Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 44(1), 2017, Pages 21–29
ISSN: 1223-6934

Existence and multiplicity of solutions for $p(x)-$ Kirchhoff-type problem

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Abstract. In the present paper, by using the Mountain Pass theorem and the Fountain theorem, we obtain the existence and multiplicity of solutions to a class of $p(x)$-Kirchhoff-type problem under Dirichlet boundary condition.

2010 Mathematics Subject Classification. 35B38,35D05,35J60,35J70.
Key words and phrases. $p(x)$-Kirchhoff-type equation; Mountain-Pass theorem; Nonlocal problem; Fountain theorem.

1. Introduction

In this paper, we are concerned with the following problem

$$\begin{cases} -M\left(A\left(x,\nabla u\right)\right)\ \operatorname{div}\left(a\left(x,\nabla u\right)\right)=f\left(x,u\right) & \text{in } \Omega\ , \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^N$ $(N\geq 3)$ is a smooth bounded domain, $p\in C\left(\overline{\Omega}\right)$ for any $x\in\overline{\Omega}$ and $\operatorname{div}(a\left(x,\nabla u\right))$ is a $p\left(x\right)-$Laplace type operator. Moreover $M:\mathbb{R}^+\to\mathbb{R}^+$ is a continuous function and $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function, satisfying some certain conditions.

The nonlinear problems involving the $p\left(x\right)$-Laplace type operator are extremely attractive because they can be used to model dynamical phenomena which arise from the study of electrorheological fluids or elastic mechanics. Problems with variable exponent growth conditions also appear in the modelling of stationary thermo-rheological viscous flows of non-Newtonian fluids and in the mathematical description of the processes filtration of an ideal barotropic gas through a porous medium. The detailed application backgrounds of the $p(x)$-Laplace type operators can be found in [3, 5, 7, 10, 13, 20, 18] and references therein.

Problem (1.1) is related to the stationary version of a model, the so-called Kirchhoff equation, introduced by Kirchhoff [15]. To be more precise, Kirchhoff established a model given by the equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2dx\right)\frac{\partial^2 u}{\partial x^2}=0, \tag{1.2}$$

where ρ, P_0, h, E, L are constants, which extends the classical D'Alambert's wave equation, by considering the effects of the changes in the length of the strings during the vibrations. For some interesting results we refer to [4, 6, 9, 11, 14]. Moreover,

Received March 19, 2015. Accepted December, 2016.

nonlocal boundary value problems like (1.2) can be used for modelling several physical and biological systems where u describes a process which depend on the average of itself, such as the population density [1, 2, 8].

In the present paper, we deal a more general Kirchhoff function M, and as a consequence the operator $div\,(a\,(x,\nabla u))$ appears in problem (1.1), a more general operator than $p\,(x)$ −Laplace operator $\Delta_{p(x)}u := div(|\nabla u|^{p(x)-2}\,\nabla u)$ where $p\,(x) > 1$. This caused some difficulties in calculations and required more general conditions. Moreover, thanks to the Mountain-Pass theorem and Fountain theorem, we show the existence and multiplicity of nontrivial weak solutions in the present paper. To our best knowledge, the present papers results are not covered in the literature.

This paper is organized as follows. In Section 2, we present some necessary preliminary results. In Section 3, using the variational method, we give the existence results of problem (1.1).

2. Preliminaries

We recall some basic properties of variable exponent Lebesgue-Sobolev spaces $L^{p(x)}\,(\Omega)$, $W^{1,p(x)}\,(\Omega)$ (for details, see e.g., [12, 16, 17])

Set,

$$C_+\left(\overline{\Omega}\right) = \left\{p;\ p \in C\left(\overline{\Omega}\right),\ \ \min p\,(x) > 1, \forall x \in \overline{\Omega}\right\}.$$

For any $p\,(x) \in C_+\left(\overline{\Omega}\right)$, denote $p^- := \min\limits_{x\in\overline{\Omega}} p\,(x)$, $p^+ := \max\limits_{x\in\overline{\Omega}} p\,(x) < \infty$, and define the variable exponent Lebesgue space by

$$L^{p(x)}\,(\Omega) = \left\{u\,|u : \Omega \to \mathbb{R} \text{ is measurable, } \int_\Omega |u\,(x)|^{p(x)}\,dx < \infty\right\}.$$

We define a norm, the so-called Luxemburg norm, on this space by the formula

$$|u|_{p(x)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \le 1\right\},$$

and $\left(L^{p(x)}\,(\Omega)\,,\ |.|_{p(x)}\right)$ becomes a Banach spaces.

Proposition 2.1 [12, 16] *The conjugate space of* $L^{p(x)}\,(\Omega)$ *is* $L^{p'(x)}\,(\Omega)$, *where* $\frac{1}{p'(x)} + \frac{1}{p(x)} = 1$. *For any* $u \in L^{p(x)}\,(\Omega)$ *and* $v \in L^{p'(x)}\,(\Omega)$, *we have*

$$\left|\int_\Omega uvdx\right| \le (\frac{1}{p^-} + \frac{1}{(p^-)'})\,|u|_{p(x)}\,|v|_{p'(x)}\,.$$

Proposition 2.2 [12, 16]*Denote* $\rho\,(u) = \int_\Omega |u\,(x)|^{p(x)}\,dx, \forall u, u_n \in L^{p(x)}(\Omega)$, *then*

$$\text{(i) } |u|_{p(x)} > 1 \implies |u|_{p(x)}^{p^-} \le \rho\,(u) \le |u|_{p(x)}^{p^+};$$

$$\text{(ii) } |u|_{p(x)} < 1 \implies |u|_{p(x)}^{p^+} \le \rho\,(u) \le |u|_{p(x)}^{p^-};$$

$$\text{(iii) } \lim_{n\to\infty} |u_n|_{p(x),\Omega} = 0 \Leftrightarrow \lim_{n\to\infty} \rho(u_n) = 0;$$

$$\text{(iv) } \lim_{n\to\infty} |u_n|_{p(x),\Omega} \to \infty \Leftrightarrow \lim_{n\to\infty} \rho(u_n) \to \infty.$$

Proposition 2.3 [12, 16] *If $u, u_n \in L^{p(x)}(\Omega)$, then the following statements are equivalent:*

$$(i) \lim_{n\to\infty} |u_n - u|_{p(x)} = 0; \quad (ii) \lim_{n\to\infty} \rho(u_n - u) = 0;$$

$$(iii)\ u_n \to u \text{ measure in } \Omega \text{ and } \lim_{n\to\infty} \rho(u_n) = \rho(u).$$

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) \mid \ |\nabla u| \in L^{p(x)}(\Omega) \right\},$$

with the norm $\|u\|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}, \forall u \in W^{1,p(x)}(\Omega)$. The space $W_0^{1,p(x)}(\Omega)$ is denoted by the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ with respect to the norm $\|u\|_{1,p(x)}$. We can define an equivalent norm $\|u\| = |\nabla u|_{p(x)}$, since Poincaré inequality holds [13], i.e. there exists a positive constant $C > 0$ such that

$$|u|_{p(x)} \le C|\nabla u|_{p(x)}, \text{ for all } u \in W_0^{1,p(x)}(\Omega).$$

Proposition 2.4 [12, 16](i) *If $1 < p^- \le p^+ < \infty$, then the spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces,*

(ii) *If $q \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$, for all $x \in \overline{\Omega}$, then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact and continuous, where*

$$p^*(x) := \begin{cases} \frac{Np(x)}{N-p(x)}, & \text{if } p(x) < N, \\ \infty, & \text{if } N \le p(x). \end{cases}$$

3. The main results

Let X denote the variable exponent Sobolev space $W_0^{1,p(x)}(\Omega)$.

We say that $u \in X$ is a weak solution of (1.1) if

$$M\left(\int_\Omega A(x, \nabla u)\right) \int_\Omega a(x, \nabla u) \nabla \varphi dx = \int_\Omega f(x, u) \varphi dx,$$

for all $\varphi \in X$.

Define the energy functional $I : X \to \mathbb{R}$ associated with (1.1) by

$$I(u) = \widehat{M}\left(\int_\Omega A(x, \nabla u) dx\right) - \int_\Omega F(x, u) dx := \widehat{M}(\Lambda(u)) - J(u),$$

where $\Lambda(u) = \int_\Omega A(x, \nabla u) dx$ and $J(u) = \int_\Omega F(x, u) dx$. Moreover, $\widehat{M}(t) = \int_0^t M(s) ds$ and $F(x, u) = \int_0^u f(x, t) dt$.

It is well known that standart arguments imply that $J \in C^1(X, \mathbb{R})$ and the derivate of J is

$$\langle J'(u), \upsilon \rangle = \int_\Omega f(x, u) \upsilon dx, \text{ for all } u, \upsilon \in X.$$

In this article, we assume that $a(x, \xi) : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is the continuous derivative with respect to ξ of the mapping $A : \Omega \times \mathbb{R}^N \to \mathbb{R}$, $A = A(x, \xi)$, i.e. $a(x, \xi) = \nabla_\xi A(x, \xi)$. Suppose that the following hypotheses:

(**A1**) For all $x \in \Omega$ and $\xi \in \mathbb{R}^N, |a(x, \xi)| \le c_0(h_0(x) + |\xi|^{p(x)-1})$, where $h_0(x) \in L^{p'(x)}(\Omega)$ is a nonnegative measurable function.

(**A2**) A is $p(x)$-uniformly convex: There exists a constant $k>0$ such that $A(x,\frac{\xi+\psi}{2})\leq\frac{1}{2}A(x,\xi)+\frac{1}{2}A(x,\psi)-k\left|\xi-\psi\right|^{p(x)}$, for all $x\in\Omega$ and $\xi,\psi\in\mathbb{R}^N$.
(**A3**) For all $x\in\Omega$ and $\xi\in\mathbb{R}^N$, $\left|\xi\right|^{p(x)}\leq a(x,\xi)\cdot\xi\leq p\left(x\right)A(x,\xi)$.
(**A4**) $A\left(x,0\right)=0$, for all $x\in\Omega$.
(**A5**) $A(x,-\xi)=A(x,\xi)$, for all $x\in\Omega$ and $\xi\in\mathbb{R}^N$.

Lemma 3.1. [17]
(i) A verifies the growth condition; $\left|A\left(x,\xi\right)\right|\leq c_0(h_0\left(x\right)\left|\xi\right|+\left|\xi\right|^{p(x)})$, *for all* $x\in\Omega$ *and* $\xi\in\mathbb{R}^N$;
(ii) A is $p(x)-$*homogeneous;* $A\left(x,z\xi\right)\leq A\left(x,\xi\right)z^{p(x)}$, *for all* $z\geq 1$, $\xi\in\mathbb{R}^N$ *and* $x\in\Omega$.

Lemma 3.2. *(i) The functional* Λ *is well-defined on* X;
(ii) The functional Λ *is of class* $C^1\left(X,\mathbb{R}\right)$ *and*

$$\left\langle\Lambda'\left(u\right),v\right\rangle=\int_\Omega a\left(x,\nabla u\right)\cdot\nabla v dx,\ \text{for all } u,v\in X;$$

(iii) The functional Λ *is weakly lower semi-continuos on* X;
(iv) For all $u,v\in X$

$$\Lambda(\frac{u+v}{2})\leq\frac{1}{2}\Lambda\left(u\right)+\frac{1}{2}\Lambda\left(v\right)-k\left\|u-v\right\|^{p^-}\ ;$$

(v) For all $u,v\in X$

$$\Lambda\left(u\right)-\Lambda\left(v\right)\geq\left\langle\Lambda'\left(v\right),u-v\right\rangle;$$

(vi) I is weakly lower semi-continuos on X;
(vii) I is well-defined on X *and of class* $C^1(X,\mathbb{R})$, *and its derivative given by*

$$\left\langle I'\left(u\right),v\right\rangle=M\left(\int_\Omega A\left(x,\nabla u\right)dx\right)\int_\Omega a\left(x,\nabla u\right)\nabla v dx-\int_\Omega f\left(x,u\right)v dx;$$

for all $u,v\in X$.

Since the proof of Lemma 3.2 is very similar to the proof of Lemma 2.2 and Lemma 2.7 given in [17], we omit it.

Theorem 3.3. *Assume that* (**A3**) *and the following conditions hold:*
(M_1) $M:\mathbb{R}^+\to\mathbb{R}^+$ *is a continuos function and satisfies the condition*

$$m_0 s^{\alpha-1}\leq M(s)\leq m_1 s^{\alpha-1},$$

for all $s>0$ *and* m_0,m_1 *real numbers such that* $0<m_0\leq m_1$ *and* $\alpha\geq 1$.
$(\mathbf{f}_1)$ $f:\Omega\times\mathbb{R}\to\mathbb{R}$ *is a Carathéodory condition and satisfies the growth condition*

$$\left|f(x,t)\right|\leq c_0\left(1+\left|t\right|^{\delta(x)-1}\right),\ \forall\left(x,t\right)\in\Omega\times\mathbb{R},$$

where c_0 *is positive constant and* $\delta(x)\in C_+\left(\overline{\Omega}\right)$ *such that* $\delta^+<\alpha p^-<p^*\left(x\right)$ *for all* $x\in\Omega$.
Then problem (1.1) *has a weak solution.*

Proof. Let $\|u\| > 1$. By $(M_1), (\mathbf{f}_1), (\mathbf{A3})$ and Proposition 2.2 (i), we get

$$\begin{aligned} I(u) &\geq \frac{m_0}{\alpha}\left(\int_\Omega A(x,\nabla u)\,dx\right)^\alpha - c_0\int_\Omega |u|^{\delta(x)}\,dx - c_0\int_\Omega |u|\,dx \\ &\geq \frac{m_0}{\alpha(p^+)^\alpha}\|u\|^{\alpha p^-} - c_1\|u\|^{\delta^+} - c_2\|u\| \to +\infty, \text{ as } \|u\| \to +\infty, \end{aligned}$$

Thus, I is coercive. Since I is weakly lower semi-continuous, I has a minimum point u in X, and u is a weak solution of problem (1.1). The proof is completed. □

Theorem 3.4. *Assume that* (M_1) *and the following conditions hold:*

$(\mathbf{f}_2)$ $f : \Omega \times \mathbb{R} \to \mathbb{R}$ *is a Carathéodory condition and satisfies the growth condition;*

$$|f(x,t)| \leq c\left(1 + |t|^{\eta(x)-1}\right), \forall (x,t) \in \Omega \times \mathbb{R},$$

$(\mathbf{f}_3)$ $f(x,t) = o\left(|t|^{\alpha p^+ - 1}\right)$, $t \to 0$, *for* $x \in \Omega$ *uniformly,*

where c *is positive constant and* $\eta(x) \in C_+\left(\overline{\Omega}\right)$ *such that* $\alpha p^+ < \eta^- \leq \eta^+ < p^*(x)$ *for all* $x \in \Omega$,

(AR) : $\exists t_* > 0$, $\theta > \frac{m_1}{m_0}\alpha p^+$ *such that*

$$0 < \theta F(x,t) \leq f(x,t)t, \ \ |t| \geq t_*, \ a.e. \ x \in \Omega.$$

Then problem (1.1) *has a nontrivial weak solution.*

Definition 3.1. We say that I satisfies Palais-Smale condition in X ((PS) condition for short) if if any sequence $\{u_n\}$ in X such that $\{I(u_n)\}$ is bounded and $I'(u_n) \to 0$ as $n \to \infty$, has a convergent subsequence.

Lemma 3.5. *Suppose* $(M_1), (\mathbf{f}_1), (\mathbf{A3})$ *and* (AR) *hold. Then,* I *satisfies* (PS) *condition.*

Proof. Let assume that there exists a sequence $\{u_n\} \subset X$ such that

$$|I(u_n)| \leq c \ \text{ and } I'(u_n) \to 0 \ \text{ as } \ n \to \infty. \tag{3.1}$$

Then, by $(M_1), (\mathbf{A3})$ and (AR), we have

$$\begin{aligned} c + \|u_n\| &\geq I(u_n) - \frac{1}{\theta}\langle I'(u_n), u_n\rangle \\ &\geq \frac{m_0}{\alpha}\left(\int_\Omega A(x,\nabla u_n)\,dx\right)^\alpha - \frac{m_1 p^+}{\theta}\left(\int_\Omega A(x,\nabla u_n)\,dx\right)^{\alpha-1}\int_\Omega A(x,\nabla u_n)\,dx \\ &\geq \left(\frac{m_0}{\alpha} - \frac{m_1 p^+}{\theta}\right)\left(\int_\Omega A(x,\nabla u_n)\,dx\right)^\alpha \end{aligned}$$

By $(\mathbf{A3})$ and Proposition 2.2 (ii), we can write

$$c + \|u_n\| \geq \left(\frac{m_0}{\alpha} - \frac{m_1 p^+}{\theta}\right)\|u_n\|^{\alpha p^-}.$$

Since $\alpha p^- > 1$, $\{u_n\}$ is bounded in X. Therefore, there exists $u \in X$, up to a subsequence, such that $u_n \rightharpoonup u$ in X.

Moreover, since we have the compact embedding $X \hookrightarrow L^{\eta(x)}(\Omega)$, we get

$$u_n \to u \text{ in } L^{\eta(x)}(\Omega) \ \text{ and } u_n \to u \text{ a.e in } \Omega \tag{3.2}$$

By (3.1) , we have

$$\begin{aligned}\langle I'(u_n), u_n - u\rangle &= M\left(\int_\Omega A\left(x, \nabla u_n\right) dx\right) \int_\Omega a\left(x, \nabla u_n\right)\left(\nabla u_n - \nabla u\right) dx \\ &\quad - \int_\Omega f\left(x, u_n\right)\left(u_n - u\right) dx \to 0.\end{aligned}$$

By using $(\mathbf{f}_1)$ and Proposition 2.1, it follows

$$\left|\int_\Omega f\left(x, u_n\right)\left(u_n - u\right) dx\right| \leq c_3 \left|\left|u_n\right|^{\eta(x)-1}\right|_{\eta'(x)} \left|u_n - u\right|_{\eta(x)}.$$

If we consider the relations given in (3.2), we get $\int_\Omega f\left(x, u_n\right)\left(u_n - u\right) dx \to 0$. Then, we have

$$M\left(\int_\Omega A\left(x, \nabla u_n\right) dx\right) \int_\Omega a\left(x, \nabla u_n\right)\left(\nabla u_n - \nabla u\right) dx \to 0.$$

From (M_1) , it follows

$$\int_\Omega a\left(x, \nabla u_n\right)\left(\nabla u_n - \nabla u\right) dx \to 0.$$

that is, $\lim\limits_{n\to\infty} \left\langle \Lambda'\left(u_n\right), u_n - u\right\rangle = 0$. By using Lemma 3.2 (v), we get

$$0 = \lim_{n\to\infty} \left\langle \Lambda'\left(u_n\right), u - u_n\right\rangle \leq \lim_{n\to\infty} \left(\Lambda\left(u\right) - \Lambda\left(u_n\right)\right) = \Lambda\left(u\right) - \lim_{n\to\infty} \Lambda\left(u_n\right)$$

or $\lim\limits_{n\to\infty} \Lambda\left(u_n\right) \leq \Lambda\left(u\right)$. This fact and from Lemma 3.2 (iii) imply $\lim\limits_{n\to\infty} \Lambda\left(u_n\right) = \Lambda\left(u\right)$.

Now, we assume by contradiction that $\{u_n\}$ does not converge strongly to u in X.Then, there exists $\varepsilon > 0$ and a subsequence $\{u_{n_m}\}$ of $\{u_n\}$ such that $\|u_{n_m} - u\| \geq \varepsilon$. On the other hand, by from Lemma 3.2 (iv), we have

$$\frac{1}{2}\Lambda\left(u\right) + \frac{1}{2}\Lambda\left(u_{n_m}\right) - \Lambda(\frac{u_{n_m} + u}{2}) \geq k \left\|u_{n_m} - u\right\|^{p^-} \geq k\varepsilon^{p^-}.$$

Letting $m \to \infty$ in the above inequality, we obtain

$$\limsup_{n\to\infty} \Lambda(\frac{u_{n_m} + u}{2}) \leq \Lambda\left(u\right) - k\varepsilon^{p^-}.$$

Moreover, we have $\{\frac{u_{n_m} + u}{2}\}$ converges weakly to u in X. Using Lemma 3.2 (iii), we obtain

$$\Lambda\left(u\right) \leq \liminf_{n\to\infty} \Lambda(\frac{u_{n_m} + u}{2}),$$

which is a contradiction. Therefore, it follows that $\{u_n\}$ converges strongly to u in X. The proof of Lemma 3.5 is complete. □

Lemma 3.6. *Suppose* $(M_1), (\mathbf{f}_1), (\mathbf{f}_3), (\mathbf{A3})$ *and* (AR) *hold. Then the following statements hold:*

(i) There exist two positive real numbers γ *and* a *such that* $I\left(u\right) \geq a > 0$, $u \in X$ *with* $\|u\| = \gamma$.

(ii) There exists $u \in X$ *such that* $\|u\| > \gamma$, $I\left(u\right) < 0$.

Proof. (i) Let $\|u\| < 1$. Then by (M_1), we have

$$I\left(u\right) \geq \frac{m_0}{\alpha\left(p^+\right)^\alpha} \left\|u\right\|^{\alpha p^+} - \int_\Omega F(x,u)dx.$$

Using the continuous embeddings $X \hookrightarrow L^{\alpha p^+}(\Omega)$ and $X \hookrightarrow L^{\eta(x)}(\Omega)$, there exist positive constants c_4 and c_5 such that

$$|u|_{\eta(x)} \leq c_4 \|u\| \quad \text{and} \quad |u|_{\alpha p^+} \leq c_5 \|u\| \ , \ \forall \ u \in X.$$

Let $\varepsilon > 0$ be small enough such that $\varepsilon c_4^{\alpha p^+} \leq \frac{m_0}{2\alpha(p^+)^\alpha}$. By $(\mathbf{f}_1)$ and $(\mathbf{f}_3)$, we get $F(x,t) \leq \varepsilon |t|^{\alpha p^+} + c_\varepsilon |t|^{\eta(x)}, \forall (x,t) \in \Omega \times \mathbb{R}$. Therefore, Proposition 2.2 (ii), we have

$$\begin{aligned} I(u) &\geq \frac{m_0}{\alpha(p^+)^\alpha} \|u\|^{\alpha p^+} - \varepsilon \int_\Omega |u|^{\alpha p^+} dx - c_\varepsilon \int_\Omega |u|^{\eta(x)} dx \\ &\geq \frac{m_0}{\alpha(p^+)^\alpha} \|u\|^{\alpha p^+} - \varepsilon c_4^{\alpha p^+} \|u\|^{\alpha p^+} - c_\varepsilon c_5^{\eta^-} \|u\|^{\eta^-} \\ &\geq \frac{m_0}{2\alpha(p^+)^\alpha} \|u\|^{\alpha p^+} - c_\varepsilon c_5^{\eta^-} \|u\|^{\eta^-} . \end{aligned}$$

Since $\|u\| < 1$ and $\alpha p^+ < \eta^-$, there exist two positive real numbers γ and a such that $I(u) \geq a > 0$, $u \in X$ with $\|u\| = \gamma \in (0,1)$.

(ii) From (AR), one easily deduces

$$F(x,t) \geq c_6 |t|^\theta, \quad |t| \geq t_*, \ a.e. \ x \in \Omega.$$

In the other hand, when $t > t_* > 1$, from (M_1) we can easily obtain

$$\widehat{M}(t) \leq \frac{m_1}{\alpha} t^\alpha \leq \frac{m_1}{\alpha} t^{\frac{m_1}{m_0}\alpha}.$$

Thus, for any fixed $\omega \in X\backslash\{0\}$, $t > 1$ and from Lemma 3.1 (ii), we have

$$\begin{aligned} I(t\omega) &= \widehat{M}\left(\int_\Omega A(x, \nabla t\omega)\, dx\right) - \int_\Omega F(x,t\omega)dx \\ &\leq \frac{m_1}{\alpha}\left(\int_\Omega A(x, \nabla t\omega)\, dx\right)^{\frac{m_1}{m_0}\alpha} - \int_\Omega F(x,t\omega)dx \\ &\leq \frac{m_1}{\alpha(p^-)^{\frac{m_1}{m_0}\alpha p^+}} t^{\frac{m_1}{m_0}\alpha p^+} \int_\Omega A(x, \nabla\omega)\, dx - c_6 t^\theta \int_\Omega |\omega|^\theta dx. \end{aligned}$$

From (AR) ,it can be obtained that $\theta > \frac{m_1}{m_0}\alpha p^+$. Hence, $I(t\omega) \to -\infty$ as $t \to +\infty$. □

Proof of Theorem 3.3. From Lemma 3.5, Lemma 3.6, Lemma 3.2 (vii), $(\mathbf{A4})$ and the fact that $I(0) = 0$, I satisfies the Mountain Pass Theorem [19]. Therefore, I has at least one nontrivial critical point, i.e., problem (1.1) has a nontrivial weak solution. The proof of Theorem 3.3 is complete. □

Theorem 3.7. *Assume that* $(M_1), (\mathbf{f}_1), (AR)$ *and the following*

$(\mathbf{f}_4) : f(x,-t) = -f(x,t)$, *for* $(x,t) \in \Omega \times \mathbb{R}$,

then I *has a sequence of critical points* $\{u_n\}$*such that* $I(u_n) \to +\infty$ *and* (1.1) *has infinite many pairs of solutions.*

In order to prove Theorem 3.7, we need Lemma 3.8.

Since X be a reflexive and separable Banach space, then there are $\{e_j\} \subset X$ and $\{e_j^*\} \subset X^*$ such that

$$X = \overline{span\{e_j \mid j = 1, 2, ...\}}, \qquad X^* = \overline{span\{e_j^* \mid j = 1, 2, ...\}},$$

and

$$\langle e_i^*, e_j \rangle = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$$

For convenience, we write $X_j = span\{e_j\}$, $Y_k = \oplus_{j=1}^{k} X_j$, $Z_k = \oplus_{j=k}^{\infty} X_j$.

Lemma 3.8. *If $\eta(x) \in C_+\left(\overline{\Omega}\right)$, $\eta(x) < p^*(x)$ for any $x \in \overline{\Omega}$, denote*

$$\beta_k = \sup\left\{ |u|_{\eta(x)} : \|u\| = 1, u \in Z_k \right\}.$$

Then $\lim_{k\to\infty} \beta_k = 0$.

Since the proof of Lemma 3.8 is similar to that of Lemma 4.9 in [13], we omit it.

Proof of Theorem 3.7. According to (M_1), $(\mathbf{f}_4)$ and (AR), I satisfies (PS) condition and from (**A5**) it is an even functional. We only need to prove that if k is large enough, then there exist $\rho_k > \gamma_k > 0$ such that

(**A6**) $b_k := \inf\{I(u) \mid u \in Z_k, \|u\| = \gamma_k\} \to \infty$ as $k \to \infty$;

(**A7**) $a_k := \max\{I(u) \mid u \in Y_k, \|u\| = \rho_k\} \leq 0$.

Thus, the conclusion of Theorem 3.7 can be obtained by Fountain Theorem [19].

(**A6**) For any $u \in Z_k$, $\|u\| = \gamma_k = \left(c_8 \eta^+ \beta_k^{\eta^+} m_0^{-1}\right)^{\frac{1}{\alpha p^- - \eta^+}}$, we have

$$\begin{aligned}
I(u) &\geq \frac{m_0}{\alpha}\left(\int_\Omega A(x, \nabla u)\, dx\right)^\alpha - c_0 \int_\Omega |u|^{\eta(x)}\, dx - c_0 \int_\Omega |u|\, dx \\
&\geq \frac{m_0}{\alpha (p^+)^\alpha} \|u\|^{\alpha p^-} - c_0 |u|_{\eta(x)}^{\eta(\zeta)} - c_0 \|u\|, \text{ where } \zeta \in \Omega \\
&\geq \begin{cases} \frac{m_0}{\alpha(p^+)^\alpha} \|u\|^{\alpha p^-} - c_0 - c_0 \|u\|, & \text{if } |u|_{\eta(x)} \leq 1 \\ \frac{m_0}{\alpha(p^+)^\alpha} \|u\|^{\alpha p^-} - c_0 \beta_k^{\eta^+} \|u\|^{\eta^+} - c_0 \|u\|, & \text{if } |u|_{\eta(x)} > 1 \end{cases} \\
&\geq \frac{m_0}{\alpha (p^+)^\alpha} \|u\|^{\alpha p^-} - c_0 \beta_k^{\eta^+} \|u\|^{\eta^+} - c_0 \|u\| - c_7 \\
&= \frac{m_0}{\alpha (p^+)^\alpha} \left(c_8 \eta^+ \beta_k^{\eta^+} m_0^{-1}\right)^{\frac{\alpha p^-}{\alpha p^- - \eta^+}} - c_0 \beta_k^{\eta^+} \left(c_8 \eta^+ \beta_k^{\eta^+} m_0^{-1}\right)^{\frac{\eta^+}{\alpha p^- - \eta^+}} - c_0 \|u\| - c_7 \\
&\geq \frac{m_0}{(p^+)^\alpha}\left(\frac{1}{\alpha} - \frac{1}{\eta^+}\right)\left(c_8 \eta^+ \beta_k^{\eta^+} m_0^{-1}\right)^{\frac{\alpha p^-}{\alpha p^- - \eta^+}} - c_0 \left(c_8 \eta^+ \beta_k^{\eta^+} m_0^{-1}\right)^{\frac{1}{\alpha p^- - \eta^+}} - c_7
\end{aligned}$$

Because $\beta_k \to 0$ and $\alpha < \alpha p^- < \eta^+$, we have $I(u) \to \infty$ as $k \to \infty$

(**A7**) From (AR) ,we get $F(x,t) \geq c_9 |t|^\theta - c_{10}$. Therefore, for any $w \in Y_k$ with $\|w\| = 1$ and $1 < t = \rho_k$, we have

$$\begin{aligned}
I(t\omega) &\leq \frac{m_1}{\alpha}\left(\int_\Omega A(x, \nabla t\omega)\, dx\right)^{\frac{m_1}{m_0}\alpha} - c_9 t^\theta \int_\Omega |\omega|^\theta\, dx - c_{10} \\
&\leq \frac{m_1}{\alpha (p^-)^{\frac{m_1}{m_0}\alpha p^+}} t^{\frac{m_1}{m_0}\alpha p^+} \int_\Omega A(x, \nabla \omega)\, dx - c_9 t^\theta \int_\Omega |\omega|^\theta\, dx - c_{10}.
\end{aligned}$$

By $\theta > \frac{m_1}{m_0}\alpha p^+$ and $\dim Y_k = k$, it is easy see that $I(t\omega) \to -\infty$ as $t \to +\infty$ for $u \in Y_k$. $\square$

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