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$p(.)$-parabolic capacity and decomposition of measures

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Abstract. In this paper, we develop a concept of $p(.)$ −parabolic capacity in order to give a result of decomposition of measures(in space and time) which does not charge the sets of null capacity.

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1. Introduction and main result

The concept of capacity play an important role in the study of solutions of partial differential equations; it permits to see that the functions in the Sobolev spaces are defined better than almost everywhere. In the elliptic case, the notion of capacity is related to the Sobolev spaces (see [4]). More precisely, let $\Omega \subset \mathbb{R}^N$, be open bounded, for $E \subset \Omega$, the Sobolev $p(.)$-capacity of E is defined by

$$C_{p(.)}(E) := \inf_{u \in S_{p(.)}(E)} \int_\Omega \left(|u|^{p(x)} + |\nabla u|^{p(x)} \right) dx, \tag{1}$$

where

$$S_{p(.)}(E) := \left\{ u \in W^{1,p(.)}(\Omega) : \ u \geq 1 \text{ in an open set containing } E \text{ and } u \geq 0 \right\}. \tag{2}$$

In the case where $S_{p(.)}(E) = \emptyset$, we set $C_{p(.)}(E) = \infty$. One of the properties of the elliptic capacity is the following: for every $u \in W^{1,p(.)}(\Omega)$, there exists a $p(.)$-quasicontinuous function $v \in W^{1,p(.)}(\Omega)$ such that $u = v$ almost everywhere in Ω i.e $u = v$ a.e. Ω and for every $\varepsilon > 0$, there exists an open set $U_\varepsilon \subset \Omega$ with $C_{p(.)}(U_\varepsilon) < \varepsilon$ such that v restricted to $\Omega \setminus U_\varepsilon$ is continuous.
The theory of capacity is an essential tool in the study of the existence and uniqueness of the solution of some elliptic and parabolic problems with measures data. Let's recall that in the context of constant exponent, the authors in [3] proved that every diffuse measure μ i.e. a measure which does not charge the sets of null $p-$capacity belongs to $L^1(\Omega) + W^{-1,p'}(\Omega)$ with p' the conjuguate of p, that permit them to prove the existence and uniqueness of entropy solution for the following problem.

$$\begin{cases} A(u) = \mu & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3}$$

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where A is a Leray-Lions type operator.
In the context of variable exponent, a similar approach is used in [12] for the elliptic problem

$$\begin{cases} \nabla.a(x,\nabla u)+\beta(u)\ni\mu & \text{in} \quad \Omega \\ u=0 & \text{on} \quad \partial\Omega, \end{cases} \tag{4}$$

where μ is a diffuse measure. In [12], the authors used the ideas of [3] to prove that for every diffuse measure μ, there exists $f\in L^1(\Omega)$ and $g\in W^{-1,p'(.)}(\Omega)$ such that $\mu=f+g$, that permits them to prove the existence and uniqueness of entropy solution of (4).
The notion of parabolic capacity have been introduced firstly in the quadratic case $p\equiv 2$. The thermal capacity related to the heat equation, and its generalizations have been studied, for example, by Lanconelli [9] and Watson [18]. In the papers [1, 6, 7], the concept of parabolic capacities for constant exponent are defined in terms of function spaces. Droniou, Porretta and Prignet in [6], introduced and studied the notion of parabolic capacity associated with the initial boundary valued problem

$$\begin{cases} u_t+A(u)=\mu & \text{in} \quad Q=(0,T)\times\Omega \\ u=u_0 & \text{on} \quad \{0\}\times\Omega \\ u=0 & \text{on} \quad (0,T)\times\partial\Omega. \end{cases} \tag{5}$$

They worked with the space

$$W=\left\{u\in L^p\left(0,T;W^{1,p}(\Omega)\cap L^2(\Omega)\right);u_t\in L^{p'}\left(0,T;\left(W_0^{1,p}(\Omega)\cap L^2(\Omega)\right)'\right)\right\},$$

to get a representation theorem for measures that are zero on subsets of Q of null capacity, more precisely they proved the following result (see [6]).

Theorem 1.1. *Let μ be a bounded measure on Q which does not charge the sets of null capacity. Then there exists $g_1\in L^{p'}\left(0,T;W^{-1,p'}(\Omega)\right)$, $g_2\in L^p\left(0,T;W^{1,p}(\Omega)\cap L^2(\Omega)\right)$ and $h\in L^1(\Omega)$ such that*

$$\int_Q\varphi d\mu=\int_0^T\langle g_1,\varphi\rangle dt-\int_0^T\langle g_2,\varphi_t\rangle dt+\int_Q h\varphi dxdt, \tag{6}$$

for all $\varphi\in C_c^\infty([0,T]\times\Omega)$, where $\langle .,.\rangle$ denote the duality between $\left(W^{1,p}(\Omega)\cap L^2(\Omega)\right)'$ and $W^{1,p}(\Omega)\cap L^2(\Omega)$.

In this paper, we extend the theory developed in [6] in the case of variable exponents (see [14, 15] for the theory of PDEs with variable exponents). The paper is organized as follows: in Section 2, we recall some basic notations and properties of Lebesgue and Sobolev spaces with variable exponents. In Section 3, we introduce and study the notion of $p(.)-$parabolic capacity. In the last section, we show the connection between measures defined on the σ-algebra of borelians of Q and the notion of $p(.)-$ parabolic capacity and, we prove a theorem of decomposition of measures.

2. Preliminary

In this paper, we assume that

$$\begin{cases} p(.):\overline{\Omega} & \rightarrow & \mathbb{R} \quad \text{is a continuous function such that} \\ 1<p_- & \leq & p_+<+\infty, \end{cases} \tag{7}$$

where $p_- := \inf_{x\in\Omega} p(x)$ and $p_+ := \sup_{x\in\Omega} p(x)$.
We denote the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ (see [4]) as the set of all measurable function $u : \Omega \to \mathbb{R}$ for which the convex modular

$$\rho_{p(.)}(u) := \int_\Omega |u|^{p(x)}\, dx$$

is finite.
If the exponent is bounded, i.e., if $p_+ < +\infty$, then the expression

$$\|u\|_{p(.)} := \inf\left\{\lambda > 0 : \rho_{p(.)}(u/\lambda) \leq 1\right\}$$

defines a norm in $L^{p(\cdot)}(\Omega)$, called the Luxembourg norm.
The space $\left(L^{p(\cdot)}(\Omega), \|.\|_{p(.)}\right)$ is a separable Banach space. Moreover, if $1 < p_- \leq p_+ < +\infty$, then $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$, for $x \in \Omega$.
Finally, we have the Hölder type inequality

$$\left|\int_\Omega uv dx\right| \leq \left(\frac{1}{p_-} + \frac{1}{(p')_-}\right) \|u\|_{p(.)} \|v\|_{p'(.)}, \tag{8}$$

for all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$.
Let

$$W^{1,p(\cdot)}(\Omega) := \left\{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\right\},$$

which is Banach space equiped with the following norm

$$\|u\|_{1,p(.)} := \|u\|_{p(.)} + \|\nabla u\|_{p(.)}.$$

The space $\left(W^{1,p(\cdot)}(\Omega), \|.\|_{1,p(.)}\right)$ is a separable and reflexive Banach space.
An important role in manipulating the generalized Lebesgue and Sobolev spaces is played by the modular $\rho_{p(.)}$ of the space $L^{p(\cdot)}(\Omega)$. We have the following result.

Proposition 2.1. *(see* [8, 21]*) If $u_n, u \in L^{p(\cdot)}(\Omega)$ and $p_+ < \infty$, the following properties hold true.*
(i) $\|u\|_{p(.)} > 1 \Rightarrow \|u\|_{p(.)}^{p_-} < \rho_{p(.)}(u) < \|u\|_{p(.)}^{p_+}$;
(ii) $\|u\|_{p(.)} < 1 \Rightarrow \|u\|_{p(.)}^{p_+} < \rho_{p(.)}(u) < \|u\|_{p(.)}^{p_-}$;
(iii) $\|u\|_{p(.)} < 1$ (*respectively* $= 1; > 1$) $\Leftrightarrow \rho_{p(.)}(u) < 1$ (*respectively* $= 1; > 1$);
(iv) $\|u_n\|_{p(.)} \to 0$ (*respectively* $\to +\infty$) $\Leftrightarrow \rho_{p(.)}(u_n) < 1$ (*respectively* $\to +\infty$);
(v) $\rho_{p(.)}\left(u/\|u\|_{p(.)}\right) = 1$.

For a measurable function $u : \Omega \to \mathbb{R}$, we introduce the following notation.

$$\rho_{1,p(.)}(u) = \int_\Omega |u|^{p(x)}\, dx + \int_\Omega |\nabla u|^{p(x)}\, dx.$$

Proposition 2.2. *(see* [17, 19]*) If $u \in W^{1,p(\cdot)}(\Omega)$, the following properties hold true.*
(i) $\|u\|_{1,p(.)} > 1 \Rightarrow \|u\|_{1,p(.)}^{p_-} < \rho_{1,p(.)}(u) < \|u\|_{1,p(.)}^{p_+}$;
(ii) $\|u\|_{1,p(.)} < 1 \Rightarrow \|u\|_{1,p(.)}^{p_+} < \rho_{p(.)}(u) < \|u\|_{1,p(.)}^{p_-}$;
(iii) $\|u\|_{1,p(.)} < 1$ (*respectively* $= 1; > 1$) $\Leftrightarrow \rho_{1,p(.)}(u) < 1$ (*respectively* $= 1; > 1$).

Following [2], we extend a variable exponent $p : \overline{\Omega} \to [1, +\infty)$ to $\overline{Q} = [0, T] \times \overline{\Omega}$ by setting $p(t, x) = p(x)$ for all $(t, x) \in \overline{Q}$.
We may also consider the generalized Lebesgue space

$$L^{p(.)}(Q) = \left\{ u : Q \to \mathbb{R} \text{ measurable such that } \int\int_Q |u(t,x)|^{p(x)} \, d(t,x) < \infty \right\}$$

endowed with the norm

$$\|u\|_{L^{p(.)}(Q)} := \inf \left\{ \lambda > 0, \int\int_Q \left| \frac{u(t,x)}{\lambda} \right|^{p(x)} d(t,x) < 1 \right\},$$

which share the same properties as $L^{p(.)}(\Omega)$.

3. Parabolic capacity and measures

3.1. Capacity. In this part, we introduce our notion of capacity, following the approach developed in [6].

Definition 3.1. Let us define $V = W_0^{1,p(.)}(\Omega) \cap L^2(\Omega)$, endowed with its natural norm $\|.\|_{W_0^{1,p(.)}(\Omega)} + \|.\|_{L^2(\Omega)}$ and the space

$$W_{p(.)}(0, T) = \left\{ u \in L^{p_-}(0, T; V) ; \nabla u \in (L^{p(.)}(Q))^N, \; u_t \in L^{(p_-)'}(0, T; V') \right\}$$

endowed with its natural norm $\|u\|_{W_{p(.)}(0,T)} = \|u\|_{L^{p_-}(0,T;V)} + \|\nabla u\|_{(L^{p(.)}(Q))^N} + \|u_t\|_{L^{(p_-)'}(0,T;V')}$.

Since $W_0^{1,p(.)}(\Omega)$ and $L^2(\Omega)$ are separables and reflexives Banach spaces, it follows that V is a separable and reflexive Banach space. Consequently, the following result can be proved similarly to that in [5]; thus, we omit its proof.

Theorem 3.1. *The space $W_{p(.)}(0, T)$ is a separable and reflexive Banach space.*

We also have the following result.

Proposition 3.2. *i) $W_{p(.)}(0, T)$ is continuously embedded in $C\left(0, T; L^2(\Omega)\right)$.*
ii) For all $\theta \in C^\infty\left(\mathbb{R} \times \mathbb{R}^N\right)$ and $u \in W_{p(.)}(0, T)$, $\theta u \in W_{p(.)}(0, T)$ and there exists $C(\theta)$ not depending on u such that $\|\theta u\|_{W_{p(.)}(0,T)} \le C(\theta) \|u\|_{W_{p(.)}(0,T)}$.

Proof. *i)* Since $V \hookrightarrow L^2(\Omega) \hookrightarrow V'$, thanks to [5], $W_{p(.)}(0, T)$ is continuously embedded in $C\left(0, T; L^2(\Omega)\right)$ i.e. there exists $C > 0$ such that, for all $u \in W_{p(.)}(0, T)$,

$$\|u\|_{C(0,T;L^2(\Omega))} \le C \|u\|_{W_{p(.)}(0,T)}.$$

ii) The fact that θ is a smooth function implies that $\theta u \in L^{p_-}(0, T; V)$ and there exists $C(\theta) > 0$ such that $\|\theta u\|_{L^{p_-}(0,T;V)} \le C(\theta) \|u\|_{L^{p_-}(0,T;V)}$. We know that $\nabla(\theta u) = u \nabla \theta + \theta \nabla u$. Since θ is a smooth function, there exists $C(\theta) > 0$ such that $\|\theta \nabla u\|_{\left(L^{p(.)}(Q)\right)^N} \le C(\theta) \|\nabla u\|_{\left(L^{p(.)}(Q)\right)^N}$; moreover, using Poincaré type inequality, one shows that $\|u \nabla \theta\|_{\left(L^{p(.)}(Q)\right)^N} \le C(\theta) \|\nabla u\|_{\left(L^{p(.)}(Q)\right)^N}$. Therefore, we can write $\|\nabla(u\theta)\|_{\left(L^{p(.)}(Q)\right)^N} \le C(\theta) \|\nabla u\|_{\left(L^{p(.)}(Q)\right)^N}$. We have, in the sense of distributions, $(\theta u)_t = u \theta_t + \theta u_t$. The second term belongs to $L^{(p_-)'}(0, T; V')$ and

$\|\theta u_t\|_{L^{(p_-)'}(0,T;V')} \leq C(\theta)\|u_t\|_{L^{(p_-)'}(0,T;V')}$. Since $W_{p(.)}(0,T) \hookrightarrow C\left(0,T;L^2(\Omega)\right) \hookrightarrow L^{(p_-)'}\left(0,T;L^2(\Omega)\right)$, then $u\theta_t \in L^{(p_-)'}\left(0,T;L^2(\Omega)\right)$ and $\|u\theta_t\|_{L^{(p_-)'}(0,T;L^2(\Omega))} \leq C(\theta)\|u\|_{W_{p(.)}(0,T)}$. We know that $L^2(\Omega) \hookrightarrow V'$, so $L^{(p_-)'}\left(0,T;L^2(\Omega)\right) \hookrightarrow L^{(p_-)'}(0,T;V')$, which implies that
$u\theta_t \in L^{(p_-)'}(0,T;V')$ and $\|u\theta_t\|_{L^{(p_-)'}(0,T;V')} \leq C(\theta)\|u\|_{W_{p(.)}(0,T)}$ □

Remark 3.1. Since $L^{(p_-)'}(0,T;V') = (L^{p_-}(0,T;V))'$ (since V is a separable reflexive space), and since $L^{p_-}(0,T;V) = L^{p_-}\left(0,T;W_0^{1,p(.)}(\Omega)\right) \cap L^{p_-}\left(0,T;L^2(\Omega)\right) = E \cap F$, with $E\cap F$ being dense both in E and F, we have $L^{(p_-)'}(0,T;V') = E' + F' = L^{(p_-)'}\left(0,T;W^{-1,p'(.)}(\Omega)\right) + L^{(p_-)'}\left(0,T;L^2(\Omega)\right)$ and the norms of these spaces are equivalent.

We introduce the space $\widetilde{W}_{p(.)}(0,T)$ by

$$\widetilde{W}_{p(.)}(0,T) = \left\{u \in L^{p_-}\left(0,T;W_0^{1,p(.)}(\Omega)\right) \cap L^\infty\left(0,T;L^2(\Omega)\right); \nabla u \in \left(L^{p(.)}(Q)\right)^N, \right.$$
$$\left. u_t \in L^{(p_-)'}\left(0,T;W^{-1,p'(.)}(\Omega)\right)\right\}$$

Remark 3.2. $W^{-1,p'(.)}(\Omega) \hookrightarrow V'$, then $\widetilde{W}_{p(.)}(0,T)$ is continuously embedded in $W_{p(.)}(0,T)$.

Now, we give the definition and some properties of capacity.

Definition 3.2. If $U \subset Q$ is an open set, we define the parabolic capacity of U as

$$\mathrm{Cap}_{p(.)}(U) = \inf\left\{\|u\|_{W_{p(.)}(0,T)} : u \in W_{p(.)}(0,T), u \geq \chi_U \text{ almost everywhere in } Q\right\}. \tag{9}$$

Remark 3.3. We will use the convention that $\inf\emptyset = +\infty$ and for any borelian subset $B \subset Q$ the definition of capacity is extended by setting

$$\mathrm{Cap}_{p(.)}(B) = \inf\left\{\mathrm{Cap}_{p(.)}(U), U \text{ open subset of } Q, B \subset U\right\}. \tag{10}$$

Proposition 3.3. *The set function $E \mapsto Cap_{p(.)}(E)$ has the following properties.*
i) If $E_1 \subset E_2$, then

$$Cap_{p(.)}(E_1) \leq Cap_{p(.)}(E_2). \tag{11}$$

ii) For $E_i \subset Q$, $i \in \mathbb{N}$, we have

$$Cap_{p(.)}\left(\bigcup_{i=1}^\infty E_i\right) \leq \sum_{i=1}^\infty Cap_{p(.)}(E_i). \tag{12}$$

Proof. $i)$ Firstly, we consider the case where E_1 and E_2 are open sets of Q.
Since $E_1 \subset E_2$, we have

$$\left\{u \in W_{p(.)}(0,T), u \geq \chi_{E_1} \text{ a.e. } Q\right\} \supset \left\{u \in W_{p(.)}(0,T), u \geq \chi_{E_2} \text{ a.e. } Q\right\}.$$

Hence,

$$\begin{aligned} Cap_{p(.)}(E_1) &= \inf\left\{\|u\|_{W_{p(.)}(0,T)} : u \in W_{p(.)}(0,T), u \geq \chi_{E_1} \text{ a.e. } Q\right\} \\ &\leq \inf\left\{\|u\|_{W_{p(.)}(0,T)} : u \in W_{p(.)}(0,T), u \geq \chi_{E_2} \text{ a.e. } Q\right\} \\ &\leq Cap_{p(.)}(E_2). \end{aligned} \tag{13}$$

Now, we suppose that E_1 and E_2 are two borelians subsets of Q such that $E_1 \subset E_2$, then we have

$$\{U \text{ open set of } Q/E_2 \subset U\} \subset \{U \text{ open set of } Q/E_1 \subset U\}$$

Then, it follows that

$$\begin{aligned} Cap_{p(.)}(E_1) &= \inf\{U \text{ open set of } Q/E_1 \subset U\} \\ &\leq \inf\{U \text{ open set of } Q/E_2 \subset U\} \\ &\leq Cap_{p(.)}(E_2). \end{aligned} \tag{14}$$

ii) If $\sum_{i=1}^{\infty} Cap_{p(.)}(E_i) = +\infty$, then we have

$$Cap_{p(.)}\left(\bigcup_{i=1}^{\infty} E_i\right) = Cap_{p(.)}\left(\bigcup_{i=1}^{\infty} \{E_i/E_i \neq \emptyset\}\right) < +\infty = \sum_{i=1}^{\infty} Cap_{p(.)}(E_i). \tag{15}$$

Assuming that $\sum_{i=1}^{\infty} Cap_{p(.)}(E_i) < \infty$. Let U_i be open set containing E_i such that $Cap_{p(.)}(U_i) \leq Cap_{p(.)}(E_i) + \frac{\varepsilon}{2^i}$ and u_i be such that $u_i \geq \chi_{U_i}$ a.e. in Q with $\|u_i\|_{W_{p(.)}(0,T)} \leq Cap_{p(.)}(U_i) + \frac{\varepsilon}{2^i}$. Then,

$$\left\|\sum_{i=1}^{n} u_i\right\|_{W_{p(.)}(0,T)} \leq \sum_{i=1}^{n} \|u_i\|_{W_{p(.)}(0,T)} \leq \sum_{i=1}^{\infty} Cap_{p(.)}(E_i) + \varepsilon;$$

i.e. $\sum_{i=1}^{n} u_i$ converges strongly in $W_{p(.)}(0,T)$.

Let's now consider $u = \sum_{i=1}^{\infty} u_i$; we have $u \geq \chi_U$ a.e. in Q, where $U = \bigcup_{i=1}^{\infty} U_i$, so that, U being open,

$$Cap_{p(.)}(U) \leq \|u\|_{W_{p(.)}(0,T)} \leq \sum_{i=1}^{\infty} \|u_i\|_{W_{p(.)}(0,T)} \leq \sum_{i=1}^{\infty} Cap_{p(.)}(E_i) + \varepsilon. \tag{16}$$

Since $\bigcup_{i=1}^{\infty} E_i \subset U$, from (16) we get (12). □

The notion of capacity can be defined alternatively using compact sets of Q. Before that, we introduce the following density result(for the proof, we refer the reader to the proof of Theorem 2.11 in [6]).

Lemma 3.4. *Let* Ω *be a bounded subset of* $\mathbb{R}^N$ *and* $1 < p_- \leq p_+ < \infty$. *Then,* $C_c^\infty([0,T] \times \Omega)$ *is dense in* $W_{p(.)}(0,T)$.

Definition 3.3. Let K be a compact subset of Q. The capacity of K is defined as

$$cap(K) = \inf\left\{\|u\|_{W_{p(.)}(0,T)} : u \in C_c^\infty([0,T] \times \Omega),\ u \geq \chi_K\right\}.$$

The capacity of any open subset U of Q is then defined by

$$cap(U) = \sup\{cap(K), K\text{compact},\ K \subset U\}$$

and the capacity of any Borelian set $B \subset Q$ by

$$cap(B) = \inf\{cap(U), U\text{open subset of } Q,\ B \subset U\}.$$

We have the following result.

Proposition 3.5. *i) The capacity cap satisfies the subadditivity property.*
ii) Let B *be a borelian subset of* Q. *Then,* $cap(B) = 0$ *if and only if* $Cap_{p(.)}(B) = 0$.

Proof. The proof is similar to the proofs of Proposition 2.13 and 2.14 in [6]. □

Now, we give a characterization of null capacity.

Theorem 3.6. *Let* B *be a borelian set in* Ω . *Let* $t_0 \in (0,T)$ *fixed. One has* $Cap_{p(.)}(\{t_0\} \times B) = 0$ *if and only if* $meas(B) = 0$.

Proof. Assume first that $Cap_{p(.)}(\{t_0\} \times B) = 0$ and let K be any compact set contained in B, so that $Cap_{p(.)}(\{t_0\} \times K) = 0$. Since, by Proposition 3.5, we also have that $cap(\{t_0\} \times B) = 0$, then, for all $\varepsilon > 0$, there exists a function $\psi_\varepsilon \in C_c^\infty([0,T] \times \Omega)$ such that $\|\psi_\varepsilon\|_{W_{p(.)}(0,T)} \leq \varepsilon$ and $\psi_\varepsilon(t_0) \geq 1$ on K. Since $W_{p(.)}(0,T)$ is embedded in $C\left([0,T]; L^2(\Omega)\right)$, ones has then

$$\text{meas}(K) \leq \int_K |\psi_\varepsilon(t_0)|^2\, dx \leq \|\psi_\varepsilon\|^2_{L^\infty(0,T;L^2(\Omega))} \leq C\|\psi_\varepsilon\|^2_{W_{p(.)}(0,T)} \leq C\varepsilon^2.$$

So, we deduce that $\text{meas}(K) \leq C\varepsilon^2$, and from the arbitrariness of ε, we get that $\text{meas}(K) = 0$. Since this is true for any compact subset contained in B, by regularity of the Lebesgue measure we conclude that $\text{meas}(B) = 0$.
Conversely, if $\text{meas}(B) = 0$, then there exists, for all $\varepsilon > 0$, an open set A_ε such that $B \subset A_\varepsilon$ and $\text{meas}(A_\varepsilon) < \varepsilon$.
Let us consider an $\varepsilon > 0$ fixed in what follows and, let K_n be a sequence of compact sets contained in A_ε such that $K_n \subset K_{n+1}$, for all $n \geq 1$ and $\displaystyle\bigcup_{n=1}^\infty K_n = A_\varepsilon$.
Let $\varphi_n \in C_c(A_\varepsilon)$ (the space of continuous functions with compact support in A_ε) be such that $0 \leq \varphi_n \leq 1$, $\varphi_n \equiv 1$ on K_n and $\varphi_n \leq \varphi_{n+1}$. Then, we consider for $t_0 \in [0,T]$, the problem

$$\begin{cases} (\psi_n)_t - div\left(|\nabla\psi_n|^{p(x)-2}\nabla\psi_n\right) = 0 & \text{in} \quad (t_0,T) \times \Omega \\ \psi_n(t_0) = \varphi_n & \text{in} \quad \Omega \\ \psi_n = 0 & \text{on} \quad (t_0,T) \times \partial\Omega, \end{cases} \tag{17}$$

which admits (see [20]) a unique weak solution

$$\psi_n \in L^{p_-}\left(t_0,T; W_0^{1,p(.)}(\Omega)\right) \cap C\left([t_0,T]; L^2(\Omega)\right)$$

and $(\psi_n)_t \in L^{(p_-)'}\left(t_0, T; W^{-1,p'(.)}(\Omega)\right)$ with $\nabla\psi_n \in \left(L^{p(.)}((t_0,T)\times\Omega)\right)^N$ such that for all $v \in C^1\left([t_0,T]\times\overline{\Omega}\right)$ with $v(.,T) = 0$,

$$-\int_\Omega \varphi_n(x)\, v(t_0,x)\, dx - \int_{t_0}^T \int_\Omega \psi_n v_t dxdt + \int_{t_0}^T \int_\Omega |\nabla\psi_n|^{p(x)-2} \nabla\psi_n.\nabla v dxdt = 0 \tag{18}$$

holds true.

It's not difficult to see that $\psi_n \in L^{p_-}(t_0,T;V)$ and by Remark 3.1 we have $(\psi_n)_t \in L^{(p_-)'}(t_0,T;V')$ hence, $\psi_n \in W_{p(.)}(t_0,T)$. We know that $(\psi_n(s), v(s)) \in V^2$ for all $s \in [t_0,T]$ and $V \hookrightarrow L^2(\Omega) \hookrightarrow V'$, then thanks to [6] we have

$$\int_{t_0}^T \int_\Omega \psi_n v dxdt = \int_{t_0}^T \langle \psi_n, v\rangle_{L^2(\Omega),L^2(\Omega)} dt = \int_{t_0}^T \langle \psi_n, v\rangle_{V',V} dt. \tag{19}$$

Moreover, (ψ_n, v) satisfies the following integration by part formula

$$\begin{aligned}\int_{t_0}^T \langle v_t, \psi_n\rangle dt &= \langle \psi_n(T), v(T)\rangle_{L^2(\Omega),L^2(\Omega)} - \langle \psi_n(t_0), v(t_0)\rangle_{L^2(\Omega),L^2(\Omega)} \\ &\quad - \int_{t_0}^T \langle (\psi_n)_t, v\rangle_{L^2(\Omega),L^2(\Omega)}.\end{aligned} \tag{20}$$

Therefore, using (19), (20) and the fact that $v(.,T) = 0$, we can rewrite (18) as follows.

$$\int_{t_0}^T (\psi_n)_t v dxdt + \int_{t_0}^T |\nabla\psi_n|^{p(x)-2} \nabla\psi_n.\nabla v dxdt = 0. \tag{21}$$

Since $C_c^\infty([t_0,T]\times\Omega)$ is dense in $W_{p(.)}(t_0,T)$, we can choose ψ_n as a test function in (21) to obtain

$$\int_{t_0}^T \int_\Omega \psi_n (\psi_n)_t + \int_{t_0}^T \int_\Omega |\nabla\psi_n|^{p(x)} dxdt = 0, \tag{22}$$

which is equivalent to

$$\frac{1}{2}\int_\Omega \psi_n(.,T)^2 + \int_{t_0}^T \int_\Omega |\nabla\psi_n|^{p(x)} dxdt = \frac{1}{2}\int_\Omega \varphi_n^2 dx. \tag{23}$$

So,

$$\int_{t_0}^T \int_\Omega |\nabla\psi_n|^{p(x)} dxdt \le \frac{1}{2}\int_\Omega \varphi_n^2 dx. \tag{24}$$

Therefore, using Proposition 2.1 we obtain

$$\begin{aligned}\|\nabla\psi\|_{\left(L^{p(.)}((t_0,T)\times\Omega)\right)} &\le \max\left\{\left(\int_{t_0}^T\int_\Omega |\nabla\psi_n|^{p(x)} dxdt\right)^{\frac{1}{p_-}}, \left(\int_{t_0}^T\int_\Omega |\nabla\psi_n|^{p(x)} dxdt\right)^{\frac{1}{p_+}}\right\} \\ &\le \max\left\{\left(\frac{1}{2}\int_\Omega \varphi_n^2 dx\right)^{\frac{1}{p_-}}, \left(\frac{1}{2}\int_\Omega \varphi_n^2 dx\right)^{\frac{1}{p_+}}\right\}\end{aligned} \tag{25}$$

and

$$\begin{aligned}\int_{t_0}^{T}\|\nabla\psi\|_{p(.)}^{p_-}dt &\le \int_{t_0}^{T}\max\left\{\int_\Omega|\nabla\psi_n|^{p(x)}dx,\left(\int_\Omega|\nabla\psi_n|^{p(x)}dx\right)^{\frac{p_-}{p_+}}\right\}dt\\ &\le \int_{t_0}^{T}\int_\Omega|\nabla\psi_n|^{p(x)}dxdt+T^{1-\frac{p_-}{p_+}}\left(\int_{t_0}^{T}\int_\Omega|\nabla\psi_n|^{p(x)}dxdt\right)^{\frac{p_-}{p_+}} \quad (26)\end{aligned}$$

then it follows that

$$\int_{t_0}^{T}\|\psi_n\|_{W_0^{1,p(.)}(\Omega)}^{p_-}dt\le\frac{1}{2}\int_\Omega\varphi_n^2dx+T^{1-\frac{p_-}{p_+}}\left(\frac{1}{2}\int_\Omega\varphi_n^2dx\right)^{\frac{p_-}{p_+}}. \quad (27)$$

Hence,

$$\|\psi_n\|_{L^{p_-}\left(t_0,T;W_0^{1,p(.)}(\Omega)\right)}\le\left(\frac{1}{2}\int_\Omega\varphi_n^2dx+T^{1-\frac{p_-}{p_+}}\left(\frac{1}{2}\int_\Omega\varphi_n^2dx\right)^{\frac{p_-}{p_+}}\right)^{\frac{1}{p_-}}. \quad (28)$$

In (21), we take $v=\psi_n\chi_{(t_0,t)}$ as a test function, where $\chi_{(t_0,t)}$ is defined as the characteristic function of (t_0,t), $t\in[t_0,T]$ then, using the integration by part formula, we get

$$\frac{1}{2}\int_\Omega\psi_n(.,t)^2dx+\int_{t_0}^{T}\int_\Omega|\nabla\psi_n|^{p(x)}dxdt=\frac{1}{2}\int_\Omega\varphi_n^2dx, \quad (29)$$

which implies that

$$\frac{1}{2}\int_\Omega\psi_n(.,t)^2dx\le\frac{1}{2}\int_\Omega\varphi_n^2dx, \quad (30)$$

Consequently,

$$\|\psi_n\|_{L^\infty((t_0,t);L^2(\Omega))}\le\left(\int_\Omega\varphi_n^2dx\right)^{\frac{1}{2}}. \quad (31)$$

Let $v\in L^{p_-}(0,T;V)$ such that $\|v\|_{L^{p_-}(0,T;V)}\le 1$, for every $k\ge 1$,
$A_k=\{t\in[0,T]:\|v\|_V\ge k\}$ and $\mathcal{A}=\bigcup_{k\ge1}A_k$.

We have

$$\begin{aligned}\text{meas}(\mathcal{A}) &= \frac{1}{k}\int_{\mathcal{A}}kdt\le\frac{1}{k}\int_{\mathcal{A}}\|v\|_V dt\le\frac{1}{k}\int_{\mathcal{A}}\|v\|_V^{p_-}dt\\ &\le \frac{1}{k}\int_0^T\|v\|_V^{p_-}dt\le\frac{1}{k}\|v\|_{L^{p_-}(0,T;V)}^{p_-}\le\frac{1}{k}. \quad (32)\end{aligned}$$

Hence, we deduce by letting $k\to\infty$ that $\text{meas}(\mathcal{A})=0$.
We use (22) and the Hölder type inequality to get

$$\begin{aligned}\left|\langle(\psi_n)_t,v\rangle_{L^{(p_-)'}(t_0,T;V'),L^{p_-}(t_0,T;V)}\right| &= \left|\int_{t_0}^{T}\langle(\psi_n)_t,v\rangle_{V',V}dt\right|\\ =\left|\int_{t_0}^{T}\int_\Omega(\psi_n)_t vdxdt\right| &\le \int_{t_0}^{T}\int_\Omega\left||\nabla\psi_n|^{p(x)-2}\nabla\psi_n.\nabla vdxdt\right|\\ \le 2\int_{t_0}^{T}\int_\Omega\left\||\nabla\psi_n|^{p(.)-1}\right\|_{p'(.)}\|\nabla v\|_{p(.)} &\le \int_{t_0}^{T}\left\||\nabla\psi_n|^{p(.)-1}\right\|_{p'(.)}\|v\|_V. \quad (33)\end{aligned}$$

Since meas $(\mathcal{A})=0$, we deduce that

$$\begin{aligned}
&\int_{t_0}^{T}\left\||\nabla\psi_n|^{p(.)-1}\right\|_{p'(.)}\|v\|_V\,dt\\
&=\int_{A_1}\left\||\nabla\psi_n|^{p(.)-1}\right\|_{p'(.)}\|v\|_V\,dt+\int_{[[t_0,T]\setminus A_1]}\left\||\nabla\psi_n|^{p(.)-1}\right\|_{p'(.)}\|v\|_V\,dt\\
&\leq\int_{\mathcal{A}}\left\||\nabla\psi_n|^{p(.)-1}\right\|_{p'(.)}\|v\|_V\,dt+\int_{[[t_0,T]\setminus A_1]}\left\||\nabla\psi_n|^{p(.)-1}\right\|_{p'(.)}\|v\|_V\,dt\\
&\leq\int_{[[t_0,T]\setminus A_1]}\left\||\nabla\psi_n|^{p(.)-1}\right\|_{p'(.)}\|v\|_V\,dt, \qquad (34)
\end{aligned}$$

which implies that

$$\begin{aligned}
&\left|\langle(\psi_n)_t\,,v\rangle_{L^{(p_-)'}(t_0,T;V'),L^{p_-}(t_0,T;V)}\right|\leq\int_{[[t_0,T]\setminus A_1]}\left\||\nabla\psi_n|^{p(.)-1}\right\|_{p'(.)}\|v\|_V\,dt\\
&\leq\int_{[[t_0,T]\setminus A_1]}\left\||\nabla\psi_n|^{p(.)-1}\right\|_{p'(.)}dt\leq\int_{t_0}^{T}\left\||\nabla\psi_n|^{p(.)-1}\right\|_{p'(.)}dt \qquad (35)\\
&\leq(T-t_0)^{1-\frac{1}{(p')_-}}\left(\int_{t_0}^{T}\left\||\nabla\psi_n|^{p(.)-1}\right\|_{p'(.)}^{(p')_-}dt\right)^{\frac{1}{(p')_-}}.
\end{aligned}$$

Hence, we get

$$\|(\psi_n)_t\|_{L^{(p_-)'}(t_0,T;V')}\leq T^{1-\frac{1}{(p')_-}}\left(\int_{t_0}^{T}\left\||\nabla\psi_n|^{p(.)-1}\right\|_{p'(.)}^{(p')_-}dt\right)^{\frac{1}{(p')_-}}. \qquad (36)$$

Consequently, we use Proposition 2.1 to get

$$\begin{aligned}
&\int_{t_0}^{T}\left\||\nabla\psi|^{p(.)-1}\right\|_{p'(.)}^{(p')_-}dt\leq\int_{t_0}^{T}\max\left\{\int_\Omega|\nabla\psi_n|^{p(x)}\,dx,\left(\int_\Omega|\nabla\psi_n|^{p(x)}\,dx\right)^{\frac{(p')_-}{(p')_+}}\right\}dt\\
&\leq\int_{t_0}^{T}\int_\Omega|\nabla\psi_n|^{p(x)}\,dxdt+(T-t_0)^{1-\frac{(p')_-}{(p')_+}}\left(\int_{t_0}^{T}\int_\Omega|\nabla\psi_n|^{p(x)}\,dxdt\right)^{\frac{(p')_-}{(p')_+}}. \qquad (37)
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\|(\psi_n)_t\|_{L^{(p_-)'}(t_0,T;V')}\\
&\leq T^{1-\frac{1}{(p')_-}}\left(\int_{t_0}^{T}\!\!\int_\Omega|\nabla\psi_n|^{p(x)}\,dxdt+T^{1-\frac{(p')_-}{(p')_+}}\left(\int_{t_0}^{T}\!\!\int_\Omega|\nabla\psi_n|^{p(x)}\,dxdt\right)^{\frac{(p')_-}{(p')_+}}\right)^{\frac{1}{(p')_-}}\\
&\leq T^{1-\frac{1}{(p')_-}}\left(\frac{1}{2}\int_\Omega\varphi_n^2dx+T^{1-\frac{(p')_-}{(p')_+}}\left(\frac{1}{2}\int_\Omega\varphi_n^2dx\right)^{\frac{(p')_-}{(p')_+}}\right)^{\frac{1}{(p')_-}}. \qquad (38)
\end{aligned}$$

Finally, combining (25), (26), (31) and (38), we conclude that

$$\|\nabla\psi_n\|_{L^{p(\cdot)}((t_0,T)\times\Omega)}+\|\psi_n\|_{L^{p_-}(t_0,T;W_0^{1,p(\cdot)}(\Omega))}+\|\psi_n\|_{L^\infty(t_0,t;L^2(\Omega))}+\|(\psi_n)_t\|_{L^{(p_-)'}(t_0,T;V')}$$
$$\leq\left(\frac{1}{2}\int_\Omega\varphi_n^2dx\right)^{\frac{1}{p_-}}+\left(\frac{1}{2}\int_\Omega\varphi_n^2dx\right)^{\frac{1}{p_+}}+\left(\frac{1}{2}\int_\Omega\varphi_n^2dx+T^{1-\frac{p_-}{p_+}}\left(\frac{1}{2}\int_\Omega\varphi_n^2dx\right)^{\frac{p_-}{p_+}}\right)^{\frac{1}{p_-}}$$
$$+\left(\int_\Omega\varphi_n^2dx\right)^{\frac{1}{2}}+T^{1-\frac{1}{(p')_-}}\left(\frac{1}{2}\int_\Omega\varphi_n^2dx+T^{1-\frac{(p')_-}{(p')_+}}\left(\frac{1}{2}\int_\Omega\varphi_n^2dx\right)^{\frac{(p')_-}{(p')_+}}\right)^{\frac{1}{(p')_-}} \quad (39)$$

Let us now construct a function $\tilde{\psi_n}$ defined on $[0,T]$ by setting

$$\begin{cases}\tilde{\psi_n}=\psi_n & \text{in } \;]t_0,T]\times\Omega\\ \tilde{\psi_n}=\psi_n\left(T-\dfrac{t(T-t_0)}{t_0}\right) & \text{in } \;[0,t_0]\times\Omega.\end{cases}$$

By (39), we have

$$\left\|\nabla\tilde{\psi_n}\right\|_{(L^{p(\cdot)}((t_0,T)\times\Omega)}+\left\|\tilde{\psi_n}\right\|_{L^{p_-}(t_0,T;W_0^{1,p(\cdot)}(\Omega)}+\left\|\tilde{\psi_n}\right\|_{L^\infty(t_0,t;L^2(\Omega))}+\left\|\left(\tilde{\psi_n}\right)_t\right\|_{L^{(p_-)'}(t_0,T;V')}$$
$$\leq\left(\frac{1}{2}\|\varphi_n\|^2_{L^2(\Omega)}\right)^{\frac{1}{p_-}}+\left(\|\varphi_n\|^2_{L^2(\Omega)}\right)^{\frac{1}{p_+}}+\left(\frac{1}{2}\|\varphi_n\|^2_{L^2(\Omega)}+T^{1-\frac{p_-}{p_+}}\left(\frac{1}{2}\|\varphi_n\|^2_{L^2(\Omega)}\right)^{\frac{p_-}{p_+}}\right)^{\frac{1}{p_-}}$$
$$+\|\varphi_n\|_{L^2(\Omega)}+T^{1-\frac{1}{(p')_-}}\left(\frac{1}{2}\|\varphi_n\|^2_{L^2(\Omega)}+T^{1-\frac{(p')_-}{(p')_+}}\left(\frac{1}{2}\|\varphi_n\|^2_{L^2(\Omega)}\right)^{\frac{(p')_-}{(p')_+}}\right)^{\frac{1}{(p')_-}}. \quad (40)$$

Since $\varphi_n\in C_c(A_\varepsilon)$ and $0\leq\varphi_n\leq 1$, we deduce that $\|\varphi_n\|^2_{L^2(\Omega)}\leq meas(A_\varepsilon)\leq\varepsilon$, then, it follows that

$$\left\|\nabla\tilde{\psi_n}\right\|_{\left(L^{p(\cdot)}((t_0,T)\times\Omega)\right)}+\left\|\tilde{\psi_n}\right\|_{L^{p_-}\left(t_0,T;W_0^{1,p(\cdot)}(\Omega)\right)}+\left\|\tilde{\psi_n}\right\|_{L^\infty(t_0,t;L^2(\Omega))}$$
$$+\left\|\left(\tilde{\psi_n}\right)_t\right\|_{L^{(p_-)'}(t_0,T;V')}\leq\left(\frac{1}{2}\varepsilon\right)^{\frac{1}{p_-}}+(\varepsilon)^{\frac{1}{p_+}}+\left(\frac{1}{2}\varepsilon+T^{1-\frac{p_-}{p_+}}\left(\frac{1}{2}\varepsilon\right)^{\frac{p_-}{p_+}}\right)^{\frac{1}{p_-}}+\varepsilon^{\frac{1}{2}}$$
$$+T^{1-\frac{1}{(p')_-}}\left(\frac{1}{2}\varepsilon+T^{1-\frac{(p')_-}{(p')_+}}\left(\frac{1}{2}\varepsilon\right)^{\frac{(p')_-}{(p')_+}}\right)^{\frac{1}{(p')_-}}. \quad (41)$$

The fact that ψ_n belongs to $C\left([t_0,T],L^2(\Omega)\right)$, implies that $\psi_n\in C([t_0,T]\times\Omega)$, then it follows that $\tilde{\psi_n}\in C([t_0,T]\times\Omega)$. Therefore, the set $U_n=\left\{\tilde{\psi_n}>\dfrac{1}{2}\right\}$ is open.

Since U_n is open and $2\tilde{\psi}_n > \chi_{U_n}$, we have

$$\begin{aligned} Cap_{p(.)}(U_n) &\le 2\|\psi_n\|_{W_{p(.)}(0,T)} \qquad (42)\\ &\le \left(\frac{1}{2}\varepsilon\right)^{\frac{1}{p_-}} + (\varepsilon)^{\frac{1}{p_+}} + \left(\frac{1}{2}\varepsilon + T^{1-\frac{p_-}{p_+}}\left(\frac{1}{2}\varepsilon\right)^{\frac{p_-}{p_+}}\right)^{\frac{1}{p_-}} + \varepsilon^{\frac{1}{2}}\\ &\quad + T^{1-\frac{1}{(p')_-}}\left(\frac{1}{2}\varepsilon + T^{1-\frac{(p')_-}{(p')_+}}\left(\frac{1}{2}\varepsilon\right)^{\frac{(p')_-}{(p')_+}}\right)^{\frac{1}{(p')_-}}. \qquad (43)\end{aligned}$$

Since the sequence φ_n is nondecreasing, we have that the sequence $\tilde{\psi}_n$ is nondecreasing as well, hence $U_n \subset U_{n+1}$, $Cap_{p(.)}(U_n)$ is also a nondecreasing sequence and bounded too. Let's show that

$$Cap_{p(.)}(U_\infty) = \lim_{n\to\infty} Cap_{p(.)}(U_n), \qquad (44)$$

where $U_\infty = \bigcup_{n=1}^{\infty} U_n$.
In fact, we have $U_n \subset U_\infty$, then

$$\lim_{n\to\infty} Cap_{p(.)}(U_n) \le Cap_{p(.)}(U_\infty). \qquad (45)$$

Now, we take $(u_n)_{n\in\mathbb{N}} \subset W_{p(.)}(0,T)$ such that

$$u_n \ge \chi_{U_n} \text{ a.e. in } Q \text{ and } \|u_n\|_{W_{p(.)}(0,T)} \le Cap_{p(.)}(U_n) + \frac{1}{n}.$$

Thanks to (42), $(u_n)_{n\in\mathbb{N}}$ is bounded in $W_{p(.)}(0,T)$, then we can extract a subsequence still denoted by $(u_n)_{n\in\mathbb{N}}$ such that $u_n \to u$ weakly in $W_{p(.)}(0,T)$ and a.e. in Q. Since U_n is nondecreasing and $(u_n)_{n\in\mathbb{N}}$ converges almost everywhere to u, we deduce that $u \ge \chi_{U_\infty}$ a.e. in Q, hence it follows that

$$Cap_{p(.)}(U_\infty) \le \|u\|_{W_{p(.)}(0,T)} \le \liminf_{n\to\infty} \|u_n\|_{W_{p(.)}(0,T)} \le \lim_{n\to\infty} Cap_{p(.)}(U_n). \qquad (46)$$

Combining (45) and (46), we obtain (44).
Since $\varphi_n = 1$ on K_n for each n and $\{t_0\}\times A_\varepsilon \supset \{t_0\}\times B$ then, we conclude from (44) and (45) that

$$\begin{aligned} Cap_{p(.)}(\{t_0\}\times B) &\le Cap_{p(.)}(U_\infty) = \lim_{n\infty} Cap_{p(.)}(U_n)\\ &\le \left(\frac{1}{2}\varepsilon\right)^{\frac{1}{p_-}} + (\varepsilon)^{\frac{1}{p_+}} + \left(\frac{1}{2}\varepsilon + T^{1-\frac{p_-}{p_+}}\left(\frac{1}{2}\varepsilon\right)^{\frac{p_-}{p_+}}\right)^{\frac{1}{p_-}} + \varepsilon^{\frac{1}{2}}\\ &\quad + T^{1-\frac{1}{(p')_-}}\left(\frac{1}{2}\varepsilon + T^{1-\frac{(p')_-}{(p')_+}}\left(\frac{1}{2}\varepsilon\right)^{\frac{(p')_-}{(p')_+}}\right)^{\frac{1}{(p')_-}}. \qquad (47)\end{aligned}$$

Hence, letting $\varepsilon \to 0$ in (47), we deduce that $Cap_{p(.)}(\{t_0\}\times B) = 0$ □

3.2. Quasicontinuous function.

Definition 3.4. A claim is said to hold $Cap_{p(.)}$-quasi everywhere if it holds everywhere except on a set of zero $p(.)$-capacity. A function $u : Q \to \mathbb{R}$ is said to be $Cap_{p(.)}$-quasi continuous if for every $\varepsilon > 0$, there exists an open set U_ε with $Cap_{p(.)}(U_\varepsilon) < \varepsilon$ such that u restricted to $Q \setminus U_\varepsilon$ is continuous.

In this section, we prove that every element of $W_{p(.)}(0,T)$ admits cap-quasi continuous representative. We recall that the approach developed in elliptic case (see [4]) cannot extend in our situation since if $u \in W_{p(.)}(0,T)$, one may have $|u| \notin W_{p(.)}(0,T)$ (see [6]).

Lemma 3.7. *(i) Let u belongs to $W_{p(.)}(0,T)$; then there exists a function z in $\widetilde{W}_{p(.)}(0,T)$ such that $|u| < z$ and*

$$\|z\|_{\widetilde{W}_{p(.)}(0,T)} \leq C\left([u]_*^{\frac{1}{2}} + [u]_*^{\frac{1}{p_-}} + [u]_*^{\frac{1}{p_+}} + [u]_*^{\frac{1}{(p')_-}} + [u]_*^{\frac{1}{(p')_+}}\right), \tag{48}$$

where

$$\begin{aligned} [u]_* &= \rho_{p(.)}(|\nabla u|) + \|u_t\|^2_{L^{(p_-)'}(0,T;V')} + \|u\|^2_{L^\infty(0,t;L^2(\Omega))} + \|u_t\|^{(p_-)'}_{L^{(p_-)'}(0,T;V')} \\ &\quad + \|u_t\|_{L^{(p_-)'}(0,T;V')} + \|u_t\|_{L^{(p_-)'}(0,T;V')}\|u\|_{L^\infty(0,t;L^2(\Omega))}. \end{aligned} \tag{49}$$

(ii) If u belongs to $L^{p_-}(0,T;W_0^{1,p(.)}(\Omega)) \cap L^\infty(Q)$ and u_t in $L^{(p_-)'}(0,T;W^{-1,p'(.)}(\Omega)) + L^1(Q)$, then there exists $z \in \widetilde{W}_{p(.)}(0,T)$ such that $|u| < z$ and

$$[z] \leq C\left([u]_{**} + [u]_{**}^{\frac{1}{p_-}} + [u]_{**}^{\frac{1}{p_+}} + [u]_{**}^{\frac{1}{(p')_-}} + [u]_{**}^{\frac{1}{(p')_+}}\right), \tag{50}$$

where

$$\begin{aligned} [u]_{**} &= \rho_{p(.)}(|\nabla u|) + \|u\|^2_{L^\infty(0,T;L^2(\Omega))} + \|u_t\|^{(p_-)'}_{L^{(p_-)'}\left(0,T;W^{-1,p'(.)}(\Omega)\right)+L^1(Q)} \\ &\quad + \|u_t\|_{L^{(p_-)'}\left(0,T;W^{-1,p'(.)}(\Omega)\right)+L^1(Q)} + \|u_t\|_{L^{(p_-)'}\left(0,T;W^{-1,p'(.)}(\Omega)\right)+L^1(Q)}\|u\|_{L^\infty(Q)} \end{aligned} \tag{51}$$

and

$$[z] = \|z\|^{p_-}_{L^{p_-}\left(0,T;W_0^{1,p(.)}(\Omega)\right)} + \|z_t\|^{(p_-)'}_{L^{(p_-)'}(0,T;V')} + \|z\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\nabla z\|_{p(.)}. \tag{52}$$

Proof. We divide the proof in two steps.
Step 1. Let us consider the penalized problem

$$\begin{cases} (u_\varepsilon)_t - \Delta_{p(.)}u_\varepsilon = \dfrac{1}{\varepsilon}(u_\varepsilon - u)^- & \text{in } (0,T)\times\Omega \\ u_\varepsilon(0) = u^+(0) & \text{on } \Omega \\ u_\varepsilon = 0 & \text{on } (0,T)\times\partial\Omega. \end{cases} \tag{53}$$

According to [11], we can prove that this problem admits a nonnegative solution u_ε in $C([0,T];L^2(\Omega)) \cap L^{p_-}\left(0,T;W_0^{1,p(.)}(\Omega)\right)$.

Taking $u_\varepsilon - u$ as a test function in (53) then, for every t in $[0,T]$ we have

$$\int_0^t \langle (u_\varepsilon - u)_t, u_\varepsilon - u\rangle \, ds + \int_0^t \int_\Omega |\nabla u_\varepsilon|^{p(x)} \, dxds = \frac{1}{\varepsilon} \int_0^t \int_\Omega (u_\varepsilon - u)^- (u_\varepsilon - u) \, dxds$$
$$+ \int_0^t \int_\Omega |\nabla u_\varepsilon|^{p(x)-2} \nabla u_\varepsilon . \nabla u dxds - \int_0^t \langle u_t, u_\varepsilon - u\rangle \, dxds.$$

By integration by parts formula and the fact that $(u_\varepsilon - u)^- (u_\varepsilon - u) \le 0$, we deduce that

$$\frac{1}{2} \int_\Omega |u_\varepsilon - u|^2 (t) \, dx + \int_0^t \int_\Omega |\nabla u_\varepsilon|^{p(x)} \, dxds \le \int_0^t \int_\Omega |\nabla u_\varepsilon|^{p(x)-2} \nabla u_\varepsilon . \nabla u dxds$$
$$+ \frac{1}{2} \int_\Omega |u_\varepsilon(0) - u(0)|^2 \, dx - \int_0^t \langle u_t, u_\varepsilon - u\rangle \, dxds;$$

which implies that

$$\frac{1}{2} \int_\Omega |u_\varepsilon|^2 (t) \, dx + \int_0^t \int_\Omega |\nabla u_\varepsilon|^{p(x)} \, dxds \le \int_0^t \int_\Omega |\nabla u_\varepsilon|^{p(x)-1} |\nabla u| \, dxds$$
$$+ \frac{1}{2} \int_\Omega |u(0)|^2 \, dx + \int_\Omega |u_\varepsilon(t)| \, |u(t)| \, dx - \int_0^t \langle u_t, u_\varepsilon - u\rangle \, ds.$$

Now, we use the Young inequality to obtain

$$\int_0^t \int_\Omega |\nabla u_\varepsilon|^{p(x)-1} |\nabla u| \, dxdt \le 2^{p^+} \int_Q |\nabla u|^{p(x)} \, dxdt + \frac{1}{2} \int_0^t \int_\Omega |\nabla u_\varepsilon|^{p(x)} \, dxds$$

and

$$\int_\Omega |u_\varepsilon(t)| \, |u(t)| \, dx \le \frac{1}{4} \int_\Omega |u_\varepsilon(t)|^2 \, dx + 2 \int_\Omega |u(t)|^2 \, dx.$$

Thus,

$$\frac{1}{4} \int_\Omega |u_\varepsilon|^2 (t) \, dx + \frac{1}{2} \int_0^t \int_\Omega |\nabla u_\varepsilon|^{p(x)} \, dxds \le 2^{p_+} \int_Q |\nabla u|^{p(x)} \, dxds \tag{54}$$
$$+ \frac{5}{2} \|u\|^2_{L^\infty(0,T;L^2(\Omega))} - \int_0^t \langle u_t, u_\varepsilon - u\rangle \, ds.$$

If we are in case $i)$, u is in $W_{p(.)}(0,T)$ and we have

$$\left| \int_0^t \langle u_t, u_\varepsilon - u\rangle \, dt \right| \le \int_0^t \|u_t\|_{V'} \|u_\varepsilon - u\|_V \, dt$$
$$\le \int_0^t \|u_t\|_{V'} \|u_\varepsilon - u\|_{W_0^{1,p(.)}(\Omega)} \, dt + \int_0^t \|u_t\|_{V'} \|u_\varepsilon - u\|_{L^2(\Omega)} \, dt \tag{55}$$
$$\le \|u_t\|_{L^{(p_-)'}(0,t;V')} \|u_\varepsilon - u\|_{L^{p_-}\left(0,t;W_0^{1,p(.)}(\Omega)\right)} + \|u_t\|_{L^1(0,t;V')} \|u_\varepsilon - u\|_{L^\infty(0,t;L^2(\Omega))}$$
$$\le \|u_t\|_{L^{(p_-)'}(0,T;V')} \|u_\varepsilon - u\|_{L^{p_-}\left(0,t;W_0^{1,p(.)}(\Omega)\right)} + C \|u_t\|_{L^{(p_-)'}(0,T;V')} \|u\|_{L^\infty(0,t;L^2(\Omega))}.$$

Thanks to Proposition 2.1 and Hölder inequality, we have

$$\begin{aligned} &\|u_\varepsilon - u\|^{p_-}_{L^{p_-}\left(0,t;W_0^{1,p(.)}(\Omega)\right)} \\ &\leq \int_0^t \max\left\{\int_\Omega |\nabla(u_\varepsilon - u)|^{p(x)}\,dx, \left(\int_\Omega |\nabla(u_\varepsilon - u)|^{p(x)}\,dx\right)^{\frac{p_-}{p_+}}\right\} ds \qquad (56) \\ &\leq \int_0^t\int_\Omega |\nabla(u_\varepsilon - u)|^{p(x)}\,dxds + t^{1-(p_-/p_+)}\left(\int_0^t\int_\Omega |\nabla(u_\varepsilon - u)|^{p(x)}\,dxds\right)^{\frac{p_-}{p_+}}. \end{aligned}$$

Hence, if $\int_0^t\int_\Omega |\nabla(u_\varepsilon - u)|^{p(x)}\,dxds > 1$, we deduce that

$$\begin{aligned} &\|u_\varepsilon - u\|^{p_-}_{L^{p_-}\left(0,t;W_0^{1,p(.)}(\Omega)\right)} \\ &\leq \int_0^t\int_\Omega |\nabla(u_\varepsilon - u)|^{p(x)}\,dxds + T^{1-(p_-/p_+)}\int_0^t\int_\Omega |\nabla(u_\varepsilon - u)|^{p(x)}\,dxds \\ &\leq \left(1 + T^{1-(p_-/p_+)}\right)\int_0^t\int_\Omega (|\nabla u_\varepsilon| + |\nabla u|)^{p(x)}\,dxds \\ &\leq \left(1 + T^{1-(p_-/p_+)}\right)\int_0^t\int_\Omega 2^{p_+-1}\left(|\nabla u_\varepsilon|^{p(x)} + |\nabla u|^{p(x)}\right) dxds \\ &\leq \left(1 + T^{1-(p_-/p_+)}\right)2^{p_+}\left(\int_0^t\int_\Omega |\nabla u_\varepsilon|^{p(x)}\,dxds + \int_0^t\int_\Omega |\nabla u|^{p(x)}\,dxds\right). \qquad (57) \end{aligned}$$

Since from the Young inequality, we have

$$\begin{aligned} &\|u_t\|_{L^{(p_-)'}(0,T;V')}\|u_\varepsilon - u\|_{L^{p_-}\left(0,t;W_0^{1,p(.)}(\Omega)\right)} \qquad (58) \\ &= 2^{\frac{p_++2}{p_-}}\left(1 + T^{1-(p_-/p_+)}\right)^{\frac{1}{p_-}}\|u_t\|_{L^{(p_-)'}(0,T;V')}\frac{\|u_\varepsilon - u\|_{L^{p_-}\left(0,t;W_0^{1,p(.)}(\Omega)\right)}}{2^{\frac{p_++2}{p_-}}\left(1 + T^{1-(p_-/p_+)}\right)^{\frac{1}{p_-}}} \\ &\leq 2^{\frac{(p_-)'(p_++2)}{p_-}}\left(1 + T^{1-(p_-/p_+)}\right)^{\frac{(p_-)'}{p_-}}\|u_t\|^{(p_-)'}_{L^{(p_-)'}(0,T;V')} + \frac{\|u_\varepsilon - u\|^{p_-}_{L^{p_-}\left(0,t;W_0^{1,p(.)}(\Omega)\right)}}{2^{p_++2}\left(1 + T^{1-(p_-/p_+)}\right)}. \end{aligned}$$

Then, if $\int_0^t\int_\Omega |\nabla(u_\varepsilon - u)|^{p(x)}\,dxds > 1$, by (57) and (58) we deduce that

$$\begin{aligned} &\|u_t\|_{L^{(p_-)'}(0,T;V')}\|u_\varepsilon - u\|_{L^{p_-}\left(0,t;W_0^{1,p(.)}(\Omega)\right)} \qquad (59) \\ &\leq 2^{\frac{(p_-)'(p_++2)}{p_-}}\left(1 + T^{1-(p_-/p_+)}\right)^{\frac{(p_-)'}{p_-}}\|u_t\|^{(p_-)'}_{L^{(p_-)'}(0,T;V')} \\ &+\frac{1}{4}\int_0^t\int_\Omega |\nabla u_\varepsilon|^{p(x)}\,dxds + \frac{1}{4}\int_Q |\nabla u_\varepsilon|^{p(x)}\,dxds. \end{aligned}$$

If $\int_0^t\int_\Omega |\nabla(u_\varepsilon - u)|^{p(x)}\,dxds \leq 1$, from (56) we get

$$\|u_\varepsilon - u\|_{L^{p_-}\left(0,t;W_0^{1,p(.)}(\Omega)\right)} \leq \left(1 + T^{1-(p_-/p_+)}\right)^{\frac{1}{p_-}}; \qquad (60)$$

which implies that

$$\|u_t\|_{L^{(p_-)'}(0,T;V')}\|u_\varepsilon-u\|_{L^{p_-}\left(0,t;W_0^{1,p(.)}(\Omega)\right)}\leq\left(1+T^{1-(p_-/p_+)}\right)^{\frac{1}{p_-}}\|u_t\|_{L^{(p_-)'}(0,T;V')}. \tag{61}$$

Therefore, using (59) – (61), we deduce that

$$\begin{aligned}&\|u_t\|_{L^{(p_-)'}(0,T;V')}\|u_\varepsilon-u\|_{L^{p_-}\left(0,t;W_0^{1,p(.)}(\Omega)\right)}\\&\leq 2^{\frac{(p_-)'(p_++2)}{p_-}}\left(1+T^{1-(p_-/p_+)}\right)^{\frac{(p_-)'}{p_-}}\|u_t\|^{(p_-)'}_{L^{(p_-)'}(0,T;V')} \qquad (62)\\&+\left(1+T^{1-(p_-/p_+)}\right)^{\frac{1}{p_-}}\|u_t\|_{L^{(p_-)'}(0,T;V')}+\frac14\int_0^t\int_\Omega|\nabla u_\varepsilon|^{p(x)}\,dxds+\frac14\int_Q|\nabla u_\varepsilon|^{p(x)}\,dxds.\end{aligned}$$

Note also that, from the Young inequality, we have

$$\begin{aligned}&C\|u_t\|_{L^{(p_-)'}(0,T;V')}\|u-u_\varepsilon\|_{L^\infty(0,t;L^2(\Omega))}\\&\leq C\|u_t\|_{L^{(p_-)'}(0,T;V')}\|u\|_{L^\infty(0,t;L^2(\Omega))}+4C\|u_t\|_{L^{(p_-)'}(0,T;V')}+\frac14\|u_\varepsilon\|_{L^\infty(0,t;L^2(\Omega))}\\&\leq C\|u_t\|_{L^{(p_-)'}(0,T;V')}\|u\|_{L^\infty(0,t;L^2(\Omega))}+16C^2\|u_t\|^2_{L^{(p_-)'}(0,T;V')}+\frac1{16}\|u_\varepsilon\|^2_{L^\infty(0,t;L^2(\Omega))}\\&\leq C\|u_t\|_{L^{(p_-)'}(0,T;V')}\|u\|_{L^\infty(0,T;L^2(\Omega))}+16C^2\|u_t\|^2_{L^{(p_-)'}(0,T;V')}+\frac18\|u_\varepsilon\|^2_{L^\infty(0,t;L^2(\Omega))}.\end{aligned}$$

Consequently,

$$\begin{aligned}\left|\int_0^t\langle u_t,u_\varepsilon-u\rangle\,dt\right|\;\leq\;&2^{\frac{(p_-)'(p_++2)}{p_-}}\left(1+T^{1-(p_-/p_+)'}\right)^{\frac{(p_-)'}{p_-}}\|u_t\|^{(p_-)'}_{L^{(p_-)'}(0,T;V')}\\&+\left(1+T^{1-(p_-/p_+)'}\right)^{\frac{1}{p_-}}\|u_t\|_{L^{(p_-)'}(0,T;V')}+\frac14\int_0^t\int_\Omega|\nabla u_\varepsilon|^{p(x)}\,dxds\\&+\frac14\int_Q|\nabla u_\varepsilon|^{p(x)}\,dxds+C\|u_t\|_{L^{(p_-)'}(0,T;V')}\|u\|_{L^\infty(0,t;L^2(\Omega))}\\&+16C^2\|u_t\|^2_{L^{(p_-)'}(0,T;V')}+\frac18\|u_\varepsilon\|^2_{L^\infty(0,t;L^2(\Omega))}.\end{aligned} \tag{63}$$

Combining(54) and (63), we obtain

$$\begin{aligned}&\frac14\int_\Omega|u_\varepsilon|^2(t)\,dx-\frac18\|u_\varepsilon\|^2_{L^\infty(0,t;L^2(\Omega))}+\frac14\int_0^t\int_\Omega|\nabla u_\varepsilon|^{p(x)}\,dxdt\\&\leq C\left(\int_Q|\nabla u|^{p(x)}\,dxdt+\|u_t\|^2_{L^{(p_-)'}(0,T;V')}+\|u\|^2_{L^\infty(0,T;L^2(\Omega))}+\|u_t\|^{(p_-)'}_{L^{(p_-)'}(0,T;V')}\right.\\&\qquad\left.+\|u_t\|_{L^{(p_-)'}(0,T;V')}+\|u_t\|_{L^{(p_-)'}(0,T;V')}\|u\|_{L^\infty(0,T;L^2(\Omega))}\right),\end{aligned} \tag{64}$$

which implies that

$$\begin{aligned}&\|u_\varepsilon\|^2_{L^\infty(0,t;L^2(\Omega))}+\|u_\varepsilon\|^{p_-}_{L^{p_-}0,T;W_0^{1,p(.)}(\Omega)}\\&\leq C\left(\int_Q|\nabla u|^{p(x)}\,dxdt+\|u_t\|^2_{L^{(p_-)'}(0,T;V')}+\|u\|^2_{L^\infty(0,T;L^2(\Omega))}+\|u_t\|^{(p_-)'}_{L^{(p_-)'}(0,T;V')}\right.\\&\qquad\left.+\|u_t\|_{L^{(p_-)'}(0,T;V')}+\|u_t\|_{L^{(p_-)'}(0,T;V')}\|u\|_{L^\infty(0,T;L^2(\Omega))}\right).\end{aligned} \tag{65}$$

Now, we are in the case $ii)$ and we prove an L^∞ estimate on u_ε. Let G_k be defined on $\mathbb{R}$ by $G_k(r) = (r-k)^+$, where $k = \|u\|_{L^\infty(\Omega)}$. We take $G_k(u_\varepsilon) = (u_\varepsilon - k)^+$ as a test function in (53), and using the fact that $G'_k = (G'_k)^{p(.)}$, $u_\varepsilon \geq 0$, we obtain

$$\int_Q |\nabla G_k(u_\varepsilon)|^{p(x)} dxdt = \int_Q G'_k(u_\varepsilon) |\nabla u_\varepsilon|^{p(x)} dxdt \leq \int_Q \frac{1}{\varepsilon}(u_\varepsilon - u)^- G_k(u_\varepsilon) dxdt$$

and since $(u_\varepsilon - u) G_k(u_\varepsilon) = 0$ for $k = \|u\|_{L^\infty(Q)}$, then it follows that

$$\|u_\varepsilon\|_{L^\infty(Q)} \leq \|u\|_{L^\infty(Q)}.$$

Thus, writing $u_t = u_t^1 + u_t^2$, with $u_t^1 \in L^{(p')_-}\left(0,T; W^{-1,p'(.)}(\Omega)\right)$ and $u_t^2 \in L^1(Q)$ such that $\|u_t^1\|_{L^{(p')_-}\left(0,T;W^{-1,p'(.)}(\Omega)\right)} + \|u_t^2\|_{L^1(Q)} \leq 2\|u_t\|_{L^{(p_-)'}\left(0,T;W^{-1,p'(.)}(\Omega)\right)+L^1(Q)}$, one has

$$\begin{aligned}
\left|\int_0^t \langle u_t, u_\varepsilon - u\rangle\, ds\right| &\leq \int_0^t \|u_t^1\|_{W^{-1,p'(.)}(\Omega)} \|u - u_\varepsilon\|_{W_0^{1,p(.)}(\Omega)} dt + \|u_t^2\|_{L^1(Q)} \|u - u_\varepsilon\|_{L^\infty(Q)} \\
&\leq \|u_t^1\|_{L^{(p')_-}\left(0,t;W^{-1,p'(.)}(\Omega)\right)} \|u - u_\varepsilon\|_{L^{p_-}\left(0,t;W_0^{1,p(.)}(\Omega)\right)} + 2\|u_t^2\|_{L^1(Q)} \|u\|_{L^\infty(Q)} \\
&\leq \|u_t^1\|_{L^{(p')_-}\left(0,T;W^{-1,p'(.)}(\Omega)\right)} \|u - u_\varepsilon\|_{L^{p_-}\left(0,t;W_0^{1,p(.)}(\Omega)\right)} + 2\|u_t^2\|_{L^1(Q)} \|u\|_{L^\infty(Q)} \\
&\leq 2\|u_t\|_{L^{(p')_-}\left(0,T;W^{-1,p'(.)}(\Omega)\right)+L^1(Q)} \|u - u_\varepsilon\|_{L^{p_-}\left(0,t;W_0^{1,p(.)}(\Omega)\right)} \\
&\quad + 4\|u_t\|_{L^{(p')_-}\left(0,T;W^{-1,p'(.)}(\Omega)\right)+L^1(Q)} \|u\|_{L^\infty(Q)}. \qquad (66)
\end{aligned}$$

From (62), we get

$$\begin{aligned}
&2\|u_t\|_{L^{(p_-)'}\left(0,T;W^{-1,p'(.)}(\Omega)\right)+L^1(Q)} \|u_\varepsilon - u\|_{L^{p_-}\left(0,t;W_0^{1,p(.)}(\Omega)\right)} \\
&\leq 2^{\frac{(p_-)'(p_+ + 2)}{p_-}} \left(1 + T^{1-(p_-/p_+)}\right)^{\frac{(p_-)'}{p_-}} \|2u_t\|^{(p_-)'}_{L^{(p_-)'}\left(0,T;W^{-1,p'(.)}(\Omega)\right)+L^1(Q)} \\
&\quad + \left(1 + T^{1-(p_-/p_+)}\right)^{\frac{1}{p_-}} \|2u_t\|_{L^{(p_-)'}\left(0,T;W^{-1,p'(.)}(\Omega)\right)+L^1(Q)} \\
&\quad + \frac{1}{4}\int_0^t \int_\Omega |\nabla u_\varepsilon|^{p(x)} dxds + \frac{1}{4}\int_Q |\nabla u_\varepsilon|^{p(x)} dxds. \qquad (67)
\end{aligned}$$

Therefore, using (54) and (66)-(67), we obtain

$$\begin{aligned}
&\frac{1}{4}\int_\Omega |u_\varepsilon|^2(t)\, dx + \frac{1}{4}\int_0^t \int_\Omega |\nabla u_\varepsilon|^{p(x)} dxdt \qquad (68) \\
&\leq C\Bigg(\int_Q |\nabla u|^{p(x)} dxdt + \|u\|^2_{L^\infty(0,T;L^2(\Omega))} + \|u_t\|^{(p_-)'}_{L^{(p_-)'}\left(0,T;W^{-1,p'(.)}(\Omega)\right)+L^1(Q)} \\
&\quad + \|u_t\|_{L^{(p_-)'}\left(0,T;W^{-1,p'(.)}(\Omega)\right)+L^1(Q)} + \|u_t\|_{L^{(p_-)'}\left(0,T;W^{-1,p'(.)}(\Omega)\right)+L^1(Q)} \|u\|_{L^\infty(Q)}\Bigg),
\end{aligned}$$

which implies that

$$\begin{aligned}
&\|u_\varepsilon\|^2_{L^\infty(0,T;L^2(\Omega))} + \|u_\varepsilon\|^{p_-}_{L^{p_-}\left(0,T;W_0^{1,p(.)}(\Omega)\right)} \qquad (69)\\
&\quad \le C\left(\int_Q |\nabla u|^{p(x)}\,dxdt + \|u\|^2_{L^\infty(0,T;L^2(\Omega))} + \|u_t\|^{(p_-)'}_{L^{(p_-)'}\left(0,T;W^{-1,p'(.)}(\Omega)\right)+L^1(Q)}\right.\\
&\quad \left. + \|u_t\|_{L^{(p_-)'}\left(0,T;W^{-1,p'(.)}(\Omega)\right)+L^1(Q)} + \|u_t\|_{L^{(p_-)'}\left(0,T;W^{-1,p'(.)}(\Omega)\right)+L^1(Q)}\|u\|_{L^\infty(Q)}\right).
\end{aligned}$$

Using estimates (65) or (68), we deduce that the sequence (u_ε) is bounded in $L^\infty(0,T;L^2(\Omega))$ and in $L^{p_-}\left(0,T;W_0^{1,p(.)}(\Omega)\right)$. This implies the existence of a subsequence of (u_ε) converging to an element w weakly in $L^{p_-}\left(0,T;W_0^{1,p(.)}(\Omega)\right)$ and weakly$-*$ in $L^\infty\left(0,T;L^2(\Omega)\right)$. As in [6], ones shows that if $\varepsilon<\eta$ then, $u_\varepsilon \ge u_\eta$. Therefore, we conclude that $(u_\varepsilon)_{\varepsilon>0}$ is a nonnegative decreasing bounded sequence in $L^1(Q)$. Consequently, from the monotone convergence theorem, u_ε converges to w in $L^1(Q)$ and almost everywhere in Q.
Taking $(u_\varepsilon-u)^-$ as a test function in (53), we obtain

$$\int_0^T\left\langle (u_\varepsilon)_t,(u_\varepsilon-u)^-\right\rangle dt+\int_0^T|\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon.\nabla(u_\varepsilon-u)^-dxdt=\frac{1}{\varepsilon}\int_Q\left|(u_\varepsilon-u)^-\right|^2dxdt,$$

which implies that

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_Q\left|(u_\varepsilon-u)^-\right|^2dxdt+\frac{1}{2}\int_Q\left|(u_\varepsilon-u)^-\right|^2(T)\,dx\\
&\qquad=\int_0^T\left\langle (u_\varepsilon)_t,(u_\varepsilon-u)^-\right\rangle dt+\int_0^T|\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon.\nabla(u_\varepsilon-u)^-\,dxdt.
\end{aligned}$$

Hence, by (65) in case $i)$ or (68) and L^∞-estimates in case $ii)$, we deduce that

$$\frac{1}{\varepsilon}\int_Q\left|(u_\varepsilon-u)^-\right|^2dxdt\le M,$$

which implies, by Fatou's lemma that $w\ge u$ and $w\ge u^+$ since $w\ge 0$.
Step 2: In this step, one gives some estimates in $\widetilde{W}_{p(.)}(0,T)$. Thanks to [10], there exists a unique variational solution $z^\varepsilon\in L^\infty\left(0,T;L^2(\Omega)\right)\cap L^{p_-}\left(0,T;W_0^{1,p(.)}(\Omega)\right)$ of the problem

$$\begin{cases} -z^\varepsilon_t-\Delta_{p(.)}z^\varepsilon=-2\Delta_{p(.)}u_\varepsilon & \text{in } (0,T)\times\Omega\\ z^\varepsilon(T)=u_\varepsilon(T) & \text{on } \Omega\\ z^\varepsilon=0 & \text{on } (0,T)\times\partial\Omega.\end{cases} \qquad (70)$$

Note that $-2\Delta_{p(.)}u_\varepsilon\ge(u_\varepsilon)_t-\Delta_{p(.)}u_\varepsilon$ in the distributional sense, which implies that $z^\varepsilon\ge u_\varepsilon$.
Taking z^ε as a test function in (70) and integrating between t and T and using the Young inequality, we obtain

$$\int_\Omega(z^\varepsilon(t))^2dx+\frac{1}{2}\int_Q|\nabla z^\varepsilon|^{p(x)}dx\le\frac{1}{2}\int_\Omega|u_\varepsilon|^2dx+\frac{4^{(p')_+}}{(p')_-}\int_Q|\nabla u_\varepsilon|^{p(x)}dxdt,$$

which implies that

$$\|z^{\varepsilon}\|^2_{L^{\infty}(0,T;L^2(\Omega))}+\|z^{\varepsilon}\|^{p_-}_{L^{p_-}\left(0,T;W_0^{1,p(.)}(\Omega)\right)}\leq C\Big(\int_Q|\nabla u_{\varepsilon}|^{p(x)}dxdt+\|u_{\varepsilon}\|^2_{L^{\infty}(0,T;L^2(\Omega))}\Big). \tag{71}$$

By Proposition 2.1, we have

$$\|\nabla z^{\varepsilon}\|_{(L^{p(.)}(Q))^N}\leq \max\left\{\left(\int_Q|\nabla z^{\varepsilon}|^{p(x)}\,dx\right)^{\frac{1}{p_-}},\left(\int_Q|\nabla z^{\varepsilon}|^{p(x)}\,dx\right)^{\frac{1}{p_+}}\right\}. \tag{72}$$

Hence, using (71) if we are in the case (i) i.e $u\in W_{p(.)}(0,T)$, then we deduce from $(64)-(65)$ the following estimate.

$$\begin{aligned}
&\|z^{\varepsilon}\|^2_{L^{\infty}(0,T;L^2(\Omega))}+\|z^{\varepsilon}\|^{p_-}_{L^{p_-}\left(0,T;W_0^{1,p(.)}(\Omega)\right)}\leq C\left(\int_Q|\nabla u|^{p(x)}\,dxdt\right.\\
&\quad+\|u_t\|^2_{L^{(p_-)'}(0,T;V')}+\|u\|^2_{L^{\infty}(0,t;L^2(\Omega))}+\|u_t\|^{(p_-)'}_{L^{(p_-)'}(0,T;V')}+\|u_t\|_{L^{(p_-)'}(0,T;V')}\\
&\quad\left.+\|u_t\|_{L^{(p_-)'}(0,T;V')}\|u\|_{L^{\infty}(0,t;L^2(\Omega))}\right)
\end{aligned} \tag{73}$$

and in the case (ii), we get from (69) the following estimate.

$$\begin{aligned}
&\|z^{\varepsilon}\|^2_{L^{\infty}(0,T;L^2(\Omega))}+\|z^{\varepsilon}\|^{p_-}_{L^{p_-}\left(0,T;W_0^{1,p(.)}(\Omega)\right)}\leq C\left(\int_Q|\nabla u|^{p(x)}\,dxdt+\|u\|^2_{L^{\infty}(0,T;L^2(\Omega))}\right.\\
&\quad+\|u_t\|^{(p_-)'}_{L^{(p_-)'}\left(0,T;W^{-1,p'(.)}(\Omega)\right)+L^1(Q)}+\|u_t\|_{L^{(p_-)'}\left(0,T;W^{-1,p'(.)}(\Omega)\right)+L^1(Q)}\\
&\quad\left.+\|u_t\|_{L^{(p_-)'}\left(0,T;W^{-1,p'(.)}(\Omega)\right)+L^1(Q)}\|u\|_{L^{\infty}(Q)}\right).
\end{aligned} \tag{74}$$

For reasons of simplicity one puts

$$\begin{aligned}
[u]_*=&\int_Q|\nabla u|^{p(x)}\,dxdt+\|u_t\|^2_{L^{(p_-)'}(0,T;V')}+\|u\|^2_{L^{\infty}(0,t;L^2(\Omega))}+\|u_t\|^{(p_-)'}_{L^{(p_-)'}(0,T;V')}\\
&+\|u_t\|_{L^{(p_-)'}(0,T;V')}+\|u_t\|_{L^{(p_-)'}(0,T;V')}\|u\|_{L^{\infty}(0,t;L^2(\Omega))}
\end{aligned} \tag{75}$$

and

$$\begin{aligned}
[u]_{**}=&\int_Q|\nabla u|^{p(x)}\,dxdt+\|u\|^2_{L^{\infty}(0,T;L^2(\Omega))}+\|u_t\|^{(p_-)'}_{L^{(p_-)'}\left(0,T;W^{-1,p'(.)}(\Omega)\right)+L^1(Q)}\\
&+\|u_t\|_{L^{(p_-)'}\left(0,T;W^{-1,p'(.)}(\Omega)\right)+L^1(Q)}+\|u_t\|_{L^{(p_-)'}\left(0,T;W^{-1,p'(.)}(\Omega)\right)+L^1(Q)}\|u\|_{L^{\infty}(Q)}.
\end{aligned} \tag{76}$$

We take $v\in L^{p_-}(0,T;V)$ as a test function in (70), to obtain

$$\begin{aligned}
\left|\int_0^T\langle(z^{\varepsilon})_t\,,v\rangle\,dt\right|&\leq\left|\int_Q|\nabla z^{\varepsilon}|^{p(x)-2}\nabla z^{\varepsilon}.\nabla vdxdt\right|+\left|2\int_Q|\nabla u_{\varepsilon}|^{p(x-2)}\nabla u_{\varepsilon}.\nabla vdxdt\right|\\
&\leq 2\int_0^T\left\||\nabla z^{\varepsilon}|^{p(x)-1}\right\|_{p'(.)}\|\nabla v\|_{p(.)}\,dt+4\int_0^T\left\||\nabla u_{\varepsilon}|^{p(x)-1}\right\|_{p'(.)}\|\nabla v\|_{p(.)}\,dt\\
&\leq 4\int_0^T\left(\left\||\nabla z^{\varepsilon}|^{p(x)-1}\right\|_{p'(.)}+\left\||\nabla u_{\varepsilon}|^{p(x)-1}\right\|_{p'(.)}\right)\|v\|_V\,dt.
\end{aligned} \tag{77}$$

Therefore, by the same method as in the proof of (38), it follows that

$$\begin{aligned}
&\|(z^\varepsilon)_t\|_{L^{(p_-)'}(t_0,T;V')} \\
&\leq 4T^{1-\frac{1}{(p')_-}}\left(\int_0^T\int_\Omega |\nabla z^\varepsilon|^{p(x)}dxdt + T^{1-\frac{(p')_-}{(p')_+}}\left(\int_0^T\int_\Omega |\nabla z^\varepsilon|^{p(x)}dxdt\right)^{\frac{(p')_-}{(p')_+}}\right)^{\frac{1}{(p')_-}} \qquad (78)\\
&\quad +4T^{1-\frac{1}{(p')_-}}\left(\int_0^T\int_\Omega |\nabla u_\varepsilon|^{p(x)}dxdt + T^{1-\frac{(p')_-}{(p')_+}}\left(\int_0^T\int_\Omega |\nabla u_\varepsilon|^{p(x)}dxdt\right)^{\frac{(p')_-}{(p')_+}}\right)^{\frac{1}{(p')_-}}.
\end{aligned}$$

We can rewrite (78) as follow.

$$\begin{aligned}
&\|(z^\varepsilon)_t\|_{L^{(p_-)'}(t_0,T;V')} \\
&\leq C\left(\left(\int_0^T\int_\Omega |\nabla z^\varepsilon|^{p(x)}\,dxdt\right)^{\frac{1}{(p')_-}} + \left(\int_0^T\int_\Omega |\nabla z^\varepsilon|^{p(x)}\,dxdt\right)^{\frac{1}{(p')_+}}\right.\\
&\quad \left. + \left(\int_0^T\int_\Omega |\nabla u_\varepsilon|^{p(x)}\,dxdt\right)^{\frac{1}{(p')_-}} + \left(\int_0^T\int_\Omega |\nabla u_\varepsilon|^{p(x)}\,dxdt\right)^{\frac{1}{(p')_+}}\right). \qquad (79)
\end{aligned}$$

Finally, in the case (i), i.e $u \in W_{p(.)}(0,T)$, we deduce that

$$\|(z^\varepsilon)_t\|_{L^{(p_-)'}(t_0,T;V')} \leq C\left([u]_*^{\frac{1}{(p')_-}} + [u]_*^{\frac{1}{(p')_+}}\right). \qquad (80)$$

Hence combining (72) – (73) and (80), we obtain

$$\|z^\varepsilon\|_{W_{p(.)}(0,T)} \leq C\left([u]_*^{\frac{1}{2}} + [u]_*^{\frac{1}{p_-}} + [u]_*^{\frac{1}{p_+}} + [u]_*^{\frac{1}{(p')_-}} + [u]_*^{\frac{1}{(p')_+}}\right) \qquad (81)$$

and since $\widetilde{W}_{p(.)}(0,T) \hookrightarrow W_{p(.)}(0,T)$, then

$$\|z^\varepsilon\|_{\widetilde{W}_{p(.)}(0,T)} \leq C\left([u]_*^{\frac{1}{2}} + [u]_*^{\frac{1}{p_-}} + [u]_*^{\frac{1}{p_+}} + [u]_*^{\frac{1}{(p')_-}} + [u]_*^{\frac{1}{(p')_+}}\right). \qquad (82)$$

For the second case, i.e $u \in L^{p_-}\left(0,T;W_0^{1,p(.)}(\Omega)\right) \cap L^\infty(Q)$, we have

$$\|(z^\varepsilon)_t\|_{L^{(p_-)'}(t_0,T;V')} \leq C\left([u]_{**}^{\frac{1}{(p')_-}} + [u]_{**}^{\frac{1}{(p')_+}}\right). \qquad (83)$$

Then, from (72), (74) and (83), it follows that

$$\begin{aligned}
[z^\varepsilon] &= \|z^\varepsilon\|^2_{L^\infty(0,T;L^2(\Omega))} + \|z^\varepsilon\|^{p_-}_{L^{p_-}\left(0,T;W_0^{1,p(.)}(\Omega)\right)} + \|(z^\varepsilon)\|_{L^{p_-}(0,T;V')} + \|\nabla z^\varepsilon\|_{L^{p(.)}(Q)}\\
&\leq C\left([u]_{**} + [u]_{**}^{\frac{1}{p_-}} + [u]^{\frac{1}{p_+}}\,[u]_{**}^{\frac{1}{(p')_-}} + [u]^{\frac{1}{(p')_+}}\right). \qquad (84)
\end{aligned}$$

According to (81), z^ε is bounded in $\widetilde{W}_{p(.)}(0,T)$. Hence, there exists a subsequence, still denoted by z^ε such that z^ε converges weakly to z in $L^{p_-}\left(0,T;W_0^{1,p(.)}(\Omega)\right)$

and weakly$*$ in $L^{\infty}\left(0,T;L^{2}\left(\Omega\right)\right)$, ∇z^{ε} converges weakly to ξ in $\left(L^{p'(.)}\left(Q\right)\right)^{N}$ and z_{t}^{ε} converges to $\bar{z}$ in $L^{(p_{-})'}\left(0,T;W^{-1,p'(.)}\left(\Omega\right)\right)$. Then, it follows that $z_{t}=\bar{z}$ and $\xi=\nabla z$. Therefore, $z\in\widetilde{W}_{p(.)}\left(0,T\right)$. Hence, from [16], we deduce that z^{ε} is compact in $L^{1}\left(Q\right)$. Consequently, $z^{\varepsilon}\to z$ a.e. in Q. Moreover, we have $z^{\varepsilon}\geq u_{\varepsilon}$. Then, letting $\varepsilon\to 0$, we get

$$z\geq w\geq u^{+}\ \text{ a.e. in }\ Q. \tag{85}$$

Therefore, if $u\in W_{p(.)}\left(0,T\right)$, we deduce from (81) that

$$\|z^{\varepsilon}\|_{\widetilde{W}_{p(.)}(0,T)}\leq C\left([u]_{*}^{\frac{1}{2}}+[u]_{*}^{\frac{1}{p_{-}}}+[u]_{*}^{\frac{1}{p_{+}}}+[u]_{*}^{\frac{1}{(p')_{-}}}+[u]_{*}^{\frac{1}{(p')_{+}}}\right) \tag{86}$$

which implies that

$$\|z\|_{W_{p(.)}(0,T)}\leq C\left([u]_{*}^{\frac{1}{2}}+[u]_{*}^{\frac{1}{p_{-}}}+[u]_{*}^{\frac{1}{p_{+}}}+[u]_{*}^{\frac{1}{(p')_{-}}}+[u]_{*}^{\frac{1}{(p')_{+}}}\right) \tag{87}$$

and if $u\in L^{p_{-}}\left(0,T;W_{0}^{1,p(.)}\left(\Omega\right)\right)\cap L^{\infty}\left(Q\right)$, $u_{t}\in L^{p_{-}}\left(0,T;W^{-1,p'(.)}\left(\Omega\right)\right)+L^{1}\left(Q\right)$, we deduce from (84) that

$$[z]\leq C\left([u]_{**}+[u]_{**}^{\frac{1}{p_{-}}}+[u]_{**}^{\frac{1}{p_{+}}}+[u]_{**}^{\frac{1}{(p')_{-}}}+[u]_{**}^{\frac{1}{(p')_{+}}}\right). \tag{88}$$

Since we can obtain a similar result for the negative part u^{-}, we end the proof of the lemma by writing $u=u^{+}+u^{-}\square$

As a consequence of the Lemma 3.7 we have the following.

Corollary 3.8. *For all $u\in W_{p(.)}\left(0,T\right)$,*

$$[u]_{*}\leq C\max\left\{\|u\|_{W_{p(.)}(0,T)}^{p_{-}},\|u\|_{W_{p(.)}(0,T)}^{(p_{-})'}\right\}. \tag{89}$$

Moreover, there exists $z\in\widetilde{W}_{p(.)}\left(0,T\right)$ such that $|u|\leq z$ and

$$\|z\|_{\widetilde{W}_{p(.)}(0,T)}\leq C\max\left\{\|u\|_{W_{p(.)}(0,T)}^{\frac{p_{-}}{(p')_{-}}},\|u\|_{W_{p(.)}(0,T)}^{\frac{(p_{-})'}{p_{-}}}\right\}. \tag{90}$$

Proof. Let's recall that

$$\begin{aligned}[u]_{*}=&\rho_{p(.)}\left(\nabla u\right)+\|u_{t}\|_{L^{(p_{-})'}(0,T;V')}^{2}+\|u\|_{L^{\infty}(0,t;L^{2}(\Omega))}^{2}+\|u_{t}\|_{L^{(p_{-})'}(0,T;V')}^{(p_{-})'}\\&+\|u_{t}\|_{L^{(p_{-})'}(0,T;V')}+\|u_{t}\|_{L^{(p_{-})'}(0,T;V')}\|u\|_{L^{\infty}(0,t;L^{2}(\Omega))}.\end{aligned}$$

We have

$$\|u_{t}\|_{L^{(p_{-})'}(0,T;V')}\leq\|u\|_{W_{p(.)}(0,T)} \tag{91}$$

and by Proposition 2.1, we deduce that

$$\begin{aligned}\rho_{p(.)}\left(\nabla u\right)&\leq\max\left\{\|\nabla u\|_{L^{p(.)}(Q)}^{p_{-}},\|\nabla u\|_{L^{p(.)}(Q)}^{p_{+}}\right\}\\&\leq\|u\|_{W_{p(.)}(0,T)}^{p_{-}}+\|u\|_{W_{p(.)}(0,T)}^{p_{+}}.\end{aligned} \tag{92}$$

Using Proposition 3.2, we get

$$\|u\|_{L^{\infty}(0,T;L^{2}(\Omega))}\leq C\|u\|_{W_{p(.)}(0,T)}. \tag{93}$$

Hence, by (91)-(93), we obtain

$$[u]_* \leq C\left(\|u\|_{W_{p(.)}(0,T)} + \|u\|^2_{W_{p(.)}(0,T)} + \|u\|^{p_-}_{W_{p(.)}(0,T)} + \|u\|^{p_+}_{W_{p(.)}(0,T)} + \|u\|^{(p_-)'}_{W_{p(.)}(0,T)}\right)$$
$$\leq C\max\left\{\|u\|^{p_-}_{W_{p(.)}(0,T)}, \|u\|^{(p_-)'}_{W_{p(.)}(0,T)}\right\}. \tag{94}$$

Thanks to Lemma 3.7, there exists $z \in \widetilde{W}_{p(.)}(0,T)$ such that $|u| \leq z$ and

$$\|z\|_{W_{p(.)}(0,T)} \leq C\left([u]_*^{\frac{1}{2}} + [u]_*^{\frac{1}{p_-}} + [u]_*^{\frac{1}{p_+}} + [u]_*^{\frac{1}{(p')_-}} + [u]_*^{\frac{1}{(p')_+}}\right),$$

which implies that

$$\|z\|_{W_{p(.)}(0,T)} \leq C\max\left([u]_*^{\frac{1}{p_-}}, [u]_*^{\frac{1}{(p')_-}}\right).$$

Therefore, from (94) we obtain

$$\|z\|_{\widetilde{W}_{p(.)}(0,T)} \leq C\max\left\{\|u\|^{\frac{p_-}{(p')_-}}_{W_{p(.)}(0,T)}, \|u\|^{\frac{(p_-)'}{p_-}}_{W_{p(.)}(0,T)}\right\}. \tag{95}$$

Then, we can prove the following result which gives the connection between the notions of capacity and continuity.

Proposition 3.9. *If u is cap-quasi continuous and belongs to $W_{p(.)}(0,T)$, then for all $t > 0$,*

$$cap_{p(.)}(\{|u| > t\}) \leq \frac{C}{t} max\left\{\|u\|^{\frac{p_-}{(p')_-}}_{W_{p(.)}(0,T)}, \|u\|^{\frac{(p_-)'}{p_-}}_{W_{p(.)}(0,T)}\right\}. \tag{96}$$

Proof. We consider in the first step, the case where u belongs to $C_c([0,T]\times\Omega)$, this step is motivated by the fact that $C_c([0,T]\times\Omega)$ is dense in $W_{p(.)}(0,T)$. Thanks to Corollary 3.8, there exists $z \in \widetilde{W}_{p(.)}(0,T)$ such that $|u| \leq z$ holds true; then, since $\widetilde{W}_{p(.)}(0,T)$ is continuously embedding in $W_{p(.)}(0,T)$ and $\frac{z}{t} \geq 1$ on the set $\{|u| > t\}$, we have

$$cap_{p(.)}(\{|u| > t\}) \leq \left\|\frac{z}{t}\right\|_{W_{p(.)}(0,T)} \leq \frac{C}{t}\max\left\{\|u\|^{\frac{p_-}{(p')_-}}_{W_{p(.)}(0,T)}, \|u\|^{\frac{(p_-)'}{p_-}}_{W_{p(.)}(0,T)}\right\}.$$

For the second step, we suppose only that $u \in W_{p(.)}(0,T)$ and is $cap_{p(.)}$–quasi continuous. Let $\varepsilon > 0$ be fixed, then there exists an open set A_ε such that $cap_{p(.)}(A_\varepsilon) < \varepsilon$ and $u_{|(Q\setminus A_\varepsilon)}$ is continuous, which implies that $\{u_{|(Q\setminus A_\varepsilon)} > t\} \cap (Q \setminus A_\varepsilon)$ is an open set in $Q \setminus A_\varepsilon$. Then, there exists an open set $U \subset \mathbb{R}^N$ such that $\{u_{|(Q\setminus A_\varepsilon)} > t\} \cap (Q \setminus A_\varepsilon) = U \cap (Q \setminus A_\varepsilon)$. Consequently,

$$\{|u| > t\} \cup A_\varepsilon = \left(\{u_{|(Q\setminus A_\varepsilon)} > t\} \cap (Q \setminus A_\varepsilon)\right) \cup A_\varepsilon = (U \cup A_\varepsilon) \cap Q$$

is an open set.

Now, we consider the function z given by Corollary 3.8. Let $w \in W_{p(.)}(0,T)$ be such that $w \geq \chi_{A_\varepsilon}$ and $\|w\|_{W_{p(.)}(0,T)} \leq cap_{p(.)}(A_\varepsilon) + \varepsilon < 2\varepsilon$. Since $w + \frac{z}{t} \geq 1$ a.e. in $\{|u| > t\} \cup A_\varepsilon$, we have

$$\begin{aligned} cap_{p(.)}(\{|u|>t\}) &\leq cap_{p(.)}(\{|u|>t\} \cup A_\varepsilon) \leq \left\| w + \frac{z}{t} \right\|_{W_{p(.)}(0,T)} \\ &\leq \|w\|_{W_{p(.)}(0,T)} + \frac{1}{t}\|z\|_{W_{p(.)}(0,T)} \leq 2\varepsilon + \frac{1}{t}\|z\|_{W_{p(.)}(0,T)}. \quad (97) \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, then, we deduce that

$$cap_{p(.)}(\{|u|>t\}) \leq \frac{C}{t}\max\left\{ \|u\|_{W_{p(.)}(0,T)}^{\frac{p_-}{(p')_-}}, \|u\|_{W_{p(.)}(0,T)}^{\frac{(p_-)'}{p_-}} \right\} \square$$

As in elliptic case, we have the following result of quasicontinuity.

Lemma 3.10. *Any element v of $W_{p(.)}(0,T)$ has a cap-quasi continuous representative $\tilde{v}$ which is cap-quasi everywhere unique, in the sense that two cap-quasi continuous representatives of v are equal except on a set of null capacity.*

Proof. We adapt the proof given in [6]. Since $C_c([0,T] \times \Omega)$ is dense in $W_{p(.)}(0,T)$, there exists a sequence $(v^m) \subset C_c([0,T] \times \Omega)$ such that v^m converges to v in $W_{p(.)}(0,T)$, as $m \to \infty$. Moreover, we have

$$\sum_{m=1}^{\infty} 2^m \max\left\{ \left\|v^{m+1} - v^m\right\|_{W_{p(.)}(0,T)}^{\frac{p_-}{(p')_-}}, \left\|v^{m+1} - v^m\right\|_{W_{p(.)}(0,T)}^{\frac{(p_-)'}{p_-}} \right\} < \infty.$$

We introduce the following subsets

$$\omega^m = \left\{ \left|v^{m+1} - v^m\right| > 2^{-m} \right\}, \quad \Omega^r = \bigcup_{m \geq r} \omega^m.$$

Using the fact that $v^{m+1} - v^m$ is continuous and belongs to $W_{p(.)}(0,T)$, we apply Proposition 3.9 to obtain

$$cap_{p(.)}(\omega^m) \leq C 2^m \max\left\{ \left\|v^{m+1} - v^m\right\|_{W_{p(.)}(0,T)}^{\frac{p_-}{(p')_-}}, \left\|v^{m+1} - v^m\right\|_{W_{p(.)}(0,T)}^{\frac{(p_-)'}{p_-}} \right\}.$$

By subadditivity, we get

$$cap_{p(.)}(\Omega^r) \leq C \sum_{m \geq r} 2^m \max\left\{ \left\|v^{m+1} - v^m\right\|_{W_{p(.)}(0,T)}^{\frac{p_-}{(p')_-}}, \left\|v^{m+1} - v^m\right\|_{W_{p(.)}(0,T)}^{\frac{(p_-)'}{p_-}} \right\},$$

which implies that

$$\lim_{r \to \infty} cap_{p(.)}(\Omega^r) = 0.$$

For any r, we have

$$\text{if} \quad (x,t) \notin \Omega^r, \quad \text{then} \quad \forall m \geq r, \left|v^{m+1} - v^m\right|(z) \leq 2^{-m}.$$

Hence, v^m converges uniformly on the complement of each Ω^r and pointwise in the complement of $\bigcap_r^\infty \Omega_r$.

Moreover,

$$cap_{p(.)}\left(\bigcap_r^\infty \Omega^r \right) \leq cap_{p(.)}(\Omega^r) \to 0 \quad \text{as} \quad r \quad \text{tends to infinity},$$

which prove that $cap_{p(.)}\left(\bigcap_{r}^{\infty}\Omega^r\right)=0.$

Therefore, the limit of v^m is defined cap-quasi everywhere and is cap-quasi continuous. Let us call $\tilde{v}$ this cap-quasi continuous representative of v and assume that there exists another representative z of v which is cap-quasi continuous and coincides with v almost everywhere in Q. Then we have, thanks to Proposition 3.9,

$$cap_{p(.)}\left(\left\{|z-\tilde{v}|>\frac{1}{k}\right\}\right)\leq Ck\max\left\{\|z-\tilde{v}\|_{W_{p(.)}(0,T)}^{\frac{p_-}{(p')_-}},\|z-\tilde{v}\|_{W_{p(.)}(0,T)}^{\frac{(p_-)'}{p_-}}\right\},$$

since $\tilde{v}=z$ in $W_{p(.)}(0,T)$. This being true for any k, we obtain that $\tilde{v}=z$ cap-quasi everywhere, so that the cap-quasi continuous representative of v is unique up to sets of zero capacity □

In what follows, we need the following results.

Lemma 3.11. *Let $(v_n)_{n\in\mathbb{N}}$ be a sequence in $W_{p(.)}(0,T)$ which converges to v in $W_{p(.)}(0,T)$, then there exists a subsequence $(\tilde{v}_{n_k})_{k\in\mathbb{N}}$ of $(v_n)_{n\in\mathbb{N}}$ which converges to $\tilde{v}$ cap-quasi everywhere.*

Proof. According to Proposition 3.9 and Lemma 3.10, the proof is similar to the proof of Lemma 2.2.1 in [6] □

4. Measures

In this part, we establish the relation between measures in Q and the notion of $p(.)$-parabolic capacity. We extend the results obtained in the case of constant exponent (see [6]) to the case of variable exponent. In the rest of the paper we denote by $\mathcal{M}_b(Q)$ the space of bounded measure in Q and $\mathcal{M}_b^+(Q)$ the subsets of nonnegative measures of $\mathcal{M}_b(Q)$. The duality between $\left(W_{p(.)}(0,T)\right)'$ and $W_{p(.)}(0,T)$ is denoted by $\langle\langle .,.\rangle\rangle$, $\left(W_{p(.)}(0,T)\right)'\cap\mathcal{M}_b(Q)$ is the set of element $\gamma\in\left(W_{p(.)}(0,T)\right)'$ such that there exists $c>0$ satisfying, for all $\varphi\in\mathcal{C}_c^\infty(Q)$, $|\langle\langle\gamma,\varphi\rangle\rangle|\leq c\|\varphi\|_{L^\infty(Q)}$. Every $\gamma\in\left(W_{p(.)}(0,T)\right)'\cap\mathcal{M}_b(Q)$ is identified by unique linear application $\varphi\in\mathcal{C}_c^\infty(Q)\mapsto\int_Q\varphi d\gamma^{meas}$ where γ^{meas} belongs to $\mathcal{M}_b(Q)$. The set of $\gamma\in\left(W_{p(.)}(0,T)\right)'\cap\mathcal{M}_b(Q)$ such that $\gamma^{meas}\in\mathcal{M}_b^+(Q)$ is denoted by $\left(W_{p(.)}(0,T)\right)'\cap\mathcal{M}_b^+(Q)$.

Definition 4.1. We define

$$\mathcal{M}_0(Q)=\left\{\mu\in\mathcal{M}_b(Q):\mu(E)=0\quad\text{for every}\quad E\subset Q\quad\text{such that}\quad cap_{p(.)}(E)=0\right\}.$$

The nonnegative measures in $\mathcal{M}_0(Q)$ will be said to belongs to $\mathcal{M}_0^+(Q)$.

Proposition 4.1. *Let μ belongs to $\mathcal{M}_0^+(Q)$. Then, there exists $\gamma\in\left(W_{p(.)}(0,T)\right)'\cap\mathcal{M}_b^+(Q)$ and a nonnegative function $f\in L^1(Q,\gamma^{meas})$ such that $\mu=f\gamma^{meas}$.*

Proof. Let $u\in W_{p(.)}(0,T)$. Since by Lemma 3.7, u admits a cap-quasi continuous representative denoted $\tilde{u}$ which is cap-quasi everywhere unique, then we can define the following functional $F:W_{p(.)}(0,T)\to\mathbb{R}$ by $F(u)=\int_Q\max\{\tilde{u},0\}\,d\mu$.

The function F is convex and lower semicontinuous on $W_{p(.)}(0,T)$ (the lower semicontinuity follows from Fatou's Lemma and Lemma 3.11). Since $W_{p(.)}(0,T)$ is separable, the function F is the supremum of a countable family of continuous affine functions. Hence, there exists a sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $\left(W_{p(.)}(0,T)\right)'$ and a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that $F(u)=\sup_{n\in\mathbb{N}}\{\langle\langle\lambda_n,u\rangle\rangle+a_n\}$.
We have $F(0)=0$, which implies that $a_n\leq 0$. Then, it follows that

$$F(u)\leq \sup_{n\in\mathbb{N}}\{\langle\langle\lambda_n,u\rangle\rangle\}. \tag{98}$$

Since for every $t>0$ and for every $u\in W_{p(.)}(0,T)$, we have

$$t\langle\langle\lambda_n,u\rangle\rangle+a_n\leq F(tu)=tF(u) \tag{99}$$

then, we get $\langle\langle\lambda_n,u\rangle\rangle\leq F(u)$; hence, by (98) we deduce that

$$F(u)=\sup_{n\in\mathbb{N}}\{\langle\langle\lambda_n,u\rangle\rangle\}. \tag{100}$$

Now, we are going to show that λ_n belongs to $\left(W_{p(.)}(0,T)\right)'$. Using (100) and the definition of F, we obtain

$$\langle\langle\lambda_n,\varphi\rangle\rangle\leq\int_Q \max\{\varphi,0\}\,d\mu\leq\|\mu\|_{\mathcal{M}_b(Q)}\|\varphi\|_{L^\infty(Q)}, \tag{101}$$

for all $\varphi\in C_c^\infty(Q)$. Since the inequality (101) remains true for $-\varphi$, we deduce that $|\langle\langle\lambda_n,\varphi\rangle\rangle|\leq\|\mu\|_{\mathcal{M}_b(Q)}\|\varphi\|_{L^\infty(Q)}$, hence $\lambda_n\in\left(W_{p(.)}(0,T)\right)'\cap\mathcal{M}_b(Q)$.
For all $\varphi\in C_c^\infty(Q)$ such that $\varphi\geq 0$, we have

$$-\langle\langle\lambda_n,\varphi\rangle\rangle=\langle\langle\lambda_n,-\varphi\rangle\rangle\leq F(-\varphi)=0$$

which implies that

$$0\leq\langle\langle\lambda_n,\varphi\rangle\rangle=\int_Q\varphi d\lambda_n^{meas}.$$

Then, it follows that λ_n^{meas} belongs to $\mathcal{M}_b^+(Q)$, that is equivalent to say that $\lambda_n\in\left(W_{p(.)}(0,T)\right)'\cap\mathcal{M}_b^+(Q)$. By (100), for any nonnegative $\varphi\in C_c^\infty(Q)$ we have

$$\int_Q\varphi d\lambda_n^{meas}=\langle\langle\lambda_n,\varphi\rangle\rangle\leq\int_Q\varphi d\mu,$$

then

$$\lambda_n^{meas}\leq\mu, \tag{102}$$

moreover, we can write $\|\lambda_n^{meas}\|_{\mathcal{M}_b(Q)}\leq\|\mu\|_{\mathcal{M}_b(Q)}$.
We define $\gamma\in\left(W_{p(.)}(0,T)\right)'$ by

$$\gamma=\sum_{n=1}^{\infty}\frac{\lambda_n}{2^n(\|\lambda_n\|_{\left(W_{p(.)}(0,T)\right)'}+1)}. \tag{103}$$

The serie γ is absolutely convergent in $\left(W_{p(.)}(0,T)\right)'$, moreover for all $\varphi\in C_c^\infty(Q)$, we have

$$\begin{aligned}|\langle\langle\gamma,\varphi\rangle\rangle| &= \left|\sum_{n=1}^{\infty}\frac{\langle\langle\lambda_n,\varphi\rangle\rangle}{2^n(\|\lambda_n\|_{\left(W_{p(.)}(0,T)\right)'}+1)}\right|\\ &\leq \sum_{n=1}^{\infty}\frac{\|\lambda_n^{meas}\|_{\mathcal{M}_b(Q)}\|\varphi\|_{L^\infty(Q)}}{2^n}\leq\|\mu\|_{\mathcal{M}_b(Q)}\|\varphi\|_{L^\infty(Q)},\end{aligned}$$

which implies that $\gamma \in \left(W_{p(.)}(0,T)\right)' \cap \mathcal{M}_b(Q)$.
Thanks to (103), for all $\varphi \in C_c^\infty(Q)$, we have

$$\begin{aligned}\int_Q \varphi d\gamma^{meas} &= \langle\langle \gamma, \varphi\rangle\rangle = \sum_{n=1}^{\infty} \frac{\langle\langle \lambda_n, \varphi\rangle\rangle}{2^n(\|\lambda_n\|_{\left(W_{p(.)}(0,T)\right)'}+1)} \\ &= \sum_{n=1}^{\infty} \frac{1}{2^n(\|\lambda_n\|_{\left(W_{p(.)}(0,T)\right)'}+1)} \int_Q \varphi d\lambda_n^{meas},\end{aligned}$$

hence,

$$\gamma^{meas} = \frac{\lambda_n^{meas}}{2^n(\|\lambda_n\|_{\left(W_{p(.)}(0,T)\right)'}+1)} \tag{104}$$

and since $\lambda_n^{meas} \geq 0$, γ^{meas} is a nonnegative measure. For every $n \in \mathbb{N}$, the measure λ^{meas} is absolutely continuous with respect to γ^{meas} thus, there exists a nonnegative function $f_n \in L^1(Q, d\gamma^{meas})$ such that $\lambda_n^{meas} = f_n\gamma^{meas}$. Then, from (100) we get

$$\int_Q \varphi d\mu = \sup_{n\in\mathbb{N}} \int_Q f_n\varphi d\gamma^{meas}, \tag{105}$$

for any nonnegative $\varphi \in C_c^\infty(Q)$. Since by (102), we have $f_n\gamma^{meas} = \lambda_n^{meas} \leq \mu$, then

$$\int_B f_n d\gamma^{meas} \leq \mu(B), \tag{106}$$

for any borelian subset B in Q and every $n \in \mathbb{N}$. So we can write

$$\int_B \sup\{f_1, f_2, \cdots, f_k\} d\gamma^{meas} \leq \mu(B), \tag{107}$$

for any borelian subset B in Q and any $k \geq 1$. Letting k tends to infinity we deduce by the monotone convergence theorem

$$\int_B f d\gamma^{meas} \leq \mu(B), \tag{108}$$

where $f = \sup_{n\in\mathbb{N}}\{f_n\}$, hence by (104), we obtain

$$\int_B \varphi d\mu = \sup_{n\in\mathbb{N}} \int_Q f_n\varphi d\gamma^{meas} \leq \int_Q f\varphi d\gamma^{meas} \leq \int_Q \varphi d\mu, \tag{109}$$

for every nonnegative function $\varphi \in C_c^\infty(Q)$ which implies that $\mu = f\gamma^{meas}$ and from the fact that $\mu(Q) < +\infty$, we get $f \in L^1(Q, d\gamma^{meas})$ □

Lemma 4.2. *Let $g \in \left(W_{p(.)}(0,T)\right)'$. Then, there exists $g_1 \in L^{(p_-)'}\left(0,T; W^{-1,p'(.)}(\Omega)\right)$, $g_2 \in L^{p_-}(0,T;V)$, $F \in \left(L^{p'(.)}(Q)\right)^N$ and $g_3 \in L^{(p_-)'}\left(0,T;L^2(\Omega)\right)$ such that*

$$\langle\langle g, u\rangle\rangle = \int_0^T \langle g_1, u\rangle\, dt + \int_0^T \langle u_t, g_2\rangle + \int_Q F.\nabla u dxdt \int_Q g_3 u dxdt, \quad \forall u \in W_{p(.)}(0,T).$$

Moreover, we can choose (g_1, g_2, F, g_3) such that

$$\begin{aligned}\|g_1\|_{L^{(p_-)'}\left(0,T;W^{-1,p'(.)}(\Omega)\right)} + \|g_2\|_{L^{p_-}(0,T;V)} + \||F|\|_{L^{p'(.)}(Q)} + \|g_3\|_{L^{(p_-)'}(0,T;L^2(\Omega))} \\ \leq C\|g\|_{\left(W_{p(.)}(0,T)\right)'}.\end{aligned} \tag{110}$$

Proof. We introduce the following functional space

$$E = L^{p_-}(0,T;V) \times \left(L^{p(.)}(Q)\right)^N \times L^{(p_-)'}(0,T;V')$$

endowed with the norm

$$\|(v_1,v_2,v_3)\|_E = \|v_1\|_{L^{p_-}(0,T;V)} + \||v_2|\|_{L^{p(.)}(Q)} + \|v_3\|_{L^{(p_-)'}(0,T;V')}$$

and we consider the map $T : W_{p(.)}(0,T) \to E$ by $T(u) = (u, \nabla u, u_t)$.
Since

$$\|T(u)\|_E = \|(u_t, \nabla u, u)\|_E = \|u\|_{W_{p(.)}(0,T)}. \tag{111}$$

Then T is isometric from $W_{p(.)}(0,T)$ to E.
Setting $G = T\left(W_{p(.)}(0,T)\right)$, then T^{-1} is defined from G to $W_{p(.)}(0,T)$. Now, we take $g \in \left(W_{p(.)}(0,T)\right)'$ and we introduce the functional $\Phi : G \to \mathbb{R}$ by $\Phi(v_1,v_2,v_3) = \left\langle\left\langle g, T^{-1}(v_1,v_2,v_3)\right\rangle\right\rangle$.
Since Φ is a continuous linear form on G then by Hahn-Banach theorem, it can be extended to a continuous linear form on E still denoted by Φ with $\|\Phi\|_{E'} = \|g\|_{(W_{p(.)}(0,T))'}$.
Consequently, there exists $h_1 \in (L^{p_-}(0,T;V))'$, $F = (f_1, f_2, \cdots, f_N) \in \left(L^{p'(.)}(Q)\right)^N$ and $h_2 \in \left(L^{(p_-)'}(0,T;V')\right)'$ such that

$$\begin{aligned}\Phi(v_1,v_2,v_3) &= \langle h_1, v_1\rangle_{(L^{p_-}(0,T;V))', L^{p_-}(0,T;V)} + \langle F, v_2\rangle_{\left(L^{p'(.)}(Q)\right)^N, \left(L^{p(.)}(Q)\right)^N} \\ &\quad + \langle h_2, v_3\rangle_{\left(L^{(p_-)'}(0,T;V')\right)', L^{(p_-)'}(0,T;V')}.\end{aligned} \tag{112}$$

Moreover, we have

$$\|h_1\|_{(L^{p_-}(0,T;V))'} + \||F|\|_{L^{p'(.)}(Q)} + \|h_2\|_{\left(L^{(p_-)'}(0,T;V')\right)'} \le \|\Phi\|_{E'}. \tag{113}$$

Thanks to Remark 3.1, we have

$$(L^{p_-}(0,T;V))' = L^{(p_-)'}\left(0,T;W^{-1,p'(.)}(\Omega)\right) + L^{(p_-)'}\left(0,T;L^2(\Omega)\right)$$

(with equivalent norms). Then, there exists $g_1 \in L^{(p_-)'}(0,T;W^{-1,p'(.)}(\Omega))$ and $g_3 \in L^{(p_-)'}(0,T;L^2(\Omega))$ such that

$$\langle h_1, v_1\rangle_{(L^{p_-}(0,T;V))', L^{p_-}(0,T;V)} = \int_0^T \langle g_1, v_1\rangle\, dt + \int_Q g_3 v_1 dxdt. \tag{114}$$

Since $\left(L^{(p_-)'}(0,T;V')\right)' = L^{p_-}(0,T;V)$, there exists $g_2 \in L^{p_-}(0,T;V)$ such that

$$\langle h_2, v_2\rangle_{\left(L^{(p_-)'}(0,T;V')\right)', L^{(p_-)'}(0,T;V')} = \int_0^T \langle v_2, g_2\rangle\, dt. \tag{115}$$

Therefore, we have

$$\Phi(v_1,v_2,v_3) = \int_0^T \langle g_1, v_1\rangle\, dt + \int_0^T \langle v_2, g_2\rangle\, dt + \int_Q F\nabla u dxdt + \int_Q g_3 v_1 dxdt$$

with

$$\begin{aligned} &\|g_1\|_{L^{(p_+)'}\left(0,T;W^{-1,p'(.)}(\Omega)\right)} + \|g_3\|_{L^{(p_+)'}(0,T;L^2(\Omega))} + \|F\|_{\left(L^{p'(.)}(Q)\right)^N} + \|g_2\|_{L^{p_-}(0,T;V)} \\ &\leq C\left(\|h_1\|_{L^{(p_-)'}(0,T;V')} + \||F|\|_{L^{p'(.)}(Q)} + \|h_2\|_{\left(L^{(p_-)'}(0,T;V')\right)'}\right) \\ &\leq C\|g\|_{\left(W_{p(.)}(0,T)\right)'}. \end{aligned} \tag{116}$$

Then it follows that for all $u \in W_{p(.)}(0,T)$, we have

$$\begin{aligned} \langle\langle g,u\rangle\rangle &= \left\langle\left\langle g,T^{-1}(T(u))\right\rangle\right\rangle = \Phi(T(u)) \\ &= \int_0^T \langle g_1,u\rangle\,dt + \int_0^T \langle u_t,g_2\rangle\,dt + \int_Q F\nabla u\;dxdt + \int_Q g_3 u\;dxdt \end{aligned} \tag{117}$$

Since for all $\theta \in C_c^\infty(Q)$, the multiplication $\varphi \mapsto \theta\varphi$ is linear continuous from $W_{p(.)}(0,T)$ to $W_{p(.)}(0,T)$, we can define the multiplication of an element $\nu \in (W_{p(.)}(0,T))'$ by θ thanks to a duality method : $\theta\nu \in \left(W_{p(.)}(0,T)\right)'$ is defined by $\langle\theta\nu,\varphi\rangle = \langle\nu,\theta\varphi\rangle$. Then, the following result can be proved similarly to that in [6].

Lemma 4.3. *Let $\nu \in \left(W_{p(.)}(0,T)\right)' \cap \mathcal{M}_b(Q)$ and $\theta \in C_c^\infty(Q)$. We take ρ_n as a sequence of symmetric (i.e $\rho_n(.,-) = \rho_n(.)$) regularizing kernels in $\mathbb{R}\times\mathbb{R}^N$ and $\mu = \theta\nu \in \left(W_{p(.)}(0,T)\right)'$. Then, $\mu \in \left(W_{p(.)}(0,T)\right)' \cap \mathcal{M}_b(Q)$, $\mu^{meas} = \theta\nu^{meas}$, μ^{meas} has a compact support in Q and*

$$\|\mu^{meas} * \rho_n\|_{L^1(Q)} \leq \|\nu^{meas}\|_{\mathcal{M}_b(Q)} \quad \text{and} \quad \mu^{meas} * \rho_n \to \mu \quad \text{in} \quad \left(W_{p(.)}(0,T)\right)'. \tag{118}$$

Proof. Since $\theta \in C_c^\infty(Q)$ and $\nu \in \left(W_{p(.)}(0,T)\right)'$, then $\mu = \theta\nu \in \left(W_{p(.)}(0,T)\right)'$. Moreover, for all $\varphi \in C_c^\infty(Q)$, we have $|\langle\langle\mu,\varphi\rangle\rangle| = |\langle\langle\nu,\theta\nu\rangle\rangle| \leq C\|\theta\varphi\|_{L^\infty(Q)} \leq C\|\theta\|_{L^\infty(Q)}\|\varphi\|_{L^\infty(Q)}$, which implies that $\mu \in \left(W_{p(.)}(0,T)\right)' \cap \mathcal{M}_b(Q)$.
For all $\varphi \in C_c^\infty(Q)$, we have

$$\int_Q \varphi d\mu^{meas} = \langle\langle\mu,\varphi\rangle\rangle = \langle\langle\nu,\theta\varphi\rangle\rangle = \int_Q \theta\varphi d\nu^{meas},$$

hence $\mu^{meas} = \theta\nu^{meas}$ and μ^{meas} has compact support. Therefore, $\mu^{meas} * \rho_n$ is well defined and belongs to $C_c^\infty(Q)$ for n large enough. Moreover, we have $\|\mu^{meas} * \rho_n\|_{L^1(Q)} \leq \|\mu^{meas}\|_{\mathcal{M}_b(Q)}$.
Since $\nu \in (W_{p(.)}(0,T))'$, then by the Lemma 4.2, there exists $(g_1,g_2,F,g_3) \in L^{(p_-)'}(0,T;W^{-1,p'(.)}(\Omega)) \times L^{p_-}(0,T;V) \times (L^{p'}(Q))^N \times L^{(p_-)'}(0,T;L^2(\Omega))$ such that

$$\begin{aligned} \langle\langle\mu,\varphi\rangle\rangle &= \langle\langle\nu,\theta\varphi\rangle\rangle \\ &= \int_0^T \langle g_1,\theta\varphi\rangle\,dt + \int_0^T \langle(\theta\varphi)_t,g_2\rangle\,dt + \int_Q F.\nabla(\theta\varphi)\,dxdt + \int_Q g_3\theta\varphi dt \\ &= \int_0^T \langle g_1,\theta\varphi\rangle\,dt + \int_0^T \langle\varphi_t,\theta g_2\rangle\,dt + \int_0^T \langle\theta_t\varphi,g_2\rangle\,dt \\ &\quad + \int_Q F.\nabla(\theta\varphi)\,dxdt + \int_Q g_3\theta\varphi dt, \end{aligned}$$

for all $\varphi \in W_{p(.)}(0,T)$. Since by the proof of the second part of Proposition 3.2, the term $\theta_t\varphi$ belongs to $L^{(p_-)'}\left(0,T;L^2(\Omega)\right)$, then we have

$$\int_0^T \langle \theta_t\varphi, g_2\rangle\, dt = \int_Q \theta_t\varphi g_2 dxdt.$$

We have $g_1 \in L^{(p_-)'}\left(0,T;W^{-1,p'(.)}(\Omega)\right)$, then there exists $G_1 \in \left(L^{p'(.)}(Q)\right)^N$ such that $g_1 = div\,(G_1)$, so that

$$\int_0^T \langle \theta g_1, \varphi\rangle\, dt = \int_0^T \langle div\,(G_1)\,, \varphi\rangle\, dt - \int_0^T \langle G_1\nabla\theta, \varphi\rangle\, dt.$$

Moreover, we have

$$\int_Q F.\nabla\,(\theta\varphi)\, dxdt = \int_Q F.\nabla\theta\varphi dxdt + \int_Q \theta F.\nabla\varphi dxdt.$$

Thus, for all $\varphi \in W_{p(.)}(0,T)$, one has

$$\begin{aligned} \langle\langle \mu, \varphi\rangle\rangle &= \int_0^T \langle div\,(\theta G_1)\,, \varphi\rangle\, dt + \int_0^T \langle \varphi_t, \theta g_2\rangle\, dt + \int_Q F.\nabla\theta\varphi dxdt \\ &+ \int_Q \theta F.\nabla\varphi dxdt + \int_Q g_3\theta\varphi dt - \int_Q G_1\nabla\theta\varphi dxdt + \int_Q \theta_t\varphi g_2 dxdt. \quad (119) \end{aligned}$$

For n large enough, $\operatorname{supp}(\theta) \cup \operatorname{supp}(\rho_n)$ is included in a fixed compact $K \subset Q$. Then it follows that $\operatorname{supp}(\mu^{meas} * \rho_n) = \operatorname{supp}(\theta\nu^{meas} * \rho_n)$ is also contained in K. Now, we take $\xi \in C_c^\infty(Q)$ be such that $\xi \equiv 1$ on a neighborhood of K; then for n large enough, $\operatorname{supp}(\xi) \cup \operatorname{supp}(\rho_n)$ is a compact subset of Q. Since $C_c^\infty(Q) \hookrightarrow \left(W_{p(.)}(0,T)\right)'$, for all $\varphi \in W_{p(.)}(0,T)$, we have

$$\langle\langle \mu^{meas} * \rho_n, \varphi\rangle\rangle = \int_Q \varphi\mu^{meas} * \rho_n dxdt.$$

Hence, for all $\varphi \in C_c^\infty([0,T] \times \Omega)$, we have

$$\langle\langle \mu^{meas} * \rho_n, \varphi\rangle\rangle = \int_Q \xi\varphi\mu^{meas} * \rho_n dxdt = \int_Q (\xi\varphi) * \rho_n d\mu^{meas}.$$

We suppose that n is large enough, then $\operatorname{supp}((\xi\varphi) * \rho_n)$ is a compact subset of Q and since $(\xi\varphi) * \rho_n$ belongs to $C_c^\infty(Q)$, then by (119), we get

$$\begin{aligned} &\langle\langle \mu^{meas} * \rho_n, \varphi\rangle\rangle = \langle\langle \mu, (\xi\varphi) * \rho_n\rangle\rangle \\ &= \int_0^T \langle div\,(\theta G_1)\,, (\xi\varphi) * \rho_n\rangle\, dt + \int_0^T \langle ((\xi\varphi) * \rho_n)_t\,, \theta g_2\rangle\, dt \\ &\quad + \int_Q F.\nabla\theta\,(\xi\varphi) * \rho_n dxdt + \int_Q \theta F.\nabla\,(\xi\varphi) * \rho_n dxdt + \int_Q g_3\theta\,(\xi\varphi) * \rho_n dt \\ &\quad - \int_Q G_1\nabla\theta\,(\xi\varphi) * \rho_n dxdt + \int_Q \theta_t g_2\,(\xi\varphi) * \rho_n dxdt. \end{aligned}$$

According to the support of θ and ξ we can write

$$\begin{aligned}\langle\langle\mu^{meas}*\rho_n,\varphi\rangle\rangle &= \int_0^T \langle div\left((\theta G_1)*\rho_n\right),\xi\varphi\rangle\, dt + \int_0^T \langle(\xi\varphi)_t\,,(\theta g_2)*\rho_n\rangle\, dt \\ &+ \int_Q (F.\nabla\theta)*\rho_n\xi\varphi dxdt + \int_Q \theta F.\nabla\left(\xi\varphi\right)*\rho_n dxdt + \int_Q (\theta g_3)*\rho_n\xi\varphi dxdt \\ &- \int_Q (G_1\nabla\theta)\,\xi\varphi dxdt + \int_Q (\theta_t g_2)*\rho_n\xi\varphi dxdt.\end{aligned}$$

Now, using the fact that $\xi\equiv 1$ on a neighborhood of $\operatorname{supp}(\theta)\cup\operatorname{supp}(\rho_n)$, we obtain

$$\begin{aligned}\langle\langle\mu^{meas}*\rho_n,\varphi\rangle\rangle &= \int_0^T \langle div\left((\theta G_1)*\rho_n\right),\varphi\rangle\, dt + \int_0^T \langle\varphi_t,(\theta g_2)*\rho_n\rangle\, dt \\ &+ \int_Q (F.\nabla\theta)*\rho_n\varphi dxdt + \int_Q \theta F.\nabla\varphi*\rho_n dxdt + \int_Q (\theta g_3)*\rho_n\varphi dt \\ &- \int_Q (G_1\nabla\theta)*\rho_n\varphi dxdt + \int_Q (\theta_t g_2)*\rho_n\varphi dxdt,\end{aligned} \tag{120}$$

for all $\varphi\in C_c^\infty([0,T]\times\Omega)$, but since this space is dense in $W_{p(.)}(0,T)$ and both sides are continuous with respect to the norm of $W_{p(.)}(0,T)$, equality (120) remains true for $\varphi\in W_{p(.)}(0,T)$.

We have $(\theta G_1)*\rho_n\to\theta G_1$ in $\left(L^{p'(.)}(Q)\right)^N$, $(\theta g_2)*\rho_n\to\theta g_2$ in $L^{p_-}(0,T;V)$, $(F.\nabla\theta)*\rho_n\to F.\nabla\theta$ in $L^{p'(.)}(Q)$, $\nabla\varphi*\rho_n\to\nabla\varphi$ in $\left(L^{p(.)}(Q)\right)^N$, $(\theta g_3)*\rho_n\to\theta g_3$ in $L^{(p_+)'}\left(0,T;L^2(\Omega)\right)$, $(G_1\nabla\theta)*\rho_n\to G_1\nabla\theta$ in $L^{p'(.)}(Q)$ and $(\theta_t g_2)*\rho_n\to\theta_t g_2$ in $L^{p_-}\left(0,T;L^2(\Omega)\right)$, then subtracting (119) and (120), we obtain

$$\begin{aligned}|\langle\langle\mu^{meas}*\rho_n,\varphi\rangle\rangle| &= \Bigg|\int_0^T \langle div\left((\theta G_1)*\rho_n-\theta G_1\right),\varphi\rangle\, dt \\ &+ \int_0^T \langle\varphi_t,(\theta g_2)*\rho_n-\theta g_2\rangle\, dt + \int_Q ((\theta g_3)*\rho_n-\theta g_3)\,\varphi dxdt \\ &+ \int_Q ((G_1\nabla\theta)-(G_1\nabla\theta)*\rho_n)\,\varphi dxdt + \int_Q ((\theta_t g_2)*\rho_n-\theta_t g_2)\,\varphi dxdt \\ &+ \int_Q ((F.\nabla\theta)*\rho_n-F.\nabla\theta)\,\varphi dxdt + \int_Q \theta F.\left(\nabla\varphi*\rho_n-\nabla\varphi\right)dxdt\Bigg| \\ &\le \Big(\|(\theta G_1)*\rho_n-\theta G_1\|_{\left(L^{p'(.)}(Q)\right)^N}\|\nabla\varphi\|_{\left(L^{p(.)}(Q)\right)^N} + \|(\theta g_2)*\rho_n-\theta g_2\|_{L^{p_-}(0,T;V)} \\ &\times\|\varphi_t\|_{L^{(p_-)'}(0,T;V')} + \|(\theta g_3)*\rho_n-\theta g_3\|_{L^{(p_+)'}(0,T;L^2(\Omega))}\|\varphi\|_{L^{p_+}(0,T;L^2(\Omega))} \\ &+ \|G_1\nabla\theta-(G_1\nabla\theta)*\rho_n\|_{L^{p'(.)}(Q)}\|\varphi\|_{L^{p(.)}(Q)} + \|(\theta_t g_2)*\rho_n-\theta_t g_2\|_{L^{p_-}(0,T;L^2(\Omega))} \\ &\times\|\varphi\|_{L^{(p_-)'}(0,T;L^2(\Omega))} + \|(F.\nabla\theta)*\rho_n-F.\nabla\theta\|_{L^{p'(.)}(Q)}\|\varphi\|_{L^{p(.)}(Q)} \\ &+ \|\theta F\|_{\left(L^{p'(.)}(Q)\right)^N}\|\nabla\varphi*\rho_n-\nabla\varphi\|_{\left(L^{p(.)}(Q)\right)^N}\Big)\end{aligned}$$

$$\leq C\Big(\|(\theta G_1)*\rho_n-\theta G_1\|_{\left(L^{p'(.)}(Q)\right)^N}+\|(\theta g_2)*\rho_n-\theta g_2\|_{L^{p_-}(0,T;V)}$$
$$+\|(\theta g_3)*\rho_n-\theta g_3\|_{L^{(p_+)'}(0,T;L^2(\Omega))}+\|(F.\nabla\theta)*\rho_n-F.\nabla\theta\|_{L^{p'(.)}(Q)}$$
$$+\|G_1\nabla\theta-(G_1\nabla\theta)*\rho_n\|_{L^{p'(.)}(Q)}+\|(\theta_t g_2)*\rho_n-\theta_t g_2\|_{L^{p_-}(0,T;L^2(\Omega))}\Big)\|\varphi\|_{W_{p(.)}(0,T)}$$
$$+\|\theta F\|_{\left(L^{p'(.)}(Q)\right)^N}\|\nabla\varphi*\rho_n-\nabla\varphi\|_{\left(L^{p(.)}(Q)\right)^N},$$

which implies that $\mu^{meas}*\rho_n$ converges to μ in $W_{p(.)}(0,T)$ □

Theorem 4.4. *Let $\mu\in\mathcal{M}_0(Q)$ then there exists $g\in\left(W_{p(.)}(0,T)\right)'$ and $h\in L^1(Q)$ such that $\mu=g+h$ in the sense that*

$$\int_Q\varphi d\mu=\langle\langle g,\varphi\rangle\rangle+\int_Q h\varphi dxdt, \tag{121}$$

for all $\varphi\in C_c^\infty([0,T]\times\Omega)$.

Proof. Since μ belongs to $\mathcal{M}_0(Q)$, then by Hahn Banach decomposition of μ we have $\mu^+,\mu^-\in\mathcal{M}_0(Q)$, so we can assume that $\mu\in\mathcal{M}_0^+(Q)$. Hence, from the Proposition 4.1, there exists $\gamma\in\left(W_{p(.)}(0,T)\right)'\cap\mathcal{M}_0^+(Q)$ and nonnegative Borel function $f\in L^1(Q,d\gamma^{meas})$ such that

$$\mu(B)=\int_B fd\gamma^{meas}\quad\text{for all Borel set } B \quad\text{in}\quad Q.$$

Since γ^{meas} is a regular measure and $C_c^\infty(Q)$ is dense in $L^1(Q,d\gamma^{meas})$, then there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $C_c^\infty(Q)$ such that f_n converges strongly to f in $L^1(Q,d\gamma^{meas})$. Moreover, we have $\sum_{n=0}^{\infty}\|f_n-f_{n-1}\|_{L^1(Q,d\gamma^{meas})}<\infty$.

Defining ν_n by $\nu_n=(f_n-f_{n-1})\gamma\in\left(W_{p(.)}(0,T)\right)'$, then by Lemma 4.3 we get $\nu\in\left(W_{p(.)}(0,T)\right)'\cap\mathcal{M}_b(Q)$ and $\sum_{n=0}^{\infty}\nu_n^{meas}=\sum_{n=0}^{\infty}(f_n-f_{n-1})\gamma^{meas}$ strongly converges to μ in $\mathcal{M}_b(Q)$. Therefore, we can consider μ as compactly supported measure. Using the Lemma 4.3, we deduce that $\rho_l*\nu_n^{meas}$ strongly converges to ν_n in $\left(W_{p(.)}(0,T)\right)'$, hence we can extract a subsequence still denoted by l such that $\|\rho_l*\nu_n^{meas}-\nu_n\|_{\left(W_{p(.)}(0,T)\right)'}\leq\frac{1}{2^n}$.

Let us rewrite now $\sum_{k=0}^{n}\nu_k^{meas}$ as follows

$$\sum_{k=0}^{n}\nu_k^{meas}=\sum_{k=0}^{n}\rho_{l_k}*\nu_k^{meas}+\sum_{k=0}^{n}\left(\nu_k^{meas}-\rho_{l_k}*\nu_k^{meas}\right). \tag{122}$$

In the following, we denote respectively by m_n, h_n the first and second term in (122) and we define the sequence g_n by $g_n=\sum_{k=0}^{n}\left(\nu_k-\rho_{l_k}*\nu_k^{meas}\right)$, so m_n is a measure with compact support, h_n is a function in $C_c^\infty(Q)$ and g_n belongs to $\left(W_{p(.)}(0,T)\right)'$.

We have $g_n = \sum_{k=0}^{n} \left(\nu_k^{meas} - \rho_{l_k} * \nu_k^{meas}\right)$. Taking θ_n in $C_c^\infty(Q)$ be such that $\theta \equiv 1$ on a neighborhood of $(\operatorname{supp}(f_0) \cup \cdots \cup \operatorname{supp}(f_n)) \cap \operatorname{supp}\left(\sum_{k=0}^{n} \rho_{l_k} * \nu_k^{meas}\right)$, then we can write $g_n = \theta_n g_n$.

Since all terms in (122) has compact support, we can use $\varphi \in C_c^\infty([0,T] \times \Omega)$ as test function in (122) to obtain

$$\int_Q \varphi dm_n = \int_Q h_n \varphi dxdt + \langle\langle g_n, \varphi \rangle\rangle \tag{123}$$

since

$$\int_Q \varphi dg_n^{meas} = \int_Q \theta_n \varphi dg_n^{meas} = \langle\langle g_n, \theta_n \varphi \rangle\rangle = \langle\langle g_n, \varphi \rangle\rangle .$$

We have

$$\|h\|_{L^1(Q)} \leq \sum_{k=0}^{\infty} \|\rho_{l_k} * \nu_k^{meas}\|_{L^1(Q)} \leq \sum_{k=0}^{\infty} \|\nu_k^{meas}\|_{\mathcal{M}_b(Q)} < \infty,$$

which implies the existence of a subsequence of $(h_n)_{n\in\mathbb{N}}$ converging to an element h in $L^1(Q)$. We have

$$\|g_n\| \leq \sum_{k=0}^{\infty} \|\nu_k - \rho_{l_k} * \nu_n^{meas}\|_{\left(W_{p(.)}(0,T)\right)'} \leq \sum_{k=0}^{\infty} \frac{1}{2^k} < \infty,$$

hence $(h_n)_{n\in\mathbb{N}}$ converges strongly to an element g in $\left(W_{p(.)}(0,T)\right)'$. Then it follows that

$$\langle\langle g_n, \varphi \rangle\rangle + \int_Q h_n \varphi dxdt \to \langle\langle g, \varphi \rangle\rangle + \int_Q h \varphi dxdt, \tag{124}$$

for every $\varphi \in C_c^\infty([0,T] \times \Omega)$.

Now, we prove that $\int_Q \varphi dm_n$ converges to $\int_Q \varphi d\mu$. For that, we recall the following linear and continuous injection

$$\begin{cases} \mathcal{M}_b(Q) & \to \left(C\left(\bar{Q}\right)\right)' \\ \quad m & \mapsto \tilde{m} \quad \text{defined by} \quad \tilde{m}(f) = \int_Q f dm. \end{cases}$$

We know that m_n strongly converges to μ in $\mathcal{M}_b(Q)$, $\tilde{m}_n$ strongly converges to $\tilde{m}$ and since $\varphi \in C\left(\bar{Q}\right)$, we have

$$\int_Q \varphi dm_n = \tilde{m}_n(\varphi) \to \tilde{m}(\varphi) = \int_Q \varphi d\mu. \tag{125}$$

Combining $(123) - (125)$, we get (121) $\square$

As consequences of Theorem 4.4 and Lemma 4.2, we have the following decomposition theorem which is the main result of this part.

Theorem 4.5. *Let $\mu \in \mathcal{M}_0(Q)$ then there exists (f, F, g_1, g_2) such that $f \in L^1(Q)$, $F \in \left(L^{p'(.)}(Q)\right)^N$, $g_1 \in L^{(p_-)'}\left(0,T; W^{-1,p'(.)}(\Omega)\right)$, $g_2 \in L^{p_-}(0,T;V)$ such that*

$$\int_Q \varphi d\mu = \int_Q f \varphi dxdt + \int_Q F.\nabla u dxdt + \int_0^T \langle g_1, \varphi \rangle dt - \int_0^T \langle \varphi_t, g_2 \rangle dt,$$

$\forall \varphi \in C_c^\infty([0,T]\times\Omega)$. *Such a triplet* (f, F, g_1, g_2) *will be called a decomposition of* μ.

Notice that the decomposition of $\mu \in \mathcal{M}_0(Q)$ given by the previous theorem is not unique, however as in [6] the following result can be proved.

Lemma 4.6. *Let* $\mu \in \mathcal{M}_0(Q)$ *and let* (f, F, g_1, g_2), $\left(\tilde{f}, \tilde{F}, \tilde{g}_1, \tilde{g}_2\right)$ *be two different decompositions of* μ *according to Theorem* 4.5. *Then we have*

$$\int_0^T \left\langle (g_2-\tilde{g}_2)_t, \varphi\right\rangle dt = \int_Q \left(\tilde{f}-f\right)\varphi dxdt + \int_Q \left(\tilde{F}-F\right).\nabla\varphi dxdt + \int_0^T \langle \tilde{g}_1 - g_1, \varphi\rangle dt \tag{126}$$

for all $\varphi \in C_c^\infty([0,T]\times\Omega)$. *Moreover,* $g_2-\tilde{g}_2 \in C\left([0,T]; L^1(Q)\right)$ *and* $(g_2-\tilde{g}_2)(0) = 0$.

Proof. We have

$$\int_Q \varphi d\mu = \int_Q f\varphi dxdt + \int_Q F.\nabla\varphi dxdt + \int_0^T \langle g_1, \varphi\rangle dt - \int_0^T \langle \varphi_t, g_2\rangle dt \tag{127}$$

and

$$\int_Q \varphi d\mu = \int_Q \tilde{f}\varphi dxdt + \int_Q \tilde{F}.\nabla\varphi dxdt + \int_0^T \langle \tilde{g}_1, \varphi\rangle dt - \int_0^T \langle \varphi_t, \tilde{g}_2\rangle dt, \tag{128}$$

for all $\varphi \in C_c^\infty([0,T]\times\Omega)$, then subtracting (126) and (127), we get

$$\int_Q \left(\tilde{f}-f\right)\varphi dxdt + \int_Q \left(\tilde{F}-F\right).\nabla\varphi dxdt + \int_0^T \langle \tilde{g}_1 - g_1, \varphi\rangle dt = -\int_0^T \langle \varphi_t, g_2-\tilde{g}_2\rangle dt, \tag{129}$$

which is equivalent to say that

$$\int_Q \left(\tilde{f}-f\right)\varphi dxdt + \int_Q \left(\tilde{F}-F\right).\nabla\varphi dxdt + \int_0^T \langle \tilde{g}_1 - g_1, \varphi\rangle dt = \int_0^T \left\langle (g_2-\tilde{g}_2)_t, \varphi\right\rangle dt, \tag{130}$$

for all $\varphi \in C_c^\infty([0,T]\times\Omega)$.
Since $g_2-\tilde{g}_2 \in L^{p-}\left(0,T; W_0^{1,p(.)}(\Omega)\right)$, applying Theorem 1.1 in [13], we deduce that $g_2-\tilde{g}_2 \in C\left([0,T]; L^1(\Omega)\right)$.
Since, by the integration by part formula, we have

$$\int_0^T \langle \varphi_t, g_2-\tilde{g}_2\rangle dt + \int_0^T \left\langle (g_2-\tilde{g}_2)_t, \varphi\right\rangle dt = \int_\Omega \varphi(0)(g_2-\tilde{g}_2)(0)dx,$$

for all $\varphi \in C_c^\infty([0,T]\times\Omega)$, such that $\varphi(T)=0$, then, from (129), we obtain

$$\int_\Omega \varphi(0)(g_2-\tilde{g}_2)(0)dx = 0.$$

Choosing $\varphi = (T-t)\psi$ with $\psi \in C_c^\infty(\Omega)$, we get

$$T\int_\Omega (g_2-\tilde{g}_2)(0)\psi dx = 0 \quad \text{for all} \quad \psi \in C_c^\infty(\Omega),$$

which implies that $(g_2-\tilde{g}_2)(0) = 0$ □

References

[1] D.G. Aronson, Removable singularities for linear parabolic equations, *Arch. Rat. Mech. Anal.* **17** (1964), 79–84.

[2] M. Bendahmane, P. Wittbold, A. Zimmermann, Renormalized solutions for a nonlinear parabolic equation with variable exponents and L^1 data, *J. Diff. Equ.* **249** (2010), 1483–1515.

[3] L. Boccardo, T. Gallouët, L. Orsina, Existence and uniqueness of entropy solutions for nonlinear elliptic equations with measure data, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **13** (1996), 539–551.

[4] L. Diening, P. Harjulehto, P. Hästö, M. Ruzicka, *Lebesgue and Sobolev spaces with variable exponents,* Springer, 2010.

[5] J. Droniou, *Intégration et Espaces de Sobolev à valeurs vectorielles*, Lecture notes, Université de Provence, Marseille, 2001.

[6] J. Droniou, A. Porretta, A. Prignet, Parabolic Capacity and soft measures for nonlinear equations, *Potential Analysis* **19**, no. 2 (2003), 99–161.

[7] D.E. Edmunds, L.A. Peletier, Removable singularities of solutions of quasilinear parabolic equations, *J. London Math. Soc.* **2** (1970), no. 2, 273–283.

[8] X. Fan, D. Zhao, On the spaces $L^p(x)\,(\Omega)$ and $W^{m,p(x)}\,(\Omega)$, *J. Math. Anal. Appl.* **263** (2001), 424–446.

[9] E. Lanconelli, Sul problema di Dirichlet per l'equazione del calore, *Ann. Mat. Pura Appl.* **97** (1973), 83–114.

[10] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaire*, Dunod et Gauthier-Villars, (1969).

[11] J.-L. Lions, Sur certaines équations paraboliques non linéaires, *Bull. Soc. math. France* **93** (1969), 155–175.

[12] I. Nyanquini, S. Ouaro, S. Soma, Entropy solution to nonlinear multivalued elliptic problem with variable exponents and measure data, *Ann. Univ. of Craiova Math. Comp. Sci. Ser.* **40** (2013), no. 2, 174–198.

[13] A. Porretta, Existence results for nonlinear parabolic equations via strong convergence of truncations, *Ann. Mat. Pura ed Appl.* (IV) **177** (1999), 143–172.

[14] V. Rădulescu, Nonlinear elliptic equations with variable exponent: old and new, *Nonlinear Analysis TMA* **121** (2015), 336–369.

[15] V. Rădulescu, D. Repovš, *Partial Differential Equations with Variable Exponents: Variational Methods and Qualitative Analysis*, CRC Press, Taylor and Francis Group, Boca Raton FL, 2015.

[16] J. Simon, Compact sets in $L^p(0,T;B)$, *Ann. Mat. Pura Appl.* **146** (1987), no. 4, 65–96.

[17] L. Wang, Y. Fan, W. Ge, Existence and multiplicity of solutions for a Neumann problem involving the $p\,(x)-$ Laplace operator, *Nonlinear Anal.* **71** (2009), 4259–4270.

[18] N. A. Watson, Thermal capacity *Proc. London Math. Soc.* **37** (1978), 342–362.

[19] J. Yao, Solutions for Neumann boundary value problems involving $p\,(x)-$ Laplace operator, *Nonlinear Anal.* **68** (2008), 1271–1283.

[20] C. Zhang, S. Zhou, Renormalized and entropy solutions for nonlinear parabolic equations with variable exponents and L^1 data, *J. Diff. Equations* **248** (2010), 1376–1400.

[21] D. Zhao, W.J. Qiang, X.L. Fan, On generalized Orlicz spaces $L^{p(x)}$, *J. Gansu Sci.* **9** (1997), no. 2, 1–7.

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