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Renormalized solution for nonlinear elliptic problems with lower order terms and L^1 data in Musielak-Orlicz spaces

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Abstract. We prove the existence of a renormalized solution for the nonlinear elliptic problem

$$-\text{div } a(x,u,\nabla u) - \text{div } \phi(u) + g(x,u,\nabla u) = f \text{ in } \Omega,$$

in the setting of Musielak-Orlicz spaces. $\phi \in \mathcal{C}^0(\mathbb{R},\mathbb{R}^N)$, the nonlinearity g has a natural growth with respect to its third argument and satisfies the sign condition while the datum f belongs to $L^1(\Omega)$. No Δ_2-condition is assumed on the Musielak function.

Key words and phrases. Musielak-Orlicz spaces, boundary value problems, truncations, renormalized solutions.

1. Introduction

Let Ω be an open bounded subset of $\mathbb{R}^N$ $(N \geq 2)$. Consider the following non-linear Dirichlet problem

$$\begin{cases} A(u) - \text{div } \phi(u) + g(x,u,\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where $A(u) = -\text{div } a(x,u,\nabla u)$ is a Leray-Lions Operator defined on $D(A) \subset W_0^1 L_\varphi(\Omega) \to W^{-1}L_\psi(\Omega)$ with φ and ψ are two complementary Musielak-Orlicz functions, $\phi \in \mathcal{C}^0(\mathbb{R},\mathbb{R}^N)$ and g is a non-linearity which satisfies the classical sign condition: $g(x,s,\xi)s \geq 0$ and the following natural growth condition: $|g(x,s,\xi)| \leq b(|s|)(c'(x) + \varphi(x,|\xi|))$, where $b : \mathbb{R} \to \mathbb{R}$ is a continuous non-decreasing function and $c'(.)$ is a non-negative function in $L^1(\Omega)$.
The right-hand side f is assumed to belong to $L^1(\Omega)$.

In the usual Sobolev spaces, the concept of renormalized solutions was introduced by Diperna and Lions in [1] for the study of the Boltzmann equations, this notion of solutions was then adapted to the study of the problem (1) by Boccardo et al. in [2] when the right hand side is in $W^{-1,p'}(\Omega)$ and in the case where the nonlinearity g depends only on x and u, this work was then studied by Rakotoson in [3] when the right hand side is in $L^1(\Omega)$, and finally by DalMaso et al. in [4] for the case in which the right hand side is general measure data. Some elliptic boundary value problems with L^1 or Radon measure data or involving the p-Laplacian have been studied by Rădulescu et al. in [5], [6] and [7].

On Orlicz-Sobolev spaces and in variational case, Benkirane and Bennouna have studied in [8] the problem (1) where the nonlinearity g depends only on x and u under the restriction that the N-function satisfies the Δ_2-condition, this work was then

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extended in [9] by Aharouch, Bennouna and Touzani for N-function not satisfying necessarily the Δ_2-condition. If g depends also on ∇u, the problem (1) has been solved by Aissaoui Fqayeh, Benkirane, El Moumni and Youssfi in [10] without assuming the Δ_2-condition on the N-function.

In the framework of variable exponent Sobolev spaces, Bendahmane and Wittbold have treated in [11] the nonlinear elliptic equation

$$\begin{cases} -\text{div}\,(|\nabla u|^{p(x)-2}\nabla u) = f & \text{in} \quad \Omega \\ \qquad\qquad u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

where f is assumed in $L^1(\Omega)$. They proved the existence and uniqueness of a renormalized solution in Sobolev space with variable exponents $W_0^{1,p(x)}(\Omega)$. In [12] Azroul, Barbara, Benboubker and Ouaro have proved the existence of a renormalized solution for some elliptic problem involving the $p(x)$-Laplacian with Neumann nonhomogeneous boundary conditions in the case where the second member f is in $L^1(\Omega)$. Further works for nonlinear elliptic equations with variable exponent can be found in [13] and [14].

In the variational case of Musielak-Orlicz spaces and in the case where $g \equiv 0$ and $\phi \equiv 0$, an existence result for (1) has been proved by Benkirane and Sidi El Vally in [15] and then in [16] when the non-linearity g depends only on x and u. If g depends also on ∇u, the problem (1) has recently been solved by Ait Khellou, Benkirane and Douiri in [17] and then in [20] when the right hand side is in $L^1(\Omega)$.

Our main goal, in this paper, is to prove the existence of a renormalized solution for the problem (1) in Musielak-Orlicz space $W^1L_\varphi(\Omega)$ by assuming that the Musielak function φ depends only on $N-1$ coordinates of the spatial variable x. This assumption allow us to use a Poincaré inequality in Musielak-Orlicz spaces (see Lemma 2.9).

2. Preliminaries

Musielak-Orlicz function. Let Ω be an open subset of $\mathbb{R}^N$ and let φ be a real-valued function defined in $\Omega \times \mathbb{R}_+$ and satisfying the following conditions:

(a): $\varphi(x,.)$ is an N-function for all $x \in \Omega$ (i.e. convex, nondecreasing, continuous, $\varphi(x,0)=0$, $\varphi(x,t)>0$ for all $t>0$, $\lim\limits_{t\to 0}\sup\limits_{x\in\Omega}\frac{\varphi(x,t)}{t}=0$ and $\lim\limits_{t\to\infty}\inf\limits_{x\in\Omega}\frac{\varphi(x,t)}{t}=\infty$);

(b): $\varphi(.,t)$ is a measurable function for all $t \geq 0$.

A function φ which satisfies the conditions (a) and (b) is called a Musielak-Orlicz function.

For a Musielak-Orlicz function φ we put $\varphi_x(t) = \varphi(x,t)$ and we associate its nonnegative reciprocal function φ_x^{-1}, with respect to t, that is

$$\varphi_x^{-1}(\varphi(x,t)) = \varphi(x,\varphi_x^{-1}(t)) = t.$$

The Musielak-Orlicz function φ is said to satisfy the Δ_2-condition if for some $k>0$, and a non negative function h, integrable in Ω, we have

$$\varphi(x,2t) \leq \; k\;\varphi(x,t) + h(x) \quad \text{for all} \quad x \in \Omega \text{ and all } t \geq 0. \tag{2}$$

When (2) holds only for $t \geq t_0 > 0$, then φ is said to satisfy the Δ_2-condition near infinity.

Let φ and γ be two Musielak-Orlicz functions, we say that φ dominate γ, and we write $\gamma \prec \varphi$, near infinity (resp. globally) if there exists two positive constants c and t_0 such that for almost all $x \in \Omega$:

$$\gamma(x,t) \leq \varphi(x,ct) \text{ for all } t \geq t_0 \quad (\text{resp. for all } t \geq 0 \text{ i.e. } t_0 = 0).$$

We say that γ grows essentially less rapidly than φ at 0 (resp. near infinity), and we write $\gamma \prec\prec \varphi$, If for every positive constant c we have

$$\lim_{t \to 0} \left(\sup_{x \in \Omega} \frac{\gamma(x,ct)}{\varphi(x,t)} \right) = 0 \quad (\text{resp.} \lim_{t \to \infty} \left(\sup_{x \in \Omega} \frac{\gamma(x,ct)}{\varphi(x,t)} \right) = 0).$$

Remark 2.1. [16] If $\gamma \prec\prec \varphi$ near infinity, then $\forall \varepsilon > 0$ there exist $k(\varepsilon) > 0$ such that for almost all $x \in \Omega$ we have

$$\gamma(x,t) \leq k(\varepsilon)\, \varphi(x, \varepsilon t) \text{ for all } t \geq 0.$$

Musielak-Orlicz space. For a Musielak-Orlicz function φ and a measurable function $u : \Omega \to \mathbb{R}$ we define the functional

$$\varrho_{\varphi,\Omega}(u) = \int_\Omega \varphi(x, |u(x)|)\, dx.$$

The set $K_\varphi(\Omega) = \{u : \Omega \to \mathbb{R} \text{ measurable } : \varrho_{\varphi,\Omega}(u) < \infty\}$ is called the Musielak-Orlicz class (or generalized Orlicz class). The Musielak-Orlicz space (or generalized Orlicz space) $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is, $L_\varphi(\Omega)$ is the smallest linear space containing the set $K_\varphi(\Omega)$. Equivalently:

$$L_\varphi(\Omega) = \{u : \Omega \to \mathbb{R} \text{ measurable } : \varrho_{\varphi,\Omega}\left(\frac{u}{\lambda}\right) < \infty \text{ for some } \lambda > 0\}.$$

For a Musielak-Orlicz function φ we put

$$\psi(x,s) = \sup_{t \geq 0}(st - \varphi(x,t)).$$

ψ is called the Musielak-Orlicz function complementary (or conjugate) to φ in the sense of Young with respect to s.

We say that a sequence of functions $u_n \subset L_\varphi(\Omega)$ is modular convergent to $u \in L_\varphi(\Omega)$ if there exists a constant $\lambda > 0$ such that

$$\lim_{n \to \infty} \varrho_{\varphi,\Omega}\left(\frac{u_n - u}{\lambda}\right) = 0.$$

This implies convergence for $\sigma(\Pi L_\varphi, \Pi L_\psi)$ (Lemma 4.7 of [16]).

In the space $L_\varphi(\Omega)$ we define the following two norms:

$$\|u\|_{\varphi,\Omega} = \inf\{\lambda > 0 : \int_\Omega \varphi(x, \frac{|u(x)|}{\lambda})\, dx \leq 1\},$$

which is called the Luxemburg norm, and the so-called Orlicz norm by

$$|||u|||_{\varphi,\Omega} = \sup_{\|v\|_\psi \leq 1} \int_\Omega |u(x)\, v(x)|\, dx,$$

where ψ is the Musielak-Orlicz function complementary to φ. These two norms are equivalent [21]. $K_\varphi(\Omega)$ is a convex subset of $L_\varphi(\Omega)$.

The closure in $L_\varphi(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_\varphi(\Omega)$. It is a separable space and $(E_\psi(\Omega))^* = L_\varphi(\Omega)$ [21]. We have $E_\varphi(\Omega) = K_\varphi(\Omega)$ if and only if $K_\varphi(\Omega) = L_\varphi(\Omega)$ if and only if φ satisfy the Δ_2-condition (2) for large values of t or for all values of t, according to whether Ω has finite measure or not.
We define

$$W^1L_\varphi(\Omega) = \{u \in L_\varphi(\Omega) : D^\alpha u \in L_\varphi(\Omega), \quad \forall|\alpha| \leq 1\}$$

$$W^1E_\varphi(\Omega) = \{u \in E_\varphi(\Omega) : D^\alpha u \in E_\varphi(\Omega), \quad \forall|\alpha| \leq 1\},$$

where $\alpha = (\alpha_1, \dots, \alpha_N)$, $|\alpha| = |\alpha_1| + \cdots + |\alpha_N|$ and $D^\alpha u$ denote the distributional derivatives. The space $W^1L_\varphi(\Omega)$ is called the Musielak-Orlicz-Sobolev space. Let

$$\overline{\varrho}_{\varphi,\Omega}(u) = \sum_{|\alpha|\leq 1} \varrho_{\varphi,\Omega}(D^\alpha u) \text{ and } \|u\|^1_{\varphi,\Omega} = \inf\left\{\lambda > 0 : \overline{\varrho}_{\varphi,\Omega}\left(\frac{u}{\lambda}\right) \leq 1\right\} \text{ for } u \in W^1L_\varphi(\Omega).$$

These functionals are convex modular and a norm on $W^1L_\varphi(\Omega)$ respectively. The pair $\langle W^1L_\varphi(\Omega), \|u\|^1_{\varphi,\Omega}\rangle$ is a Banach space if φ satisfies the following condition [21]:

$$\text{there exists a constant } c_0 > 0 \text{ such that } \inf_{x\in\Omega} \varphi(x, 1) \geq c_0. \tag{3}$$

The space $W^1L_\varphi(\Omega)$ is identified to a subspace of the product $\Pi_{|\alpha|\leq 1}L_\varphi(\Omega) = \Pi L_\varphi$; this subspace is $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closed.
We denote by $\mathfrak{D}(\Omega)$ the Schwartz space of infinitely smooth functions with compact support in Ω and by $\mathfrak{D}(\overline{\Omega})$ the restriction of $\mathfrak{D}(\mathbb{R}^N)$ on Ω. The space $W^1_0L_\varphi(\Omega)$ is defined as the $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closure of $\mathfrak{D}(\Omega)$ in $W^1L_\varphi(\Omega)$ and the space $W^1_0E_\varphi(\Omega)$ as the (norm) closure of the Schwartz space $\mathfrak{D}(\Omega)$ in $W^1L_\varphi(\Omega)$.
For two complementary Musielak-Orlicz functions φ and ψ, we have [21]:
(i) The Young inequality: $ts \leq \varphi(x, t) + \psi(x, s)$ for all $t, s \geq 0$, $x \in \Omega$,
(ii) The Hölder inequality: $|\int_\Omega u(x)\, v(x)\, dx| \leq 2\|u\|_{\varphi,\Omega}\, \|v\|_{\psi,\Omega}$, for all $u \in L_\varphi(\Omega), v \in L_\psi(\Omega)$.

We say that a sequence of functions u_n converges to u for the modular convergence in $W^1L_\varphi(\Omega)$ (respectively in $W^1_0L_\varphi(\Omega)$) if, for some $\lambda > 0$,

$$\lim_{n\to\infty} \overline{\varrho}_{\varphi,\Omega}\left(\frac{u_n - u}{\lambda}\right) = 0.$$

The following spaces of distributions will also be used:

$$W^{-1}L_\psi(\Omega) = \{f \in \mathfrak{D}'(\Omega) : f = \sum_{|\alpha|\leq 1} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ where } f_\alpha \in L_\psi(\Omega)\}$$

$$W^{-1}E_\psi(\Omega) = \{f \in \mathfrak{D}'(\Omega) : f = \sum_{|\alpha|\leq 1} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ where } f_\alpha \in E_\psi(\Omega)\}.$$

Lemma 2.1. [22] *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and let φ and ψ be two complementary Musielak-Orlicz functions which satisfy the following conditions:*

(*i*) *There exists a constant $c_0 > 0$ such that* $\inf_{x\in\Omega} \varphi(x,1) \geq c_0$; [(2.2)]

(*ii*) *There exists a constant $A > 0$ such that for all $x, y \in \Omega$ with* $|x-y| \leq \dfrac{1}{2}$

$$\text{we have} \qquad \frac{\varphi(x,t)}{\varphi(y,t)} \leq t^{\left(\frac{A}{\log(\frac{1}{|x-y|})}\right)} \quad \text{for all} \quad t \geq 1; \tag{4}$$

(*iii*)
$$\int_\Omega \varphi(x,1)\,dx < \infty; \tag{5}$$

(*iv*) *There exists a constant $c_1 > 0$ such that $\psi(x,1) \leq c_1$ a.e in Ω.* (6)

Under these assumptions, $\mathfrak{D}(\Omega)$ is dense in $L_\varphi(\Omega)$, $\mathfrak{D}(\Omega)$ is dense in $W_0^1 L_\varphi(\Omega)$ and $\mathfrak{D}(\overline{\Omega})$ is dense in $W^1 L_\varphi(\Omega)$ for the modular convergence.

Lemma 2.2. [16] *Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let φ be an Musielak-Orlicz function and let $u \in W_0^1 L_\varphi(\Omega)$. Then $F(u) \in W_0^1 L_\varphi(\Omega)$. Moreover, if the set D of discontinuity points of F' is finite, we have*

$$\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} & a.e \text{ in } \{x \in \Omega : u(x) \notin D\} \\ 0 & a.e \text{ in } \{x \in \Omega : u(x) \in D\}. \end{cases}$$

Lemma 2.3. [16] *Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let φ be a Musielak-Orlicz function, then the mapping $T_F : W^1 L_\varphi(\Omega) \to W^1 L_\varphi(\Omega)$ defined by $T_F(u) = F(u)$ is sequentially continuous with respect to the weak* topology $\sigma(\Pi L_\varphi, \Pi E_\psi)$.*

Lemma 2.4. *Let $f_n, f \in L^1(\Omega)$ such that*
i) $f_n \geq 0$ *a.e in Ω;*
ii) $f_n \to f$ *a.e in Ω;*
iii) $\displaystyle\int_\Omega f_n(x)\,dx \to \int_\Omega f(x)\,dx$.
Then $f_n \to f$ strongly in $L^1(\Omega)$.

Recall now the following result which is proved in [17]

Lemma 2.5. ***(The Nemytskii operator)*** *Let Ω be an open susbset of $\mathbb{R}^N$ with finite measure and let φ and ψ be two Musielak-Orlicz functions. Let $f : \Omega \times \mathbb{R}^p \to \mathbb{R}^q$ be a Carathéodory function such that for a.e. $x \in \Omega$ and all $s \in \mathbb{R}^p$*

$$|f(x,s)| \leq c(x) + \alpha_1\, \psi_x^{-1}\, \varphi(x, \alpha_2|s|)$$

where α_1, α_2 are real positive constants and $c \in E_\psi(\Omega)$.
Then the Nemytskii operator N_f defined by $N_f(u)(x) = f(x,u(x))$, is continuous from $(\mathcal{P}(E_\varphi(\Omega), \frac{1}{\alpha_2}))^p = \Pi\{u \in L_\varphi(\Omega) : d(u, E_\varphi(\Omega)) < \frac{1}{\alpha_2}\}$ into $(L_\psi(\Omega))^q$ for the modular convergence. Furthermore if $c \in E_\gamma(\Omega)$ and $\gamma \prec\prec \psi$ then N_f is strongly continuous from $(\mathcal{P}(E_\varphi(\Omega), \frac{1}{\alpha_2}))^p$ into $(E_\gamma(\Omega))^q$.

We will use the following Lemma whose proof is straightforward.

Lemma 2.6. *Let Ω be an open bounded subset of $\mathbb{R}^N$ satisfying the segment property. If $u \in (W_0^1 L_\varphi(\Omega))^N$, then*

$$\int_\Omega div\, u\, dx = 0.$$

Lemma 2.7. [18] *Let Ω be a bounded Lipschitz domain of R^N and let φ be a Musielak-Orlicz function satisfying the conditions of Lemma 2.1. Assume also that the function φ depends only on $N-1$ coordinates of x. Then there exists a constant $\lambda > 0$ depending only on Ω such that*

$$\int_\Omega \varphi(x, |v|)\, dx \leq \int_\Omega \varphi(x, \lambda|\nabla v|)\, dx \quad \text{for all } v \in W_0^1 L_\varphi(\Omega).$$

Corollary 2.8. [18] ***(Poincaré Inequality)*** *Let Ω be a bounded Lipchitz domain of R^N and let φ be a Musielak-Orlicz function satisfying the same conditions of Lemma 2.7. Then there exists a constant $C > 0$ such that*

$$\|v\|_\varphi \leq C\, \|\nabla v\|_\varphi \qquad \forall v \in W_0^1 L_\varphi(\Omega).$$

The following example shows that the integral form of Poincaré inequality can not, in general, hold

Example 2.1. [19] Let $p : (-2, 2) \to [2, 3]$ be a Lipschitz continuous exponent that equals 3 in $(-2, -1) \cup (1, 2)$, 2 in $(-\frac{1}{2}, \frac{1}{2})$ and is linear elsewhere. Let u_λ be a Lipschitz function such that $u_\lambda(\pm 2) = 0, u_\lambda = \lambda$ in $(-1, 1)$ and $|u'_\lambda| = \lambda$ in $(-2, -1) \cup (1, 2)$. Then

$$\frac{\overline{\varrho}_{p(.)}(u_\lambda)}{\overline{\varrho}_{p(.)}(u'_\lambda)} = \frac{\int_{-2}^{2} |u_\lambda|^{p(x)}\, dx}{\int_{-2}^{2} |u'_\lambda|^{p(x)}\, dx} \geq \frac{\int_{-\frac{1}{2}}^{\frac{1}{2}} \lambda^2\, dx}{2\int_{-2}^{-1} |\lambda|^3\, dx} = \frac{1}{2\lambda} \to \infty$$

as $\lambda \to 0^+$.

3. Main result

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N, N \geq 2$ and let φ and γ be two Musielak-Orlicz functions such that φ and its complementary ψ satisfies the conditions of Lemma 2.2 and $\gamma \prec\prec \varphi$.
Let $A : D(A) \subset W_0^1 L_\varphi(\Omega) \to W^{-1} L_\psi(\Omega)$ be a mapping given by

$$A(u) = -\operatorname{div} a(x, u, \nabla u),$$

where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying, for a.e $x \in \Omega$ and for all $s \in \mathbb{R}$ and all $\xi\ ,\xi_* \in \mathbb{R}^N$, $\xi \neq \xi_*$:

$$|a(x, s, \xi)| \leq k_1\, \left(c(x) + \psi_x^{-1}(\gamma(x, k_2|s|)) + \psi_x^{-1}\left(\varphi(x, k_3|\xi|)\right)\right) \tag{7}$$

$$\left(a(x, s, \xi) - a(x, s, \xi_*)\right)(\xi - \xi_*) > 0 \tag{8}$$

$$a(x, s, \xi).\xi \geq \alpha\, \varphi(x, |\xi|) \tag{9}$$

where $c(.)$ belongs to $E_\psi(\Omega)$, $c \geq 0$ and $k_i > 0$, $i = 1, 2, 3$, $\alpha \in \mathbb{R}_+^*$.
Furthermore, let $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function such that, for a.e $x \in \Omega$ and for all $s \in \mathbb{R}$

$$g(x, s, \xi)\, s \geq 0 \tag{10}$$

$$|g(x, s, \xi)| \leq b(|s|)\, (c'(x) + \varphi(x, |\xi|)) \tag{11}$$

where $b : \mathbb{R} \to \mathbb{R}$ is a continuous and non-decreasing function and $c'(.)$ is a given non-negative function in $L^1(\Omega)$.
Consider the nonlinear elliptic problem

$$\begin{cases} A(u) - \operatorname{div} \phi(u) + g(x,u,\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{12}$$

where

$$f \in L^1(\Omega) \tag{13}$$

and

$$\phi \in \mathcal{C}^0(\mathbb{R}, \mathbb{R}^N) \tag{14}$$

Note that no growth hypothesis is assumed on the function ϕ, which implies that for a solution $u \in W_0^1 L_\varphi(\Omega)$ the term div $\phi(u)$ may be meaningless, even as a distribution.

Remark 3.1. A consequence of (9) and the continuity of a with respect to ξ, is that, for almost every x in Ω and s in $\mathbb{R}$

$$a(x,s,0) = 0.$$

Definition 3.1. A measurable function $u : \Omega \to \mathbb{R}$ is called renormalized solution of (12) if

$$\begin{cases} T_k(u) \in W_0^1 L_\varphi(\Omega),\ a(x, T_k(u), \nabla T_k(u)) \in (L_\psi(\Omega))^N, \\ \displaystyle\int_{\{m \le |u| \le m+1\}} a(x,u,\nabla u).\nabla u \, dx \to 0 \text{ as } m \to +\infty, \\ \displaystyle\int_\Omega a(x,u,\nabla u).\nabla(h(u)\theta)\ dx + \int_\Omega g(x,u,\nabla u) h(u)\theta \, dx \\ \displaystyle + \int_\Omega \phi(u).\nabla(h(u)\theta)\ dx = \int_\Omega f\ h(u)\theta \, dx \\ \text{for all } h \in \mathcal{C}_c^1(\mathbb{R}) \text{ and for all } \theta \in \mathcal{D}(\Omega). \end{cases} \tag{15}$$

We shall prove the following theorem

Theorem 3.1. *Assume that (7)-(11) and (13)-(14) hold true, then there exists a renormalized solution u for the problem (12) in the sense of definition 3.1.*

Proof. Step 1 : **A priori estimates**
First let us define the truncation $T_k : \mathbb{R} \to \mathbb{R}$ at height $k > 0$ by

$$T_k(s) = \begin{cases} s & \text{if } |s| \le k, \\ k\dfrac{s}{|s|} & \text{if } |s| > k. \end{cases}$$

Consider the nonlinear elliptic approximate problem

$$(\mathcal{P}_n) \begin{cases} u_n \in W_0^1 L_\varphi(\Omega) \\ -\operatorname{div} a(x,u_n,\nabla u_n) + g_n(x,u_n,\nabla u_n) = f_n + \operatorname{div}\phi_n(u_n) \ \text{ in } \mathcal{D}'(\Omega), \end{cases}$$

where $(f_n) \in W^{-1}E_\psi(\Omega)$ is a sequence of smooth functions such that $f_n \to f$ in $L^1(\Omega), \phi_n(s) = \phi(T_n(s))$ and $g_n(x,s,\xi) = T_n(g(x,s,\xi))$.
Note that $g_n(x,s,\xi)\, s \ge 0,\ |g_n(x,s,\xi)| \le |g(x,s,\xi)|$ and $|g_n(x,s,\xi)| \le n$.
Since ϕ is continuous, we have $|\phi_n(t)| = |\phi(T_n(t))| \le c_n$, then the problem $(\mathcal{P}_n)$ have

at least one solution $u_n \in W_0^1 L_\varphi(\Omega)$ (see [23], Proposition 1 and [16], Theorem 4). Using in $(\mathcal{P}_n)$, the test function $v = T_k(u_n)$, $k > 0$, we get

$$\int_\Omega a(x, u_n, \nabla u_n).\nabla T_k(u_n)\,dx + \int_\Omega g_n(x, u_n, \nabla u_n)\; T_k(u_n)\,dx$$
$$+ \int_\Omega \phi(T_n(u_n))\nabla T_k(u_n)\,dx = \int_\Omega f_n\; T_k(u_n)\,dx$$

Remark that, by Lemma 2.6

$$\int_\Omega \phi(T_n(u_n))\nabla T_k(u_n)\,dx = \int_\Omega \operatorname{div}(\tilde{\phi_n}(u_n))\,dx = 0$$

where $\tilde{\phi_n}(s) = \int_0^{T_k(s)} \phi(T_n(\tau))\,d\tau$, ($\tilde{\phi_n}(u_n) \in W_0^1 L_\varphi(\Omega)^N$ by Lemma 2.2) which implies, by using the fact that $g_n(x, u_n, \nabla u_n)\; T_k(u_n) \geq 0$,

$$\int_{\{|u_n| \leq k\}} a(x, u_n, \nabla u_n).\nabla u_n\,dx \leq Ck,$$

where C is a constant such that $\|f_n\|_{1,\Omega} \leq C,\ \forall n$.
Thanks to (9) one easily has

$$\int_\Omega \varphi(x, |\nabla T_k(u_n)|)\,dx \leq \frac{1}{\alpha}\int_\Omega a(x, u_n, \nabla u_n).\nabla T_k(u_n)\,dx \leq C_1 k. \tag{16}$$

On the other hand, by using Lemma 2.7, there exists a positive constant λ such that

$$\int_\Omega \varphi(x, v)\,dx \leq \int_\Omega \varphi(x, \lambda|\nabla v|)\,dx \quad \text{for all } v \in W_0^1 L_\varphi(\Omega). \tag{17}$$

Taking $v = \frac{1}{\lambda}|T_k(u_n)|$ in (17) and using (16) gives

$$\int_\Omega \varphi(x, \frac{1}{\lambda}|T_k(u_n)|)\,dx \leq \int_\Omega \varphi(x, |\nabla T_k(u_n)|)\,dx \leq C_1 k,$$

which implies that

$$\begin{aligned} meas\{|u_n| > k\} &\leq \frac{1}{\inf_{x\in\Omega} \varphi(x, \frac{k}{\lambda})} \int_{\{|u_n|>k\}} \varphi(x, \frac{k}{\lambda})\,dx \\ &\leq \frac{1}{\inf_{x\in\Omega} \varphi(x, \frac{k}{\lambda})} \int_\Omega \varphi(x, \frac{1}{\lambda}|T_k(u_n)|)\,dx \\ &\leq \frac{C_1 k}{\inf_{x\in\Omega} \varphi(x, \frac{k}{\lambda})}, \quad \forall n,\ \forall k > 0. \end{aligned}$$

For any $\beta > 0$, we have

$$meas\{|u_n - u_m| > \beta\} \leq meas\{|u_n| > k\} + meas\{|u_m| > k\} + meas\{|T_k(u_n) - T_k(u_m)| > \beta\}$$

and so that

$$meas\{|u_n - u_m| > \beta\} \le \frac{2C_1 k}{\inf\limits_{x\in\Omega} \varphi(x, \frac{k}{\lambda})} + meas\{|T_k(u_n) - T_k(u_m)| > \beta\}. \tag{18}$$

By using (16) and Corollary 2.8, we deduce that $(T_k(u_n))$ is bounded in $W_0^1 L_\varphi(\Omega)$, and then there exists $\omega_k \in W_0^1 L_\varphi(\Omega)$ such that $T_k(u_n) \rightharpoonup \omega_k$ weakly in $W_0^1 L_\varphi(\Omega)$ for $\sigma(\Pi L_\varphi, \Pi E_\psi)$, strongly in $E_\varphi(\Omega)$ and a.e. in Ω.
Consequently, we can assume that $T_k(u_n)$ is a cauchy sequence in measure in Ω.
Let $\varepsilon > 0$, then by (18) and the fact that $\frac{2C_1 k}{\inf\limits_{x\in\Omega} \varphi(x, \frac{k}{\lambda})} \to 0$ as $k \to +\infty$, there exists some $k(\varepsilon) > 0$ such that

$$meas\{|u_n - u_m| > \beta\} \le \varepsilon, \quad \text{for all } n, m \ge n_0\ (k(\varepsilon), \beta).$$

This proves that (u_n) is a cauchy sequence in measure, thus, u_n converges almost everywhere to some measurable function u.
Finally, by Lemma 4.4 of [24], we obtain for all $k > 0$

$$\begin{aligned} T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } W_0^1 L_\varphi(\Omega) \text{ for } \sigma(\Pi L_\varphi, \Pi E_\psi), \\ \text{strongly in } E_\varphi(\Omega) \text{ and a.e. in } \Omega. \end{aligned} \tag{19}$$

Now, we shall prove that $(a(x, T_k(u_n), \nabla T_k(u_n)))_n$ is bounded in $(L_\psi(\Omega))^N$ for all $k > 0$, by using the dual norm of $(L_\psi(\Omega))^N$.
Let $\vartheta \in (E_\varphi(\Omega))^N$ such that $\|\vartheta\|_{\varphi,\Omega} = 1$. We have from (8)

$$\int_\Omega \left(a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \frac{\vartheta}{k_3}) \right) \cdot \left(\nabla T_k(u_n) - \frac{\vartheta}{k_3} \right) dx \ge 0$$

this implies by (16)

$$\begin{aligned} \int_\Omega \frac{1}{k_3} a(x, T_k(u_n), \nabla T_k(u_n))\, \vartheta\, dx \quad &\le \quad \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)).\nabla T_k(u_n)\, dx \\ &\quad - \int_\Omega a(x, T_k(u_n), \frac{\vartheta}{k_3}).(\nabla T_k(u_n) - \frac{\vartheta}{k_3})\, dx \\ &\le \quad C\, k - \int_\Omega a(x, T_k(u_n), \frac{\vartheta}{k_3}).\nabla T_k(u_n)\, dx \\ &\quad + \frac{1}{k_3} \int_\Omega a(x, T_k(u_n), \frac{\vartheta}{k_3})\, \vartheta\, dx. \end{aligned}$$

By using Young's inequality in the last two terms of the last side and (16) we have

$$\begin{aligned}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\,\vartheta\,dx &\le Ckk_3+3k_1(1+k_3)\int_\Omega \psi\left(x,\frac{|a(x,T_k(u_n),\frac{\vartheta}{k_3})|}{3k_1}\right)dx\\ &\quad+3k_1k_3\int_\Omega \varphi(x,|\nabla T_k(u_n)|)\,dx+3k_1\int_\Omega \varphi(x,|\vartheta|)\,dx\\ &\le Ckk_3+3C_1kk_1k_3+3k_1\\ &\quad+3k_1(1+k_3)\int_\Omega \psi\left(x,\frac{|a(x,T_k(u_n),\frac{\vartheta}{k_3})|}{3k_1}\right)dx.\end{aligned}$$

Using (7) and the convexity of ψ yields

$$\psi\left(x,\frac{|a(x,T_k(u_n),\frac{\vartheta}{k_3})|}{3k_1}\right)\le\frac{1}{3}\left(\psi(x,c(x))\,dx+\gamma(x,k_2T_k(u_n))+\varphi(x,|\vartheta|)\right)$$

and, since γ grows essentially less rapidly than φ near infinity there exists $\zeta(k)>0$ such that $\gamma(x,k_2|T_k(u_n)|)\le\gamma(x,k_2\,k)\le\zeta(k)\varphi(x,1)$ (see Remark 2.1), then we have by integrating over Ω and using (5)

$$\begin{aligned}&\int_\Omega \psi\left(x,\frac{|a(x,T_k(u_n),\frac{\vartheta}{k_3})|}{3k_1}\right)dx\\ &\qquad\le\frac{1}{3}\left(\int_\Omega \psi(x,c(x))\,dx+\zeta(k)\int_\Omega \varphi(x,1)\,dx+\int_\Omega \varphi(x,|\vartheta|)\,dx\right)\le C_k\end{aligned}$$

where C_k is a constant depending on k, we deduce that

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\,\vartheta\,dx\le C_k\quad\forall\vartheta\in(E_\varphi(\Omega))^N\ \text{ with }\ \|\vartheta\|_{\varphi,\Omega}=1,$$

which shows that $(a(x,T_k(u_n),\nabla T_k(u_n)))_n$ is bounded in $(L_\psi(\Omega))^N$.
Step 2: **Almost everywhere convergence of the gradients**.
Let $\mu(t)=te^{\delta t^2},\delta>0$. It is well known that for $\delta\ge(\frac{b(k)}{2\alpha})^2$ one has

$$\mu'(t)-\frac{b(k)}{\alpha}|\mu(t)|\ge\frac{1}{2}\qquad\text{for all }\ t\in\mathbb{R},\tag{20}$$

where $k>0$ is a fixed real number which will be used as a level of the truncation.
Let $v_j\in\mathfrak{D}(\Omega)$ be a sequence which converges to $T_k(u)$ for the modular convergence in $W_0^1L_\varphi(\Omega)$ and define the function

$$\rho_m(s)=\begin{cases}1 & \text{if}\quad |s|\le m\\ m+1-|s| & \text{if}\quad m\le|s|\le m+1\\ 0 & \text{if}\quad |s|\ge m+1\end{cases}$$

where $m>k$.
Let $\theta_n^j=T_k(u_n)-T_k(v_j),\ \theta^j=T_k(u)-T_k(v_j)$ and $z_{n,m}^j=\mu(\theta_n^j)\rho_m(u_n)$.

Using in $(\mathcal{P}_n)$ the test function $z^j_{n,m}$ gives

$$\int_\Omega a(x,u_n,\nabla u_n).\nabla z^j_{n,m} + \int_{\{m\le |u_n|\le m+1\}} \phi_n(u_n).\nabla u_n\ \rho'_m(u_n)\ \mu(T_k(u_n)-T_k(v_j))\,dx$$
$$+\int_\Omega \phi_n(u_n).\nabla\mu(T_k(u_n)-T_k(v_j))\ \rho_m(u_n)\,dx + \int_\Omega g_n(x,u_n,\nabla u_n)\ z^j_{n,m}\,dx$$
$$= \int_\Omega f_n\ z^j_{n,m}\,dx. \quad (21)$$

Denote by $\varepsilon_i(n,j)$ $(i=0,1,2,\dots)$ various sequences of real numbers which tend to 0 when n and $j\to\infty$, i.e. $\lim\limits_{j\to\infty}\lim\limits_{n\to\infty}\varepsilon_i(n,j)=0$.
In view of (19), we have $z^j_{n,m}\to\mu(\theta^j)\ \rho_m(u)$ weakly* in $L^\infty(\Omega)$ as $n\to\infty$ and then

$$\int_\Omega f_n\ z^j_{n,m}\,dx \to \int_\Omega f\ \mu(\theta^j)\rho_m(u)\,dx \text{ as } n\to\infty,$$

and since $\theta^j\to 0$ weakly* in $L^\infty(\Omega)$ we get $\int_\Omega f\ \mu(\theta^j)\ \rho_m(u)\,dx\to 0$ as $j\to\infty$, then

$$\int_\Omega f_n\ z^j_{n,m}\,dx = \varepsilon_0(n,j).$$

By Lemma 2.6, it's easy to see that

$$\int_{\{m\le |u_n|\le m+1\}} \phi_n(u_n).\nabla u_n\ \rho'_m(u_n)\ \mu(T_k(u_n)-T_k(v_j))\,dx = 0$$

Concerning the third term in the left-hand side of (21) we can write

$$\int_\Omega \phi_n(u_n).\nabla\mu(T_k(u_n)-T_k(v_j))\ \rho_m(u_n)\,dx = \int_\Omega \phi_n(u_n).\nabla T_k(u_n)\ \mu'(\theta^j_n)\ \rho_m(u_n)\,dx$$
$$-\int_\Omega \phi_n(u_n).\nabla T_k(v_j)\ \mu'(\theta^j_n)\ \rho_m(u_n)\,dx$$

Using again Lemma 2.6, we get

$$\int_\Omega \phi_n(u_n).\nabla T_k(u_n)\ \mu'(\theta^j_n)\ \rho_m(u_n)\,dx = 0.$$

From (19) we have

$$\phi_n(u_n)\ \mu'(\theta^j_n)\ \rho_m(u_n)\to\phi(u)\ \mu'(\theta^j)\ \rho_m(u) \text{ almost everywhere in } \Omega \text{ as } n\to\infty,$$

furthermore, we can check that

$$\|\phi_n(u_n)\ \mu'(\theta^j_n)\ \rho_m(u_n)\|_\psi \le c_m\ c_1\ \mu'(2k)\ |\Omega|$$

where $c_m = \max_{|t|\leq m+1} \phi(t)$ and c_1 is the constant defined in (6).
Applying [25, Theorem 14.6] we get

$$\lim_{n\to\infty} \int_\Omega \phi_n(u_n).\nabla T_k(v_j)\ \mu'(\theta_n^j)\ \rho_m(u_n)\, dx = \int_\Omega \phi(u).\nabla T_k(v_j)\ \mu'(\theta^j)\ \rho_m(u)\, dx$$

and by using the modular convergence of (v_j), we obtain

$$\lim_{j\to\infty} \lim_{n\to\infty} \int_\Omega \phi_n(u_n).\nabla T_k(v_j)\ \mu'(\theta_n^j)\ \rho_m(u_n)\, dx = \int_\Omega \phi(u).\nabla T_k(u)\ \rho_m(u)\, dx,$$

then, by Lemma 2.6, one has $\int_\Omega \phi(u).\nabla T_k(u)\ \rho_m(u)\, dx = 0$.

Hence

$$\int_\Omega \phi_n(u_n).\nabla \mu(T_k(u_n) - T_k(v_j))\ \rho_m(u_n)\, dx = \epsilon_1(n, j).$$

Since $g_n(x, u_n, \nabla u_n) z_{n,m}^j \geq 0$ on the subset $\{x \in \Omega : |u_n(x)| > k\}$ and $\rho_m(u_n) = 1$ on the subset $\{x \in \Omega : |u_n(x)| \leq k\}$ we have, from (21),

$$\int_\Omega a(x, u_n, \nabla u_n).\nabla z_{n,m}^j + \int_{\{|u_n|\leq k\}} g_n(x, u_n, \nabla u_n)\, \mu(\theta_n^j)\, dx \leq \varepsilon_2(n, j). \tag{22}$$

For what concerns the first term of the left-hand side of (22) we have

$$\begin{aligned}
\int_\Omega a(x, u_n, \nabla u_n).\nabla z_{n,m}^j &= \int_{\{|u_n|\leq k\}} a(x, u_n, \nabla u_n).(\nabla T_k(u_n) - \nabla T_k(v_j))\ \mu'(\theta_n^j)\ \rho_m(u_n)\, dx \\
&\quad - \int_{\{|u_n|>k\}} a(x, u_n, \nabla u_n).\nabla T_k(v_j)\ \mu'(\theta_n^j)\ \rho_m(u_n)\, dx \\
&\quad + \int_\Omega a(x, u_n, \nabla u_n).\nabla u_n\ \mu(\theta_n^j)\ \rho_m'(u_n)\, dx \\
&= \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)).(\nabla T_k(u_n) - \nabla T_k(v_j))\ \mu'(\theta_n^j)\, dx \\
&\quad - \int_{\{|u_n|>k\}} a(x, u_n, \nabla u_n).\nabla T_k(v_j)\ \mu'(\theta_n^j)\ \rho_m(u_n)\, dx \\
&\quad + \int_\Omega a(x, u_n, \nabla u_n).\nabla u_n\ \mu(\theta_n^j)\ \rho_m'(u_n)\, dx,
\end{aligned}$$

and then

$$\begin{aligned}\int_\Omega a(x,u_n,\nabla u_n).\nabla z^j_{n,m} &= \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\ T_k(v_j)\chi^s_j)] \\ &\quad \times[\nabla T_k(u_n) - \nabla T_k(v_j)\chi^s_j]\ \mu'(\theta^j_n)\,dx \\ &+\int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi^s_j)\ (\nabla T_k(u_n) - \nabla T_k(v_j)\chi^s_j)\ \mu'(\theta^j_n)\,dx \\ &-\int_{\Omega\setminus\Omega^s_j} a(x,T_k(u_n),\nabla T_k(u_n)).\nabla T_k(v_j)\ \mu'(\theta^j_n)\,dx \\ &-\int_{\{|u_n|>k\}} a(x,u_n,\nabla u_n).\nabla T_k(v_j)\ \mu'(\theta^j_n)\ \rho_m(u_n)\,dx \\ &+\int_\Omega a(x,u_n,\nabla u_n).\nabla u_n\ \mu(\theta^j_n)\ \rho'_m(u_n)\,dx, \qquad (23)\end{aligned}$$

where χ^s_j is the characteristic function of the set $\Omega^s_j = \{x\in\Omega : |\nabla T_k(v_j)| \le s\}$.
For the third term, since $(a(x,T_k(u_n),\nabla T_k(u_n)))_n$ is bounded in $(L_\psi(\Omega))^N$, we have, for a subsequence, $a(x,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup l_k$ weakly in $(L_\psi(\Omega))^N$ for $\sigma(\Pi L_\psi,\Pi E_\varphi)$, with $l_k\in(L_\psi(\Omega))^N$ and since $\nabla T_k(v_j)\chi_{\Omega\setminus\Omega^s_j}\in(E_\varphi(\Omega))^N$ we have, by letting $n\to\infty$

$$-\int_{\Omega\setminus\Omega^s_j} a(x,T_k(u_n),\nabla T_k(u_n)).\nabla T_k(v_j)\ \mu'(\theta^j_n)\,dx \to -\int_{\Omega\setminus\Omega^s_j} l_k.\nabla T_k(v_j)\ \mu'(\theta^j)\,dx,$$

Using now, the modular convergence of (v_j), we get

$$-\int_{\Omega\setminus\Omega^s_j} l_k.\nabla T_k(v_j)\ \mu'(\theta^j)\,dx \to -\int_{\Omega\setminus\Omega_s} l_k.\nabla T_k(u)\,dx \quad\text{as}\quad j\to\infty,$$

where $\Omega_s = \{x\in\Omega : |\nabla T_k(u)|\le s\}$. We have then proved that

$$-\int_{\Omega\setminus\Omega^s_j} a(x,T_k(u_n),\nabla T_k(u_n)).\nabla T_k(v_j)\ \mu'(\theta^j_n)\,dx = -\int_{\Omega\setminus\Omega_s} l_k.\nabla T_k(u)\,dx + \varepsilon_3(n,j). \qquad (24)$$

Concerning the fourth term, since $\rho_m(u_n)=0$ on the subset $\{|u_n|>m+1\}$, we have

$$\begin{aligned}&-\int_{\{|u_n|>k\}} a(x,u_n,\nabla u_n).\nabla T_k(v_j)\ \mu'(\theta^j_n)\ \rho_m(u_n)\,dx \\ &\qquad = -\int_{\{|u_n|>k\}} a(x,T_{m+1}(u_n),\nabla T_{m+1}(u_n)).\nabla T_k(v_j)\ \mu'(\theta^j_n)\ \rho_m(u_n)\,dx\end{aligned}$$

and as above

$$-\int_{\{|u_n|>k\}} a(x, T_{m+1}(u_n), \nabla T_{m+1}(u_n)).\nabla T_k(v_j)\ \mu'(\theta_n^j)\ \rho_m(u_n)\, dx$$

$$\begin{aligned} &= -\int_{\{|u|>k\}} l_{m+1}.\nabla T_k(u)\ \rho_m(u)\, dx\ +\ \varepsilon_4(n,j) \\ &= \varepsilon_4(n,j) \end{aligned} \tag{25}$$

where we have used the fact that $\nabla T_k(u) = 0$ on the subset $\{x \in \Omega : |u(x)| > k\}$.

For the second term of (23), remark that by using Lemma 2.5 and the fact that $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$ weakly in $(L_\varphi(\Omega))^N$, by (19), we have

$$a(x, T_k(u_n), \nabla T_k(v_j)\chi_j^s)\ \mu'(\theta_n^j) \to a(x, T_k(u), \nabla T_k(v_j)\chi_j^s)\ \mu'(\theta^j)$$

strongly in $(E_\psi(\Omega))^N$ as $n \to \infty$, then

$$\int_\Omega a(x, T_k(u_n), \nabla T_k(v_j)\chi_j^s).(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s)\ \mu'(\theta_n^j)\, dx$$

$$\to \int_\Omega a(x, T_k(u), \nabla T_k(v_j)\chi_j^s).(\nabla T_k(u) - \nabla T_k(v_j)\chi_j^s)\ \mu'(\theta^j)\, dx \quad \text{as} \quad n \to \infty$$

on the other hand, since $\nabla T_k(v_j)\chi_j^s \to \nabla T_k(u)\chi^s$ strongly in $(E_\varphi(\Omega))^N$ as $j \to \infty$, it is easy to see that

$$\int_\Omega a(x, T_k(u), \nabla T_k(v_j)\chi_j^s).(\nabla T_k(u) - \nabla T_k(v_j)\chi_j^s)\ \mu'(\theta^j)\, dx \to 0 \ \text{ as } \ j \to \infty,$$

where χ^s is the characteristic function of the set Ω_s, then

$$\int_\Omega a(x, T_k(u_n), \nabla T_k(v_j)\chi_j^s).(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s])\ \mu'(\theta_n^j)\, dx = \varepsilon_5(n,j). \tag{26}$$

The last term of (23) reads as

$$\int_\Omega a(x, u_n, \nabla u_n).\nabla u_n\ \mu(\theta_n^j)\ \rho_m'(u_n)\, dx = \int_{\{m \le |u_n| \le m+1\}} a(x, u_n, \nabla u_n).\nabla u_n\ \mu(\theta_n^j)\ \rho_m'(u_n)\, dx,$$

then

$$\Big|\int_\Omega a(x, u_n, \nabla u_n).\nabla u_n\ \mu(\theta_n^j)\ \rho_m'(u_n)\, dx\ \Big| \le \mu(2k) \int_{\{m \le |u_n| \le m+1\}} a(x, u_n, \nabla u_n).\nabla u_n\, dx.$$

Taking $T_1(u_n - T_m(u_n))$ as test function in $(\mathcal{P}_n)$ yields

$$\int_{\{m<|u_n|\le m+1\}} a(x, u_n, \nabla u_n).\nabla u_n\, dx + \int_{\{m<|u_n|\le m+1\}} \phi(T_n(u_n)).\nabla u_n\, dx$$

$$+ \int_{\{|u_n|>m\}} g_n(x, u_n, \nabla u_n)\ T_1(u_n - T_m(u_n))\, dx = \int_{\{|u_n|>m\}} f_n\ T_1(u_n - T_m(u_n))\, dx.$$

Thanks to Lemma 2.6 we have

$$\int_{\{m<|u_n|\leq m+1\}} \phi(T_n(u_n)).\nabla u_n\,dx = 0,$$

which implies, by using the fact that $g_n(x,u_n,\nabla u_n)\ T_1(u_n - T_m(u_n)) \geq 0$ on the subset $\{x\in\Omega : |u_n|\geq m\}$,

$$\int_{\{m\leq|u_n|\leq m+1\}} a(x,u_n,\nabla u_n).\nabla u_n\,dx \leq \int_{\{|u_n|>m\}} |f_n|\,dx, \tag{27}$$

consequently

$$\Big|\int_\Omega a(x,u_n,\nabla u_n).\nabla u_n\ \mu(\theta_n^j)\ \rho_m'(u_n)\,dx\ \Big| \leq\ \mu(2k) \int_{\{|u_n|>m\}} |f_n|\,dx.$$

Combining this inequality with (24), (25) and (26) we obtain

$$\begin{aligned}\int_\Omega a(x,u_n,\nabla u_n).\nabla z_{n,m}^j \quad \geq \quad & - \int_{\Omega\setminus\Omega_s} l_k.\nabla T_k(u)\,dx - \mu(2k)\int_{\{|u_n|\geq m\}} |f_n|\,dx \\ & + \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\ T_k(v_j)\chi_j^s)] \\ & \times[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s]\ \mu'(\theta_n^j)\,dx\ +\ \varepsilon_6(n,j). \end{aligned}\tag{28}$$

Concerning the second term of the left-hand side of (22), we have

$$\begin{aligned} \Big|\int_{\{|u_n|\leq k\}} g_n(x,u_n,\nabla u_n) z_{n,m}^j\,dx\Big| &= \Big|\int_{\{|u_n|\leq k\}} g_n(x,T_k(u_n),\nabla T_k(u_n))\ \mu(\theta_n^j)\,dx\Big| \\ &\leq \int_\Omega b(k)\ c'(x)\ |\mu(\theta_n^j)|\,dx + b(k)\int_\Omega \varphi(x,|\nabla T_k(u_n)|)\ |\mu(\theta_n^j)|\,dx \\ &\leq \varepsilon_7(n,j) + \frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n)).\nabla T_k(u_n)\ |\mu(\theta_n^j)|\,dx. \end{aligned}$$

We can write the last term of the last side of this inequality as

$$\begin{aligned} \frac{b(k)}{\alpha}\int_\Omega & [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)] \\ & \times[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s]\ |\mu(\theta_n^j)|\,dx \\ & + \frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s).(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s)\ |\mu(\theta_n^j)|\,dx \\ & - \frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n)).\nabla T_k(v_j)\chi_j^s\ |\mu(\theta_n^j)|\,dx \end{aligned}\tag{29}$$

we argue as above to show that

$$\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s).(\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^s)\ |\mu(\theta_n^j)|\,dx=\varepsilon_8(n,j)$$

and

$$\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n)).\nabla T_k(v_j)\chi_j^s\ |\mu(\theta_n^j)|\,dx=\varepsilon_9(n,j)$$

then

$$\begin{aligned}|\int_{\{|u_n|\leq k\}} & g_n(x,u_n,\nabla u_n)z_{n,m}^j\,dx\ | \\ &\leq \frac{b(k)}{\alpha}\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)] \\ &\quad\times[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^s]\ |\mu(\theta_n^j)|\,dx+\varepsilon_{10}(n,j).\end{aligned}$$

Combining this with (22) and (28), we obtain

$$\begin{aligned}&\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(v_j)\,\chi_j^s)].[\nabla T_k(u_n)-\nabla T_k(v_j)\,\chi_j^s] \\ &\times\left(\mu'(\theta_n^j)-\frac{b(k)}{\alpha}|\mu(\theta_n^j)|\right)dx\leq\varepsilon_{11}(n,j)+\int_{\Omega\setminus\Omega_s} l_k.\nabla T_k(u)\,dx+\mu(2k)\int_{\{|u_n|\geq m\}}|f_n|\,dx\end{aligned}$$

and by using (20) we deduce that

$$\begin{aligned}&\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)].[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^s]\,dx \\ &\qquad\leq 2\,\varepsilon_{11}(n,j)+2\int_{\Omega\setminus\Omega_s} l_k.\nabla T_k(u)\,dx+2\,\mu(2k)\int_{\{|u_n|\geq m\}}|f_n|\,dx. \qquad (30)\end{aligned}$$

On the other hand

$$\begin{aligned}&\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)\chi^s)]\times[\nabla T_k(u_n)-\nabla T_k(u)\chi^s]\,dx \\ &=\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(v_j)\,\chi_j^s)]\times[\nabla T_k(u_n)-\nabla T_k(v_j)\,\chi_j^s]\,dx \\ &\quad+\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n)).[\nabla T_k(v_j)\,\chi_j^s-\nabla T_k(u)\,\chi^s]\,dx \\ &\quad-\int_\Omega a(x,T_k(u_n),\nabla T_k(u)\,\chi^s).[\nabla T_k(u_n)-\nabla T_k(u)\,\chi^s]\,dx \\ &\quad+\int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\,\chi_j^s).[\nabla T_k(u_n)-\nabla T_k(v_j)\,\chi_j^s]\,dx.\end{aligned}$$

We shall pass to the limit in n and in j in the last three terms of the right-hand side of the above equality. Similar tools as in (23) and (29) gives

$$\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)).[\nabla T_k(v_j)\chi_j^s - \nabla T_k(u)\chi^s]\,dx = \varepsilon_{12}(n,j),$$

$$\int_\Omega a(x, T_k(u_n), \nabla T_k(u)\chi^s).[\nabla T_k(u_n) - \nabla T_k(u)\chi^s]\,dx = \varepsilon_{13}(n,j),$$

and

$$\int_\Omega a(x, T_k(u_n), \nabla T_k(v_j)\chi_j^s).[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s]\,dx = \varepsilon_{14}(n,j). \tag{31}$$

Which implies that

$$\begin{aligned}
&\int_\Omega [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi^s)].[\nabla T_k(u_n) - \nabla T_k(u)\chi^s]\,dx \\
&= \int_\Omega [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j)\chi_j^s)].[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s]\,dx \\
&\quad + \varepsilon_{15}(n,j).
\end{aligned}$$

For $r \leq s$, one has

$$\begin{aligned}
0 &\leq \int_{\Omega_r} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))].[\nabla T_k(u_n) - \nabla T_k(u)]\,dx \\
&\leq \int_{\Omega_s} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))].[\nabla T_k(u_n) - \nabla T_k(u)]\,dx \\
&= \int_{\Omega_s} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi^s)].[\nabla T_k(u_n) - \nabla T_k(u)\chi^s]\,dx \\
&\leq \int_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi^s)].[\nabla T_k(u_n) - \nabla T_k(u)\chi^s]\,dx \\
&= \int_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j)\chi_j^s)].[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s]\,dx \\
&\quad + \varepsilon_{15}(n,j) \\
&\leq \varepsilon_{16}(n,j) + 2 \int_{\Omega\setminus\Omega_s} l_k.\nabla T_k(u)\,dx + 2\,\mu(2k) \int_{\{|u_n|\geq m\}} |f_n|\,dx.
\end{aligned}$$

This implies that, by passing at first to the limit sup over n and then over j,

$$\begin{aligned}
0 \leq \limsup_{n\to\infty} &\int_{\Omega_r} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))].[\nabla T_k(u_n) - \nabla T_k(u)]\,dx \\
&\leq 2 \int_{\Omega\setminus\Omega_s} l_k.\nabla T_k(u)\,dx + 2\,\mu(2k) \int_{\{|u|\geq m\}} |f|\,dx.
\end{aligned}$$

Letting s and $m \to \infty$ and using the fact that $l_k.\nabla T_k(u) \in L^1(\Omega)$ we get, since $|\Omega\backslash\Omega_s| \to 0$ and $|\{|u| \geq m\}| \to 0$,

$$\int_{\Omega_r} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))].[\nabla T_k(u_n) - \nabla T_k(u)]\, dx \to 0 \text{ as } n \to \infty.$$

As in [26], we deduce that there exists a subsequence, still denoted by u_n, such that

$$\nabla u_n \to \nabla u \text{ a.e in } \Omega, \tag{32}$$

which implies that

$$a(x, T_k(u_n), \nabla T_k(u_n)) \rightharpoonup a(x, T_k(u), \nabla T_k(u)) \text{ weakly in } (L_\psi(\Omega))^N \text{ for } \sigma(\Pi L_\psi, \Pi E_\varphi), \forall k > 0. \tag{33}$$

Step 3: **Modular convergence of the truncations**.
Going back to the equation (30), we can write

$$\begin{aligned}\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)).\nabla T_k(u_n)\, dx \leq & \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)).\nabla T_k(v_j)\, \chi_j^s\, dx \\ & + \int_\Omega a(x, T_k(u_n), \nabla T_k(v_j)\, \chi_j^s).[\nabla T_k(u_n) - \nabla T_k(v_j)\, \chi_j^s]\, dx \\ & + 2\,\varepsilon_{11}(n, j) + 2 \int_{\Omega\backslash\Omega_s} l_k.\nabla T_k(u)\, dx + 2\,\mu(2k) \int_{\{|u_n| \geq m\}} |f_n|\, dx,\end{aligned}$$

then, by using (31), we have

$$\begin{aligned}\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)).\nabla T_k(u_n)\, dx \leq & \,\varepsilon_{17}(n, j) + \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)).\nabla T_k(v_j)\; \chi_j^s\, dx \\ & + 2 \int_{\Omega\backslash\Omega_s} l_k.\nabla T_k(u)\, dx + 2\,\mu(2k) \int_{\{|u_n| \geq m\}} |f_n|\, dx.\end{aligned}$$

Passing to the limit sup over n in both sides of this inequality yields

$$\begin{aligned}\limsup_{n\to\infty} \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)).\nabla T_k(u_n)\, dx \leq & \int_\Omega a(x, T_k(u), \nabla T_k(u)).\nabla T_k(v_j)\; \chi_j^s\, dx \\ & + \lim_{n\to\infty} \varepsilon_{17}(n, j) + 2 \int_{\Omega\backslash\Omega_s} l_k.\nabla T_k(u)\, dx + 2\; \mu(2k) \int_{\{|u| \geq m\}} |f|\, dx,\end{aligned}$$

in which, we can pass to the limit in j, to obtain

$$\begin{aligned}\limsup_{n\to\infty} \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)).\nabla T_k(u_n)\, dx \leq & \int_\Omega a(x, T_k(u), \nabla T_k(u)).\nabla T_k(u)\; \chi^s\, dx \\ & + 2 \int_{\Omega\backslash\Omega_s} l_k.\nabla T_k(u)\, dx + 2\; \mu(2k) \int_{\{|u| \geq m\}} |f|\, dx.\end{aligned}$$

Letting s and $m \to \infty$ gives

$$\limsup_{n\to\infty} \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)).\nabla T_k(u_n)\, dx \leq \int_\Omega a(x, T_k(u), \nabla T_k(u)).\nabla T_k(u)\, dx$$

then by using Fatou's Lemma we have

$$\int_\Omega a(x, T_k(u), \nabla T_k(u)).\nabla T_k(u)\, dx \leq \liminf_{n\to\infty} \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)).\nabla T_k(u_n)\, dx,$$

consequently

$$\lim_{n\to\infty} \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)).\nabla T_k(u_n)\, dx = \int_\Omega a(x, T_k(u), \nabla T_k(u)).\nabla T_k(u)\, dx$$

and, by using Lemma 2.4, we conclude that

$$a(x, T_k(u_n), \nabla T_k(u_n)).\nabla T_k(u_n) \to a(x, T_k(u), \nabla T_k(u)).\nabla T_k(u) \text{ in } L^1(\Omega). \qquad (34)$$

The convexity of the Musielak-Orlicz function φ and (9) allow us to get

$$\begin{aligned}\varphi\left(x, \frac{|\nabla T_k(u_n) - \nabla T_k(u)|}{2}\right) \leq & \frac{1}{2\alpha} a(x, T_k(u_n), \nabla T_k(u_n)).\nabla T_k(u_n) \\ & + \frac{1}{2\alpha} a(x, T_k(u), \nabla T_k(u)).\nabla T_k(u),\end{aligned}$$

and by (34) we obtain

$$\lim_{|E|\to 0} \sup_n \int_E \varphi\left(x, \frac{|\nabla T_k(u_n) - \nabla T_k(u)|}{2}\right) dx = 0$$

which implies, by using Vitali's theorem, that

$$T_k(u_n) \to T_k(u) \text{ in } W_0^1 L_\varphi(\Omega) \text{ for the modular convergence } \forall k > 0.$$

Step 4 : **Equi-integrability of the non-linearities**.
We shall prove that $g_n(x, u_n, \nabla u_n) \to g(x, u, \nabla u)$ strongly in $L^1(\Omega)$ by using Vitali's theorem. Thanks to (32) we have $g_n(x, u_n, \nabla u_n) \to g(x, u, \nabla u)$ a.e in Ω, so it suffices to prove that $g_n(x, u_n, \nabla u_n)$ is uniformly equi-integrable in Ω.
Let $E \subset \Omega$ be a measurable subset of Ω. We have for any $m > 1$,

$$\int_E |g_n(x, u_n, \nabla u_n)|\, dx = \int_{E\cap\{|u_n|\leq m\}} |g_n(x, u_n, \nabla u_n)|\, dx + \int_{E\cap\{|u_n|>m\}} |g_n(x, u_n, \nabla u_n)|\, dx.$$

Taking

$$T_1(u_n - T_{m-1}(u_n)) = \begin{cases} 0 & \text{if } |u_n| \leq m-1 \\ u_n - (m-1)sgn(u_n) & \text{if } m-1 \leq |u_n| \leq m \\ sgn(u_n) & \text{if } |u_n| > m \end{cases}$$

as test function in $(\mathcal{P}_n)$, gives

$$\int_{\{m-1<|u_n|\le m\}} a(x,u_n,\nabla u_n).\nabla u_n\,dx + \int_{\{m-1<|u_n|\le m\}} \phi(T_n(u_n)).\nabla u_n\,dx$$

$$+\int_{\{|u_n|>m-1\}} g_n(x,u_n,\nabla u_n)\,T_1(u_n-T_{m-1}(u_n))\,dx = \int_{\{|u_n|>m-1\}} f_n\,T_1(u_n-T_{m-1}(u_n))\,dx$$

consequently

$$\int_{\{|u_n|>m\}} |g_n(x,u_n,\nabla u_n)|\,dx \le \int_{\{|u_n|>m-1\}} |f_n|\,dx.$$

Let $\varepsilon>0$, there exists $m=m(\varepsilon)>1$ such that

$$\int_{E\cap\{|u_n|>m\}} |g_n(x,u_n,\nabla u_n)|\;dx \le \frac{\varepsilon}{2},\quad \forall n.$$

On the other hand

$$\begin{aligned}\int_{E\cap\{|u_n|\le m\}} |g_n(x,u_n,\nabla u_n)|\,dx &\le \int_E |g_n(x,T_m(u_n),\nabla T_m(u_n))|\,dx\\ &\le b(m)\int_E (c'(x)+\varphi(x,|\nabla T_m(u_n)|))\,dx\\ &\le \frac{b(m)}{\alpha}\int_E a(x,T_m(u_n),\nabla T_m(u_n))\;\nabla T_m(u_n)\,dx\\ &\quad +b(m)\int_E c'(x)\,dx.\end{aligned}$$

By virtue of the strong convergence (34) and the fact that $c'(.)\in L^1(\Omega)$, there exists $\eta>0$, such that

$$|E|<\eta \;\text{ implies } \int_{E\cap\{|u_n|\le m\}} |g_n(x,u_n,\nabla u_n)|\,dx \le \frac{\varepsilon}{2},\quad \forall n.$$

So that

$$|E|<\eta \;\text{ implies } \int_E |g_n(x,u_n,\nabla u_n)|\,dx \le \varepsilon,\quad \forall n,$$

which shows that $g_n(x,u_n,\nabla u_n)$ is uniformly equi-integrable in Ω. By Vitali's theorem, we conclude that $g(x,u,\nabla u)\in L^1(\Omega)$ and

$$g_n(x,u_n,\nabla u_n)\to g(x,u,\nabla u) \text{ strongly in } L^1(\Omega). \tag{35}$$

Step 5 : **Passage to the limit**.
Turning to the inequality (27), we have for the first term

$$\begin{aligned}\int_{\{m\le|u_n|\le m+1\}} a(x,u_n,\nabla u_n).\nabla u_n\,dx &= \int_\Omega a(x,u_n,\nabla u_n).(\nabla T_{m+1}(u_n)-\nabla T_m(u_n))\,dx\\ &= \int_\Omega a(x,T_{m+1}(u_n),\nabla T_{m+1}(u_n)).\nabla T_{m+1}(u_n)\,dx\\ &\quad -\int_\Omega a(x,T_m(u_n),\nabla T_m(u_n)).\nabla T_m(u_n)\,dx.\end{aligned}$$

then by (34) we obtain

$$\begin{aligned}\lim_{n\to\infty}\int_{\{m\le|u_n|\le m+1\}} a(x,u_n,\nabla u_n).\nabla u_n\,dx &= \int_\Omega a(x,T_{m+1}(u),\nabla T_{m+1}(u)).\nabla T_{m+1}(u)\,dx\\ &\quad -\int_\Omega a(x,T_m(u),\nabla T_m(u)).\nabla T_m(u)\,dx\\ &= \int_\Omega a(x,u,\nabla u).(\nabla T_{m+1}(u)-\nabla T_m(u))\,dx\\ &= \int_{\{m\le|u|\le m+1\}} a(x,u,\nabla u).\nabla u\,dx.\end{aligned}$$

Consequently, by letting n to infinity in (27) we get

$$\int_{\{m\le|u|\le m+1\}} a(x,u,\nabla u).\nabla u\,dx \le \int_{\{|u|\ge m\}} |f|\,dx$$

in which we can pass to the limit in m to obtain

$$\lim_{m\to\infty}\int_{\{m\le|u|\le m+1\}} a(x,u,\nabla u).\nabla u\,dx = 0. \tag{36}$$

Now, from (34) and Lemma 2.4 we deduce that

$$a(x,u_n,\nabla u_n).\ \nabla u_n \to a(x,u,\nabla u).\ \nabla u \ \text{ in } L^1(\Omega) \tag{37}$$

Let $h\in\mathcal{C}^1_c(\mathbb{R})$ and $\theta\in\mathcal{D}(\Omega)$. Taking $h(u_n)\theta$ as test function in $(\mathcal{P}_n)$, we get

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n).\nabla u_n h'(u_n)\ \theta\,dx + \int_\Omega a(x,u_n,\nabla u_n).\nabla h(u_n)\ \theta\,dx\\ &+\int_\Omega \phi_n(u_n).\nabla(h(u_n)\ \theta)\,dx + \int_\Omega g_n(x,u_n,\nabla u_n)h(u_n)\ \theta\,dx = \int_\Omega f_n h(u_n)\ \theta\,dx\end{aligned} \tag{38}$$

Since h and h' have compact support in $\mathbb{R}$, there exists $\rho>0$ such that supp$h\subset[-\rho,\rho]$ and supp$h'\subset[-\rho,\rho]$, then for $n>\rho$ we can write

$$\phi_n(t)h(t)=\phi(T_n(t))h(t)=\phi(T_\rho(t))h(t)$$
$$\phi_n(t)h'(t)=\phi(T_n(t))h'(t)=\phi(T_\rho(t))h'(t)$$

Moreover, the functions ϕh and $\phi h'$ belong to $(\mathcal{C}^0(\mathbb{R}) \cap L^\infty(\mathbb{R}))^N$.
Since $u_n \in W_0^1 L_\varphi(\Omega)$ there exists two positive constants η_1, η_2 such that

$$\int_\Omega \varphi(x, \frac{|\nabla u_n|}{\eta_1})\, dx \leq \eta_2.$$

Let τ be a positive constant such that $\|h(u_n)|\nabla\theta|\|_\infty \leq \tau$ and $\|h'(u_n)\theta\|_\infty \leq \tau$.
For μ large enough, we have

$$\begin{aligned}\int_\Omega \varphi\left(x, \frac{|\nabla(h(u_n)\theta)|}{\mu}\right) dx &\leq \int_\Omega \varphi\left(x, \frac{|h(u_n)\nabla\theta| + |h'(u_n)\theta||\nabla u_n|}{\mu}\right) dx \\ &\leq \int_\Omega \varphi(x, \frac{\tau + \frac{\tau\eta_1|\nabla u_n|}{\eta_1}}{\mu})\, dx \\ &\leq \int_\Omega \varphi(x, \frac{\tau}{\mu})\, dx + \frac{\tau\eta_1}{\mu}\int_\Omega \varphi(x, \frac{|\nabla u_n|}{\eta_1})\, dx \\ &\leq \int_\Omega \varphi(x,1)\, dx + \frac{\tau\eta_1\eta_2}{\mu} \leq C\end{aligned}$$

which implies that $h(u_n)\ \theta$ is bounded in $W_0^1 L_\varphi(\Omega)$ and then we deduce that

$$h(u_n)\ \theta \rightharpoonup h(u)\ \theta \text{ weakly in } W_0^1 L_\varphi(\Omega) \text{ for } \sigma(\Pi L_\varphi, \Pi E_\psi). \tag{39}$$

On the other hand, for any measurable subset E of Ω we have

$$\begin{aligned}\|\phi(T_\rho(u_n))\chi_E\|_\psi &= \sup_{\|v\|_\varphi \leq 1} |\int_E \phi(T_\rho(u_n))\, v\, dx| \\ &\leq c_\rho \sup_{\|v\|_\varphi \leq 1} \|\chi_E\|_\psi \|v\|_\varphi \\ &\leq c_\rho \frac{1}{M^{-1}(\frac{1}{|E|})}\end{aligned}$$

where $c_\rho = \max_{|t|\leq\rho} \phi(t)$ and M is the N-function defined by $M = \sup_{x\in\Omega} \psi(x,t)$, then

$$\lim_{|E|\to 0} \sup_n \|\phi(T_\rho(u_n))\chi_E\|_\psi = 0$$

consequently from (19) and by using [25, Lemma 11.2] we obtain

$$\phi(T_\rho(u_n)) \to \phi(T_\rho(u)) \text{ strongly in } (E_\psi(\Omega))^N. \tag{40}$$

It follows that by (39) and (40)

$$\int_\Omega \phi_n(u_n).\nabla(h(u_n)\ \theta)\, dx \to \int_\Omega \phi(u).\nabla(h(u)\ \theta)\, dx \text{ as } n \to +\infty.$$

For the first term of (38), we have

$$|a(x, u_n, \nabla u_n).\nabla u_n\ h'(u_n)\ \theta| \leq \tau\ a(x, u_n, \nabla u_n).\nabla u_n$$

So, by using Vitali's theorem and (37) we get

$$\int_\Omega a(x,u_n,\nabla u_n).\nabla u_n\ h'(u_n)\ \theta\,dx \to \int_\Omega a(x,u,\nabla u).\nabla u\ h'(u)\ \theta\,dx$$

Concerning the second term of (38), we have

$$h(u_n)\nabla\theta \to h(u)\nabla\theta \text{ strongly in } (E_\varphi(\Omega))^N$$

and

$$a(x,u_n,\nabla u_n) \rightharpoonup a(x,u,\nabla u) \text{ weakly in } (L_\psi(\Omega))^N \text{ for } \sigma(\Pi L_\psi, \Pi E_\varphi)$$

then

$$\int_\Omega a(x,u_n,\nabla u_n).\nabla\theta h(u_n)\,dx \to \int_\Omega a(x,u,\nabla u).\nabla\theta h(u)\,dx$$

Since $h(u_n)\ \theta \rightharpoonup h(u)\ \theta$ weakly in $L^\infty(\Omega)$ for $\sigma^*(L^\infty, L^1)$ and by using (35), we have

$$\int_\Omega g_n(x,u_n,\nabla u_n)\ h(u_n)\ \theta\,dx \to \int_\Omega g(x,u,\nabla u)\ h(u)\ \theta\,dx$$

and

$$\int_\Omega f_n\ h(u_n)\ \theta\,dx \to \int_\Omega f\ h(u)\ \theta\,dx.$$

Finally, we can easily pass to the limit in each term of (38) and obtain

$$\int_\Omega a(x,u,\nabla u).[h'(u)\ \theta\ \nabla u + h(u)\ \nabla\theta]\,dx + \int_\Omega \phi(u)h'(u)\ \theta.\nabla u\,dx$$
$$+ \int_\Omega \phi(u)h(u).\nabla\theta\,dx + \int_\Omega g(x,u,\nabla u)\ h(u)\ \theta\,dx = \int_\Omega f\ h(u)\ \theta\,dx$$

for all $h \in \mathcal{C}_c^1(\mathbb{R})$, and for all $\theta \in \mathcal{D}(\Omega)$, which proves the Theorem 3.1. □

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