T_k(u_n) - T_k(u)$ on $\{|u_n| \leq k\}$, we conclude that

$$\begin{aligned}\int_{\{|u_n|\leq k\}} & a(x, \nabla T_k(u_n)) \cdot (\nabla T_k(u_n) - \nabla T_k(u)) \, dx \\ & + \int_{\{k<|u_n|\leq M\}} a(x, \nabla T_M(u_n)) \cdot \nabla\omega_n \, dx \leq \int_\Omega f_n\omega_n \, dx.\end{aligned} \tag{44}$$

Concerning the second term on the left-hand side of (44), we have

$$\begin{aligned}&\int_{\{k<|u_n|\leq M\}} a(x, \nabla T_M(u_n)) \cdot \nabla\omega_n \, dx \\ &\quad = \int_{\{k<|u_n|\leq M\}\cap\{|z_n|\leq 2k\}} a(x, \nabla T_M(u_n)) \cdot \nabla(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) \, dx \\ &\quad \geq - \int_{\{k<|u_n|\leq M\}} |a(x, \nabla T_M(u_n))| \, |\nabla T_k(u)| \, dx,\end{aligned}$$

We have $\nabla T_k(u) \in (L_\varphi(\Omega))^N$, and since $(|a(x, \nabla T_M(u_n))|)_n$ is bounded in $L_\psi(\Omega) = E_\psi(\Omega)$, there exists $\zeta \in E_\psi(\Omega)$ such that $|a(x, \nabla T_M(u_n))| \rightharpoonup \zeta$ weakly in $E_\psi(\Omega)$ for $\sigma(E_\psi(\Omega), L_\varphi(\Omega))$. Therefore,

$$\int_{\{k<|u_n|\leq M\}} |a(x, \nabla T_M(u_n))| \, |\nabla T_k(u)| \, dx \longrightarrow \int_{\{k<|u|\leq M\}} \zeta \, |\nabla T_k(u)| \, dx = 0. \tag{45}$$

It follows that

$$\int_{\{k<|u_n|\leq M\}} a(x, \nabla T_M(u_n)) \cdot \nabla\omega_n \, dx \geq \varepsilon_1(n). \tag{46}$$

Then, since $f_n \to f$ in $L^1(\Omega)$ and $\omega_n \rightharpoonup T_{2k}(u - T_h(u))$ weak$-*$ in $L^\infty(\Omega)$, and using (44), we deduce that

$$\int_{\{|u_n|\leq k\}} a(x, \nabla T_k(u_n)) \cdot (\nabla T_k(u_n) - \nabla T_k(u)) \, dx \leq \int_\Omega f T_{2k}(u - T_h(u)) \, dx + \varepsilon_2(n). \tag{47}$$

We define $\Omega_s = \big\{x \in \Omega : |\nabla T_k(u(x))| \leq s\big\}$ and denote by χ_s the characteristic function of Ω_s. For the term on the left-hand side of (47), we have

$$\begin{aligned}
&\int_{\{|u_n|\leq k\}} a(x,\nabla T_k(u_n))\cdot(\nabla T_k(u_n)-\nabla T_k(u))\,dx \\
&= \int_\Omega a(x,\nabla T_k(u_n))\cdot(\nabla T_k(u_n)-\nabla T_k(u)\chi_s)\,dx \\
&\quad + \int_\Omega a(x,\nabla T_k(u_n))\cdot(\nabla T_k(u)\chi_s-\nabla T_k(u))\,dx + \int_{\{|u_n|>k\}} a(x,\nabla T_k(u_n))\cdot\nabla T_k(u)\,dx \\
&= \int_\Omega \big(a(x,\nabla T_k(u_n))-a(x,\nabla T_k(u)\chi_s)\big)\cdot(\nabla T_k(u_n)-\nabla T_k(u)\chi_s)\,dx \\
&\quad + \int_\Omega a(x,\nabla T_k(u)\chi_s)\cdot(\nabla T_k(u_n)-\nabla T_k(u)\chi_s)\,dx \\
&\quad - \int_{\Omega\setminus\Omega_s} a(x,\nabla T_k(u_n))\cdot\nabla T_k(u)\,dx + \int_{\{|u_n|>k\}} a(x,\nabla T_k(u_n))\cdot\nabla T_k(u)\,dx.
\end{aligned} \tag{48}$$

For the second term on the right-hand side of (48), we have $a(x,\nabla T_k(u)\chi_s) \in (E_\psi(\Omega))^N$, and since $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$ weakly in $(L_\varphi(\Omega))^N$ for $\sigma(\Pi L_\varphi(\Omega),\Pi E_\psi(\Omega))$, then

$$\begin{aligned}
&\lim_{n\to\infty}\int_\Omega a(x,\nabla T_k(u)\chi_s)\cdot(\nabla T_k(u_n)-\nabla T_k(u)\chi_s)\,dx \\
&\qquad = \int_\Omega a(x,\nabla T_k(u)\chi_s)\cdot(\nabla T_k(u)-\nabla T_k(u)\chi_s)\,dx \\
&\qquad = \int_{\Omega\setminus\Omega_s} a(x,0)\cdot\nabla T_k(u)\,dx.
\end{aligned} \tag{49}$$

Concerning the third term on the right-hand side of (48), since $(a(x,\nabla T_k(u_n))_n$ is bounded in $(E_\psi(\Omega))^N$, there exists $\xi \in (E_\psi(\Omega))^N$ such that $a(x,\nabla T_k(u_n)) \rightharpoonup \xi$ weakly in $(E_\psi(\Omega))^N$ for $\sigma(\Pi E_\psi(\Omega),\Pi L_\varphi(\Omega))$, it follows that

$$\lim_{n\to\infty}\int_{\Omega\setminus\Omega_s} a(x,\nabla T_k(u_n))\cdot\nabla T_k(u)\,dx = \int_{\Omega\setminus\Omega_s}\xi\cdot\nabla T_k(u)\,dx. \tag{50}$$

For the last term on the right-hand side of (48), we obtain

$$\lim_{n\to\infty}\int_{\{|u_n|>k\}} a(x,\nabla T_k(u_n))\cdot\nabla T_k(u)\,dx = \int_{\{|u|>k\}}\xi\cdot\nabla T_k(u)\,dx = 0. \tag{51}$$

By combining (48) – (51), we deduce that

$$\begin{aligned}
&\int_{\{|u_n|\leq k\}} a(x,\nabla T_k(u_n))\cdot(\nabla T_k(u_n)-\nabla T_k(u))\,dx \\
&= \int_\Omega \big(a(x,\nabla T_k(u_n))-a(x,\nabla T_k(u)\chi_s)\big)\cdot(\nabla T_k(u_n)-\nabla T_k(u)\chi_s)\,dx \\
&\qquad + \int_{\Omega\setminus\Omega_s}\big(a(x,0)-\xi\big)\cdot\nabla T_k(u)\,dx + \varepsilon_3(n)
\end{aligned} \tag{52}$$

and since $\big(a(x,0)-\eta\big)\cdot\nabla T_k(u) \in L^1(\Omega)$, then

$$\int_{\Omega\setminus\Omega_s}\big(a(x,0)-\xi\big)\cdot\nabla T_k(u)\,dx \longrightarrow 0 \qquad \text{as} \quad s\to\infty.$$

Therefore, using (47) we conclude that

$$\begin{aligned}&\int_\Omega \big(a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u)\chi_s)\big)\cdot(\nabla T_k(u_n) - \nabla T_k(u)\chi_s)\,dx\\ &\qquad\le \int_\Omega fT_{2k}(u - T_h(u))\,dx + \varepsilon_4(n,s).\end{aligned} \tag{53}$$

We have

$$\int_\Omega fT_{2k}(u - T_h(u))\,dx \longrightarrow 0 \quad \text{as} \quad h\to\infty.$$

It follows that

$$\lim_{n,s\to\infty}\int_\Omega \big(a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u)\chi_s)\big)\cdot\big(\nabla T_k(u_n) - \nabla T_k(u)\chi_s\big)\,dx = 0. \tag{54}$$

In view of Lemma 4.2, we deduce that

$$\nabla u_n \longrightarrow \nabla u \qquad \text{a.e. in} \quad \Omega, \tag{55}$$

and

$$\varphi(x,|\nabla T_k(u_n)|) \longrightarrow \varphi(x,|\nabla T_k(u)|) \qquad \text{in} \quad L^1(\Omega). \tag{56}$$

Step 5 : Passage to the limit. Let $v\in K_\Psi\cap L^\infty(\Omega)$ and $\eta>0$, we have $u_n-\eta T_k(u_n-v)\ \in K_\Psi$ is an admissible test function in (25) for η small enough, and we obtain

$$\int_\Omega a(x,\nabla u_n)\cdot\nabla T_k(u_n - v)\,dx \le \int_\Omega f_n T_k(u_n - v)\,dx. \tag{57}$$

Choosing $M = k + \|v\|_\infty$, then $\{|u_n - v|\le k\}\subseteq\{|u_n|\le M\}$. Firstly, we can write the term on the left-hand side of the above relation as

$$\begin{aligned}&\int_\Omega a(x,\nabla u_n)\cdot\nabla T_k(u_n - v)\,dx = \int_\Omega a(x,\nabla T_M(u_n))\cdot(\nabla T_M(u_n) - \nabla v)\chi_{\{|u_n - v|\le k\}}\,dx\\ &\quad= \int_\Omega (a(x,\nabla T_M(u_n)) - a(x,\nabla v))\cdot(\nabla T_M(u_n) - \nabla v)\chi_{\{|u_n - v|\le k\}}\,dx\\ &\qquad+\int_\Omega a(x,\nabla v)\cdot(\nabla T_M(u_n) - \nabla v)\chi_{\{|u_n - v|\le k\}}\,dx.\end{aligned} \tag{58}$$

We have

$$\begin{aligned}&(a(x,\nabla T_M(u_n)) - a(x,\nabla v))\cdot(\nabla T_M(u_n) - \nabla v)\chi_{\{|u_n - v|\le k\}}\\ &\longrightarrow (a(x,\nabla T_M(u)) - a(x,\nabla v))\cdot(\nabla T_M(u) - \nabla v)\chi_{\{|u - v|\le k\}} \quad \text{a.e. in} \quad \Omega.\end{aligned} \tag{59}$$

According to (9) and Fatou's lemma, we obtain

$$\begin{aligned}&\liminf_{n\to\infty}\int_\Omega a(x,\nabla u_n)\cdot\nabla T_k(u_n - v)\,dx\\ &\qquad\ge \int_\Omega (a(x,\nabla T_M(u)) - a(x,\nabla v))\cdot(\nabla T_M(u) - \nabla v)\chi_{\{|u - v|\le k\}}\,dx\\ &\qquad\quad+\lim_{n\to\infty}\int_\Omega a(x,\nabla v)\cdot(\nabla T_M(u_n) - \nabla v)\chi_{\{|u_n - v|\le k\}}\,dx.\end{aligned} \tag{60}$$

For the second term on the right-hand side of (60), we have $a(x,\nabla v)\ \in\ (E_\psi(\Omega))^N$ and $\nabla T_M(u_n)\rightharpoonup\nabla T_M(u)$ weakly in $(L_\varphi(\Omega))^N$ for $\sigma(\Pi L_\varphi(\Omega),\Pi E_\psi(\Omega))$, then

$$\begin{aligned}&\lim_{n\to\infty}\int_\Omega a(x,\nabla v)\cdot(\nabla T_M(u_n) - \nabla v)\chi_{\{|u_n - v|\le k\}}\,dx\\ &\qquad= \int_\Omega a(x,\nabla v)\cdot(\nabla T_M(u) - \nabla v)\chi_{\{|u - v|\le k\}}\,dx.\end{aligned}$$

Therefore, we get

$$\begin{aligned}\liminf_{n\to\infty}\int_\Omega a(x,\nabla u_n)\cdot\nabla T_k(u_n-v)\,dx &\geq \int_\Omega a(x,\nabla T_M(u))\cdot(\nabla T_M(u)-\nabla v)\chi_{\{|u-v|\leq k\}}\,dx\\ &= \int_\Omega a(x,\nabla u)\cdot\nabla T_k(u-v)\,dx.\end{aligned} \tag{61}$$

On the other hand, being $T_k(u_n-v)\rightharpoonup T_k(u-v)$ weak-$\star$ in $L^\infty(\Omega)$ we deduce that

$$\int_\Omega f_n T_k(u_n-v)\,dx \longrightarrow \int_\Omega f\ T_k(u-v)\,dx. \tag{62}$$

By combining (61) and (62), we conclude the existence of entropy solution for our problem.

5.2. Uniqueness of entropy solution. Let u_1,u_2 be two entropy solutions of the problems (24), we shall prove that $u_1=u_2$.
By using the test function $v=T_h(u_2)\in K_\Psi\cap L^\infty(\Omega)$ in (24) for the equation with solution u_1, we have

$$\int_\Omega a(x,\nabla u_1)\cdot\nabla T_k(u_1-T_h(u_2))\,dx \leq \int_\Omega fT_k(u_1-T_h(u_2))\,dx.$$

Similarly, by using $v=T_h(u_1)\in K_\Psi\cap L^\infty(\Omega)$ as a test function for the equation (24) with solution u_2, we obtain

$$\int_\Omega a(x,\nabla u_2)\cdot\nabla T_k(u_2-T_h(u_1))\,dx \leq \int_\Omega fT_k(u_2-T_h(u_1))\,dx.$$

By adding these two inequalities, we get

$$\begin{aligned}&\int_\Omega a(x,\nabla u_1)\cdot\nabla T_k(u_1-T_h(u_2))\,dx+\int_\Omega a(x,\nabla u_2)\cdot\nabla T_k(u_2-T_h(u_1))\,dx\\ &\leq \int_\Omega f[T_k(u_1-T_h(u_2))+T_k(u_2-T_h(u_1))]\,dx.\end{aligned} \tag{63}$$

We decompose the first integral of the left-hand side of (63) as

$$\begin{aligned}\int_\Omega a(x,\nabla u_1)\cdot\nabla T_k(u_1-T_h(u_2))\,dx &= \int_{\{|u_1-T_h(u_2)|\leq k\}} a(x,\nabla u_1)\cdot\nabla(u_1-T_h(u_2))\,dx\\ &= \int_{\{|u_1-u_2|\leq k\}\cap\{|u_2|\leq h\}} a(x,\nabla u_1)\cdot(\nabla u_1-\nabla u_2)\,dx\\ &\quad+\int_{\{|u_1-T_h(u_2)|\leq k\}\cap\{|u_2|>h\}} a(x,\nabla u_1)\cdot\nabla u_1\,dx,\\ &\geq \int_{\{|u_1-u_2|\leq k\}\cap\{|u_2|\leq h\}\cap\{|u_1|\leq h\}} a(x,\nabla u_1)\cdot(\nabla u_1-\nabla u_2)\,dx\\ &\quad+\int_{\{|u_1-u_2|\leq k\}\cap\{|u_2|\leq h\}\cap\{|u_1|>h\}} a(x,\nabla u_1)\cdot(\nabla u_1-\nabla u_2)\,dx.\end{aligned} \tag{64}$$

Similarly, we have

$$\int_\Omega a(x,\nabla u_2)\cdot\nabla T_k(u_2-T_h(u_1))\,dx \geq \int_{\{|u_2-u_1|\leq k\}\cap\{|u_1|\leq h\}\cap\{|u_2|\leq h\}} a(x,\nabla u_2)\cdot(\nabla u_2-\nabla u_1)\,dx$$
$$+\int_{\{|u_2-u_1|\leq k\}\cap\{|u_1|\leq h\}\cap\{|u_2|>h\}} a(x,\nabla u_2)\cdot(\nabla u_2-\nabla u_1)\,dx. \tag{65}$$

By combining (64) – (65), we obtain

$$\begin{aligned}
&\int_\Omega a(x,\nabla u_1)\cdot\nabla T_k(u_1-T_h(u_2))\,dx+\int_\Omega a(x,\nabla u_2)\cdot\nabla T_k(u_2-T_h(u_1))\,dx\\
&\quad\geq\int_{\{|u_1-u_2|\leq k\}\cap\{|u_2|\leq h\}\cap\{|u_1|\leq h\}}\big(a(x,\nabla u_1)-a(x,\nabla u_2)\big)\cdot(\nabla u_1-\nabla u_2)\,dx\\
&\qquad+\int_{\{|u_1-u_2|\leq k\}\cap\{|u_2|\leq h\}\cap\{|u_1|>h\}} a(x,\nabla u_1)\cdot(\nabla u_1-\nabla u_2)\,dx\\
&\qquad+\int_{\{|u_2-u_1|\leq k\}\cap\{|u_1|\leq h\}\cap\{|u_2|>h\}} a(x,\nabla u_2)\cdot(\nabla u_2-\nabla u_1)\,dx.
\end{aligned}$$

In view of (63), we conclude that

$$\begin{aligned}
&\int_{\{|u_1-u_2|\leq k\}\cap\{|u_2|\leq h\}\cap\{|u_1|\leq h\}}\big(a(x,\nabla u_1)-a(x,\nabla u_2)\big)\cdot(\nabla u_1-\nabla u_2)\,dx\\
&\quad\leq\int_\Omega f[T_k(u_1-T_h(u_2))+T_k(u_2-T_h(u_1))]\,dx\\
&\qquad-\int_{\{|u_1-u_2|\leq k\}\cap\{|u_2|\leq h\}\cap\{|u_1|>h\}} a(x,\nabla u_1)\cdot(\nabla u_1-\nabla u_2)\,dx\\
&\qquad-\int_{\{|u_2-u_1|\leq k\}\cap\{|u_1|\leq h\}\cap\{|u_2|>h\}} a(x,\nabla u_2)\cdot(\nabla u_2-\nabla u_1)\,dx.
\end{aligned}\tag{66}$$

For the first term on the right-hand side of (66), we have

$$\begin{aligned}
&\Big|\int_\Omega f[T_k(u_1-T_h(u_2))+T_k(u_2-T_h(u_1))]\,dx\Big|\\
&\quad\leq\int_{\{|u_1|\leq h,|u_2|\leq h\}}|f|\;|T_k(u_1-u_2)+T_k(u_2-u_1)|\,dx.\\
&\qquad+\int_{\{|u_1|>h\}}|f|\;|T_k(u_1-T_h(u_2))+T_k(u_2-T_h(u_1))|\,dx\\
&\qquad+\int_{\{|u_2|>h\}}|f|\;|T_k(u_1-T_h(u_2))+T_k(u_2-T_h(u_1))|\,dx\\
&\quad\leq 2k\int_{\{|u_1|>h\}}|f|\,dx+2k\int_{\{|u_2|>h\}}|f|\,dx.
\end{aligned}$$

since $f\in L^1(\Omega)$ and $\mathrm{meas}\{|u_i|\geq h\}\to 0$ when $h\to\infty$ for $i=1,2$, it follows that

$$\int_\Omega f[T_k(u_1-T_h(u_2))+T_k(u_2-T_h(u_1))]\,dx\longrightarrow 0\qquad\text{as}\quad h\to\infty. \tag{67}$$

Concerning the third term on the right-hand side of (66). By taking $T_h(u_1)$ as a test function in (24) for the equation with solution u_1, we obtain

$$\int_\Omega a(x,\nabla u_1)\cdot\nabla T_k(u_1-T_h(u_1))\,dx\leq\int_\Omega fT_k(u_1-T_h(u_1))\,dx,$$

in view of (10), we obtain

$$\begin{aligned}\alpha\int_{\{h<|u_1|\leq h+k\}}\varphi(x,|\nabla u_1|)\,dx &\leq \int_{\{h<|u_1|\leq h+k\}} a(x,\nabla u_1)\cdot\nabla u_1\,dx\\ &\leq k\int_{\{|u_1|\geq h\}}|f|\,dx\to 0 \qquad \text{as} \quad h\to\infty.\end{aligned} \tag{68}$$

Also, we prove can that

$$\alpha\int_{\{h<|u_2|\leq h+k\}}\varphi(x,|\nabla u_2|)\,dx\to 0 \quad \text{as} \quad h\to\infty. \tag{69}$$

On the other hand, we have

$\{|u_1-u_2|\leq k\}\cap\{|u_2|\leq h\}\cap\{|u_1|>h\}\subseteq\{h<|u_1|\leq h+k\}\cap\{h-k<|u_2|\leq h\}$,

In view of Young's inequality, we obtain

$$\begin{aligned}&\int_{\{|u_1-u_2|\leq k\}\cap\{|u_2|\leq h\}\cap\{|u_1|>h\}} a(x,\nabla u_1)\cdot(\nabla u_1-\nabla u_2)\,dx\\ &\leq \beta\int_{\{|u_1-u_2|\leq k\}\cap\{|u_2|\leq h\}\cap\{|u_1|>h\}}\Big(K(x)+k_1\psi_x^{-1}(\varphi(x,k_2|\nabla u_1|))\Big)(|\nabla u_1|+|\nabla u_2|\,dx\\ &\leq 2\beta\int_{\{|u_1|>h\}}\psi(x,K(x))\,dx+2\beta k_1\int_{\{h<|u_1|\leq h+k\}}\varphi(x,k_2|\nabla u_1|)\,dx\\ &\quad +\beta(k_1+1)\int_{\{h<|u_1|\leq h+k\}}\varphi(x,|\nabla u_1|)\,dx\\ &\quad +\beta(k_1+1)\int_{\{h-k<|u_2|\leq h\}}\varphi(x,|\nabla u_2|)\,dx\longrightarrow 0 \qquad \text{as} \quad h\to\infty,\end{aligned} \tag{70}$$

Similarly, we can prove that

$$\int_{\{|u_2-u_1|\leq k\}\cap\{|u_1|\leq h\}\cap\{|u_2|>h\}} a(x,\nabla u_2)\cdot(\nabla u_2-\nabla u_1)\,dx\longrightarrow 0 \qquad \text{as} \quad h\to\infty, \tag{71}$$

By combining (66), (67) and (70) – (71), we conclude that

$$\begin{aligned}&\int_{\{|u_1-u_2|\leq k\}}\big(a(x,\nabla u_1)-a(x,\nabla u_2)\big)\cdot(\nabla u_1-\nabla u_2)\,dx\\ &=\lim_{h\to\infty}\int_{\{|u_1-u_2|\leq k\}\cap\{|u_2|\leq h\}\cap\{|u_1|\leq h\}}\big(a(x,\nabla u_1)-a(x,\nabla u_2)\big)\cdot(\nabla u_1-\nabla u_2)\,dx=0,\end{aligned} \tag{72}$$

Since (72) is true for all $k>0$ and thanks to (9), we conclude that $\nabla(u_1-u_2)=0$ a.e.in Ω, and since $u_1=u_2=0$ on $\partial\Omega$, thus $u_1=u_2$ a.e. in Ω, which conclude the proof of uniqueness of entropy solutions.

Example 5.1. Taking $\varphi(x,t)=|t|^{p(x)}\log^{\sigma}(1+|t|)$ for $1\leq p(x)<\infty$ and $0<\sigma<\infty$. Let $f\in L^1(\Omega)$ and the obstacle $\Psi=0$. We consider the following Carathéodory function

$$a(x,\nabla u)=|\nabla u|^{p(x)-2}\log^{\sigma}(1+|\nabla u|)\;\nabla u.$$

It is clear that $a(x,\nabla u)$ verifies (8) – (10). In view of the Theorem 5.1, the problem

$$\begin{cases}-\mathrm{div}\Big(|\nabla u|^{p(x)-2}\log^{\sigma}(1+|\nabla u|)\;\nabla u\Big)=f & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases} \tag{73}$$

has one entropy solution, i.e.

$$u \geq 0 \quad \text{a.e. in } \ \Omega \qquad \text{and} \qquad T_k(u) \in W_0^1 L_\varphi(\Omega),$$

and

$$\int_\Omega |\nabla u|^{p(x)-2} \log^\sigma (1 + |\nabla u|) \nabla u \cdot \nabla T_k(u_n - \nu)\, dx \leq \int_\Omega f T_k(u_n - \nu)\, dx, \tag{74}$$

for any $\nu \in W_0^1 L_\varphi(\Omega) \cap L^\infty(\Omega)$ with $v \geq 0$ a.e. in Ω.

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