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Existence of renormalized solutions for a nonlinear elliptic equation in Musielak framework and L^1 data

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Abstract. In this paper, we prove existence result of renormalized solutions in the setting of Musielak-Orlicz spaces $W_0^1L_\varphi(\Omega)$ for the following strongly nonlinear Dirichlet problem

$$A(u) + g(x, u, \nabla u) = f \quad \text{in } \Omega,$$

where A is a Leray-Lions operator acting from its domain $D(A) \subset W_0^1L_\varphi(\Omega)$ into its dual, while $g(x, u, \nabla u)$ is a nonlinear term having a growth conditions with respect only to ∇u, and does not satisfy any sign condition. The right-hand side f belongs to $L^1(\Omega)$.
A modular-inequality of Poincaré type in this setting is also proved (see Lemma 2.5).

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1. Introduction

We consider a bounded open subset Ω of $\mathbb{R}^N$, $(N > 2)$. Let

$$A(u) = -\text{div}a(x, u, \nabla u),$$

be a Leray-Lions operator defined from the space $W_0^1L_\varphi(\Omega)$ into its dual $W^{-1}L_\psi(\Omega)$. Our aim is to prove the existence of renormalized solutions u to the non-linear elliptic problem

$$\begin{cases} A(u) + g(x, u, \nabla u) = f \in L^1(\Omega), & \text{in } \Omega, \\ u \equiv 0, \quad \partial\Omega, \end{cases} \tag{1}$$

where $f \in L^1(\Omega)$ and g is a non-linear lower order term satisfying a growth condition of the following from

$$|g(x, s, \xi)| \leq c(x) + b(|s|)\varphi(x, |\xi|).$$

And without any sign condition, in the setting of the Musielak-Orlicz space $W_0^1L_\varphi(\Omega)$, without any restriction on the Musielak-Orlicz function φ (i.e., without the Δ_2-Condition).

We recall that the notion of renormalized solutions was introduced by Lions and Diperna [18] for the study of Boltzmann equations. This notion was then adapted to the study of the problem (1) by Boccardo, Giachetti, Diaz and Murat in [14], Lions and Murat [25] and Murat [28,27] to non-linear elliptic problem and by Lions [26] to evolution problems in fluids mechanics. Recently we refer to [16,14,12,13,17] for more details.

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In the classical Sobolev space $W_0^{1,p}(\Omega)$, Benkirane and Youssfi have studied (1) where the non linearity term g depends only on x and u and the right hand side f belongs to the dual space, Porretta in [30] has studied the problem (1) where the right hand side is a measure, Boccardo, Murat and Puel, have studied the problem (1) without sign condition in the particular case where $g(x,s,\xi) = \lambda s - |\xi|^2$, $\lambda > 0$, also in [31] Rakotoson and Temam have proved the existence of a weak solution for the problem (1).

In the sitting of Lebesgue of variable exponent, Bendahmane and Wittbold in [6] proved the existence and uniqueness of renormalized solution to the problem (1) in the particular case $a(x,s,\xi) = |\xi|^{p(x)-2}\xi$, $g \equiv 0$, Azroul, Benboubker, and Rhoudaf in [5] have studied the problem (1) where the right hand side is measure.

In the Orlicz spaces framework, various authors have studied the existence of solution of (1). In the variational case, Gossez [21] solved the problem (1) in the case where g depends only on x and u, Benkirane and Elmahi in [9,8] have studied (1) by making some restriction and g depends also on ∇u, Elmahi and Meskine in [20] proved the existence of solutions for the problem (1), without assuming the Δ_2 condition on the N-function. In the case where $f \in L^1(\Omega)$, Aharouch, Benkirane, Rhoudaf have proved, in [2] the existence of solutions of problem (1) without assuming the Δ_2 condition and the sign condition on the non linearity g.

In Musielak-Orlicz spaces, Benkirane and Sidi El Vally in [10] have proved the existence results of (1) where the nonlinearity g depends only on x and u, recently Benkirane, Blali and Sidi El Vally in [7] have solved (1) in the case where the Musielak-Orlicz complementary function to φ satisfies the Δ_2-condition, Ait Khellou, Benkirane, Douiri in [4,3] have proved the existence of solution of (1), without assuming the Δ_2-condition.

The paper is organized as follows: after introduction in section 1, we give in section 2 some preliminaries and some technical lemmas needed in our paper, in the section 3 we state the essential assumptions and our main result and his prove.

2. Preliminary

Let Ω be an open set in $\mathbb{R}^N$ and let φ be a real-valued function defined in $\Omega \times \mathbb{R}_+$, and satisfiying the following conditions:

a): $\varphi(x,\cdot)$ is an N-function $\Big($convex, increasing, continous, $\varphi(x,0) = 0, \varphi(x,t) > 0$, $\forall t > 0,\ \lim\limits_{t\longrightarrow 0} \sup\limits_{x\in\Omega} \dfrac{\varphi(x,t)}{t} = 0,\ \lim\limits_{t\longrightarrow \infty} \inf\limits_{x\in\Omega} \dfrac{\varphi(x,t)}{t} = \infty\Big)$.

b): $\varphi(\cdot,t)$ is a measurable function.

A function φ, which satisfies the conditions *a*) and *b*) is called Musielak-Orlicz function.

For a Musielak-Orlicz function φ we put $\varphi_x(t) = \varphi(x,t)$ and we associate its nonnegative reciprocal function φ_x^{-1}, with respect to t that is

$$\varphi_x^{-1}(\varphi(x,t)) = \varphi(x,\varphi_x^{-1}(t)) = t.$$

The Musielak-Orlicz function φ is said to satisfy the Δ_2-condition if for some $k > 0$; and a non negative function h; integrable in Ω we have

$$\varphi(x,2t) \le k\varphi(x,t) + h(x) \text{ for all } x \in \Omega \text{ and } t \ge 0. \tag{2}$$

When (2) holds only for $t \geq t_0 > 0$; then φ said satisfies Δ_2 near infinity.
Let φ and γ be two Musielak-Orlicz functions, we say that φ dominate γ, and we write $\gamma \prec \varphi$, near infinity (resp. globally) if there exist two positive constants c and t_0 such that for almost all $x \in \Omega$

$$\gamma(x,t) \leq \varphi(x,ct) \text{ for all } t \geq t_0, \quad (\text{ resp. for all } t \geq 0 \text{ i.e. } t_0 = 0).$$

We say that γ grows essentially less rapidly than φ at 0 (resp. near infinity), and we write $\gamma \prec\prec \varphi$, If for every positive constant c we have

$$\lim_{t \longrightarrow 0} \left(\sup_{x \in \Omega} \frac{\gamma(x,ct)}{\varphi(x,t)} \right) = 0, \quad (\text{resp. } \lim_{t \longrightarrow \infty} \left(\sup_{x \in \Omega} \frac{\gamma(x,ct)}{\varphi(x,t)} \right) = 0).$$

Remark 2.1. [10] If $\gamma \prec\prec \varphi$ near infinity, then $\forall \varepsilon > 0$ there exist $k(\varepsilon) > 0$ such that for almost all $x \in \Omega$ we have

$$\gamma(x,t) \leq k(\varepsilon)\varphi(x,\varepsilon t), \qquad \text{for all } t \geq 0. \tag{3}$$

We define the functional

$$\rho_{\varphi,\Omega}(u) = \int_\Omega \varphi(x, |u(x)|)dx.$$

where $u : \Omega \longrightarrow \mathbb{R}$ a Lebesgue measurable function. In the following the measurability of a function $u : \Omega \longrightarrow \mathbb{R}$ means the Lebesgue measurability.
The set

$$K_\varphi(\Omega) = \Big\{ u : \Omega \longrightarrow \mathbb{R} \text{ measurable} \Big/ \rho_{\varphi,\Omega}(u) < +\infty \Big\}.$$

is called the generalized Orlicz class.
The Musielak-Orlicz space (the generalized Orlicz spaces) $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is, $L_\varphi(\Omega)$ is the smallest linear space containing the set $K_\varphi(\Omega)$.
Equivalently

$$L_\varphi(\Omega) = \Big\{ u : \Omega \longrightarrow \mathbb{R} \text{ measurable} \Big/ \rho_{\varphi,\Omega}\Big(\frac{|u(x)|}{\lambda}\Big) < +\infty, \text{ for some } \lambda > 0 \Big\}.$$

Let

$$\psi(x,s) = \sup_{t \geq 0} \{st - \varphi(x,t)\}.$$

that is, ψ is the Musielak-Orlicz function complementary to φ in the sens of Young with respect to the variable s.
In the space $L_\varphi(\Omega)$ we define the following two norms :

$$\|u\|_{\varphi,\Omega} = \inf \Big\{ \lambda > 0 / \int_\Omega \varphi\Big(x, \frac{|u(x)|}{\lambda}\Big)dx \leq 1 \Big\}.$$

which is called the Luxemburg norm and the so called Orlicz norm by :

$$|||u|||_{\varphi,\Omega} = \sup_{\|v\|_\psi \leq 1} \int_\Omega |u(x)v(x)|dx.$$

where ψ is the Musielak-Orlicz function complementary to φ. There two norms are equivalent [29].
The closure in $L_\varphi(\Omega)$ of the bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_\varphi(\Omega)$. It is a separable space.

We say that sequence of functions $u_n \in L_\varphi(\Omega)$ is modular convergent to $u \in L_\varphi(\Omega)$ if there exists a constant $k > 0$ such that

$$\lim_{n\to\infty} \rho_{\varphi,\Omega}\Big(\frac{u_n - u}{k}\Big) = 0.$$

For any fixed nonnegative integer m we define

$$W^m L_\varphi(\Omega) = \Big\{ u \in L_\varphi(\Omega) : \forall |\alpha| \le m,\ D^\alpha u \in L_\varphi(\Omega) \Big\}.$$

and

$$W^m E_\varphi(\Omega) = \Big\{ u \in E_\varphi(\Omega) : \forall |\alpha| \le m,\ D^\alpha u \in E_\varphi(\Omega) \Big\}.$$

where $\alpha = (\alpha_1, ..., \alpha_n)$ with nonnegative integers α_i, $|\alpha| = |\alpha_1| + ... + |\alpha_n|$ and $D^\alpha u$ denote the distributional derivatives. The space $W^m L_\varphi(\Omega)$ is called the Musielak-Orlicz Sobolev space.
Let

$$\overline{\rho}_{\varphi,\Omega}(u) = \sum_{|\alpha|\le m} \rho_{\varphi,\Omega}\Big(D^\alpha u\Big) \text{ and } \|u\|^m_{\varphi,\Omega} = \inf\Big\{\lambda > 0 : \overline{\rho}_{\varphi,\Omega}\Big(\frac{u}{\lambda}\Big) \le 1\Big\}.$$

For $u \in W^m L_\varphi(\Omega)$ there functionals are a convex modular and a norm on $W^m L_\varphi(\Omega)$, respectively, and the pair $\Big(W^m L_\varphi(\Omega), \|\cdot\|^m_{\varphi,\Omega}\Big)$ is a Banach space if φ satisfies the following condition [29] :

$$\text{there exist a constant } c > 0 \text{ such that } \inf_{x\in\Omega} \varphi(x, 1) \ge c. \tag{4}$$

The space $W^m L_\varphi(\Omega)$ will always be identified to a subspace of the product $\prod_{|\alpha|\le m} L_\varphi(\Omega) = \Pi L_\varphi$, this subspace is $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closed.

We denote by $\mathcal{D}(\Omega)$ the space of infinitely smooth functions with compact support in Ω and by $\mathcal{D}(\overline{\Omega}))$ the restriction of $\mathcal{D}(\mathbb{R}^N)$ on Ω.

Let $W^m_0 L\varphi(\Omega)$ be the $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closure of $D(\Omega)$ in $W^m L_\varphi(\Omega)$.
Let $W^m E\varphi(\Omega)$ the space of functions u such that u and its distribution derivatives up to order m lie in $E_\varphi(\Omega)$, and $W^m_0 E\varphi(\Omega)$ is the (norm) closure of $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$.
The following spaces of distributions will also be used :

$$W^{-m} L_\psi(\Omega) = \Big\{ f \in \mathcal{D}'(\Omega);\ f = \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in L_\psi(\Omega)\Big\}.$$

and

$$W^{-m} E_\psi(\Omega) = \Big\{ f \in \mathcal{D}'(\Omega);\ f = \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in E_\psi(\Omega)\Big\}.$$

We say that a sequence of functions $u_n \in W^m L_\varphi(\Omega)$ is modular convergent to $u \in W^m L_\varphi(\Omega)$ if there exists a constant $k > 0$ such that

$$\lim_{n\to\infty} \overline{\rho}_{\varphi,\Omega}\Big(\frac{u_n - u}{k}\Big) = 0.$$

For two Musielak-Orlicz functions φ and ψ the following inequality is called the Young inequality [29]:

$$ts \le \varphi(x, t) + \psi(x, s), \quad \forall t, s \ge 0, x \in \Omega. \tag{5}$$

This inequality implies the inequality

$$|||u|||_{\varphi,\Omega} \leq \rho_{\varphi,\Omega}(u) + 1. \tag{6}$$

In $L_\varphi(\Omega)$ we have the relation between the norm and the modular

$$\|u\|_{\varphi,\Omega} \leq \rho_{\varphi,\Omega}(u) \text{ if } \|u\|_{\varphi,\Omega} > 1. \tag{7}$$

$$\|u\|_{\varphi,\Omega} \geq \rho_{\varphi,\Omega}(u) \text{ if } \|u\|_{\varphi,\Omega} \leq 1. \tag{8}$$

For two complementary Musielak-Orlicz functions φ and ψ let $u \in L_\varphi(\Omega)$ and $v \in L_\psi(\Omega)$ we have the Holder inequality

$$\left| \int_\Omega u(x)v(x)dx \right| \leq \|u\|_{\varphi,\Omega} \|v\|_{\psi,\Omega}. \tag{9}$$

Lemma 2.1. [11] *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and let φ and ψ be two complementary Musielak-Orlicz functions which satisfy the following conditions:*

i): *There exist a constant $c > 0$ such that $\inf_{x\in\Omega} \varphi(x,1) \geq c$.*

ii): *There exist a constant $A > 0$ such that for all $x, y \in \Omega$ with $|x - y| \leq \frac{1}{2}$ we have*

$$\frac{\varphi(x,t)}{\varphi(y,t)} \leq t^{\overline{\frac{A}{log\left(\frac{1}{|x-y|}\right)}}}, \quad \forall t \geq 1. \tag{10}$$

iii):

$$\text{If } D \subset \Omega \text{ is a bounded measurable set, then } \int_D \varphi(x,1)dx < \infty. \tag{11}$$

iv): *There exist a constant $C > 0$ such that $\psi(x,1) \leq C$ a.e in Ω.*

Under this assumptions, $\mathcal{D}(\Omega)$ is dense in $L_\varphi(\Omega)$ with respect to the modular topology, $\mathcal{D}(\Omega)$ is dense in $W_0^1 L_\varphi(\Omega)$ for the modular convergence and $\mathcal{D}(\overline{\Omega})$ is dense in $W^1 L_\varphi(\Omega)$ the modular convergence.

Consequently, the action of a distribution S in $W^{-1}L_\psi(\Omega)$ on an element u of $W_0^1 L_\varphi(\Omega)$ is well defined. It will be denoted by $< S, u >$.

Truncation operator. For $k > 0$ we define the truncation at height k: $T_k : \mathbb{R} \longrightarrow \mathbb{R}$ by:

$$T_k(s) = \begin{cases} s & \text{if } |s| \leq k. \\ k\dfrac{s}{|s|} & \text{if } |s| > k. \end{cases} \tag{12}$$

Lemma 2.2. [10] *Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let φ be a Musielak- Orlicz function and let $u \in W_0^1 L_\varphi(\Omega)$. Then $F(u) \in W_0^1 L_\varphi(\Omega)$. Moreover, if the set D of discontinuity points of F' is finite, we have*

$$\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(u)\dfrac{\partial u}{\partial x_i} & a.e \text{ in } \{x \in \Omega : u(x) \in D\}. \\ 0 & a.e \text{ in } \{x \in \Omega : u(x) \notin D\}. \end{cases}$$

Lemma 2.3. [4] *Let $(f_n), f \in L^1(\Omega)$ such that*

i): $f_n \geq 0$ *a.e in* Ω.

ii): $f_n \longrightarrow f$ *a.e in* Ω.

iii): $\int_\Omega f_n(x)dx \longrightarrow \int_\Omega f(x)dx$.

then $f_n \longrightarrow f$ *strongly in* $L^1(\Omega)$.

Lemma 2.4. [10] *If a sequence* $g_n \in L_\varphi(\Omega)$ *converges in measure to a measurable function* g *and if* g_n *remains bounded in* $L_\varphi(\Omega)$, *then* $g \in L_\varphi(\Omega)$ *and* $g_n \rightharpoonup g$ *for* $\sigma(\Pi L_\varphi, \Pi E_\psi)$.

Lemma 2.5. [19] *Under the assumptions of Lemma 2.1, that exists a constant* $c > 0$ *depends only of* Ω *such that*

$$\int_\Omega \varphi(x, |u(x)|)dx \leq \int_\Omega \varphi(x, c|\nabla u(x)|)dx. \tag{13}$$

Proof. The proof is more detailed in [19]. It suffices to show that

$$\int_\Omega \varphi(x, |u(x)|)dx \leq \int_\Omega \varphi\Big(x, 2d\Big|\frac{\partial u}{\partial x_1}(x)\Big|\Big)dx, \quad \forall u \in W_0^1 L_\varphi(\Omega). \tag{14}$$

where $d = \max\Big(\text{diam}(\Omega), \dfrac{1}{\text{diam}(\Omega)}\Big)$ and $\text{diam}(\Omega)$ is the diameter of Ω.
First suppose that $u \in \mathcal{D}(\Omega)$, then

$$\begin{aligned}\varphi(x, |u(x_1, ..., x_n)|) &= \varphi\Big(x, \int_{-\infty}^{x_1} \Big|\frac{\partial u}{\partial x_1}\Big|(\sigma, x_2, ..., x_n)d\sigma\Big), \\ &\leq \frac{1}{d}\int_{-\infty}^{+\infty} \varphi\Big(x, d\Big|\frac{\partial u}{\partial x_1}\Big|(\sigma, x_2, ..., x_n)\Big)d\sigma,\end{aligned}$$

and thus

$$\int_\Omega \varphi(x, |u(x)|)dx \leq \int_\Omega \varphi\Big(x, d\Big|\frac{\partial u}{\partial x_1}(x)\Big|\Big)dx, \quad \forall u \in \mathcal{D}(\Omega). \tag{15}$$

For $u \in W_0^1 L_\varphi(\Omega)$ according to Lemma 2.1, we have that exists $u_n \in \mathcal{D}(\Omega)$ and $\lambda > 0$ such that

$$\overline{\varrho}_{\varphi,\Omega}\Big(\frac{u_n - u}{\lambda}\Big) = 0, \quad \text{as } n \longrightarrow +\infty,$$

hence

$$\begin{cases} \displaystyle\int_\Omega \varphi\Big(x, \frac{|u_n - u|}{\lambda}\Big)dx \longrightarrow 0, \quad \text{as } n \longrightarrow +\infty, \\ \displaystyle\int_\Omega \varphi\Big(x, \frac{|\nabla u_n - \nabla u|}{\lambda}\Big)dx \longrightarrow 0, \quad \text{as } n \longrightarrow +\infty, \\ u_n \longrightarrow u \quad \text{a.e in } \Omega, \quad (\text{ for a subsequence still denote } u_n). \end{cases}$$

Then, we have

$$\begin{aligned}\int_\Omega \varphi\Big(x, \frac{|u(x)|}{2d\lambda}\Big)dx &\leq \liminf_{n\longrightarrow+\infty} \int_\Omega \varphi\Big(x, \frac{|u_n(x)|}{2d\lambda}\Big)dx \\ &\leq \liminf_{n\longrightarrow+\infty} \int_\Omega \varphi\Big(x, \frac{1}{2\lambda}\Big|\frac{\partial u_n}{\partial x_1}(x)\Big|\Big)dx \\ &= \liminf_{n\longrightarrow+\infty} \int_\Omega \varphi\Big(x, \frac{1}{2\lambda}\Big|\frac{\partial u_n}{\partial x_1}(x) - \frac{\partial u}{\partial x_1}(x) + \frac{\partial u}{\partial x_1}(x)\Big|\Big)dx \\ &\leq \frac{1}{2}\liminf_{n\longrightarrow+\infty} \int_\Omega \varphi\Big(x, \frac{1}{\lambda}\Big|\frac{\partial u_n}{\partial x_1}(x) - \frac{\partial u}{\partial x_1}(x)\Big|\Big)dx \\ &\quad +\frac{1}{2}\int_\Omega \varphi\Big(x, \frac{1}{\lambda}\Big|\frac{\partial u}{\partial x_1}(x)\Big|\Big)dx \\ &\leq \int_\Omega \varphi\Big(x, \frac{1}{\lambda}\Big|\frac{\partial u}{\partial x_1}(x)\Big|\Big)dx.\end{aligned}$$

Hence

$$\int_\Omega \varphi\big(x,|u(x)|\big)dx \le \int_\Omega \varphi\Big(x, 2d\Big|\frac{\partial u}{\partial x_1}(x)\Big|\Big)dx, \quad \forall u \in W_0^1 L_\varphi(\Omega).$$

□

Lemma 2.6. [The Nemytskii Operator] [4] *Let Ω be an open subset of $\mathbb{R}^N$ with finite measure and let φ and ψ be two Musielak-Orlicz functions. Let $f : \Omega \times \mathbb{R}^p \longrightarrow \mathbb{R}^q$ be a Carathéodory function such that for a.e. $x \in \Omega$ and all $s \in \mathbb{R}^p$:*

$$|f(x,s)| \le c(x) + k_1 \psi_x^{-1} \varphi(x, k_2|s|). \tag{16}$$

where k_1 and k_2 are real positives constants and $c(\cdot) \in E_\psi(\Omega)$.
Then the Nemytskii Operator N_f defined by $N_f(u)(x) = f(x,u(x))$ is continuous from

$$\Big(\mathcal{P}(E_\varphi(\Omega), \frac{1}{k_2}\Big)^p = \prod \Big\{u \in L_\varphi(\Omega) : d(u, E_\varphi(\Omega)) < \frac{1}{k_2}\Big\}.$$

into $(L_\psi(\Omega))^q$ for the modular convergence.

Furthermore if $c(\cdot) \in E_\gamma(\Omega)$ and $\gamma \prec\prec \psi$ then N_f is strongly continuous from $\Big(\mathcal{P}(E_\varphi(\Omega), \frac{1}{k_2}\Big)^p$ to $(E_\gamma(\Omega))^q$.

3. Essential assumptions and some main results

Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \ge 2$, satisfying the segment property. Let

$$A : D(A) \subset W_0^1 L_\varphi(\Omega) \longrightarrow W^{-1} L_\psi(\Omega)$$

be a mapping given by $A(u) = -\mathrm{div}(a(x,u,\nabla u))$, where a is a function satisfying the following conditions :

$$a(x,s,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N \text{ is a Carathéodory function.} \tag{17}$$

There exist two Musielak-Orlicz functions φ and γ such that $\gamma \prec\prec \varphi$, a positive function $d(\cdot) \in E_\psi(\Omega)$ and positive constants ν, β such that for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$

$$|a(x,s,\xi)| \le \beta\big(d(x) + \psi_x^{-1}\gamma(x, \nu|s|) + \psi_x^{-1}\varphi(x, \nu|\xi|)\big). \tag{18}$$

$$\big(a(x,s,\xi) - a(x,s,\xi')\big)(\xi - \xi') > 0. \tag{19}$$

$$a(x,s,\xi).\xi \ge \alpha\varphi(x,|\xi|). \tag{20}$$

Furthermore, let $g(x,s,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ be a Carathéodory function such that for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, the following growth condition

$$|g(x,s,\xi)| \le c(x) + b(|s|)\varphi(x,|\xi|) \tag{21}$$

is satisfied, where $b : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a continuous positive function which belongs to $L^1(\mathbb{R}^+)$ and $c(\cdot) \in L^1(\Omega)$.

We consider the following boundary value problem

$$(\mathcal{P})\begin{cases} A(u) + g(.,u,\nabla u) = f \in L^1(\Omega), & \text{in } \Omega \\ u \equiv 0, & \partial\Omega. \end{cases}$$

Lemma 3.1. [Technical Lemma] *Assume that (17)...(20) are satisfies and let $(z_n)_n$ be a sequence in $W_0^1 L_\varphi(\Omega)$ such that*

$i)$: *$z_n \rightharpoonup z$ in $W_0^1 L_\varphi(\Omega)$ for $\sigma(\Pi L_\varphi, \Pi E_\psi)$.*

ii): $(a(\cdot, z_n, \nabla z_n))_n$ *is bounded in* $(L_\psi(\Omega))^N$.

iii): $\displaystyle\int_\Omega \big(a(x, z_n, \nabla z_n) - a(x, z_n, \nabla z\chi_s)\big)(\nabla z_n - \nabla z\chi_s)dx \longrightarrow 0$ *as* $n, s \longrightarrow \infty$.

where χ_s *is the characteristic function of* $\Omega_s = \{x \in \Omega : |\nabla z| \le s\}$.

Then, we have

$$z_n \longrightarrow z \text{ for the modular convergence in } W_0^1 L_\varphi(\Omega).$$

Proof. Let $s > 0$ and $\Omega_s = \{x \in \Omega : |\nabla z| \le s\}$ and denote by χ_s the Characteristic function of Ω_s.

Fix $r > 0$ and let $s > r$, we have

$$\begin{aligned} 0 &\le \int_{\Omega_r} \big(a(x, z_n, \nabla z_n) - a(x, z_n, \nabla z)\big)(\nabla z_n - \nabla z)dx \\ &\le \int_{\Omega_s} \big(a(x, z_n, \nabla z_n) - a(x, z_n, \nabla z)\big)(\nabla z_n - \nabla z)dx \\ &= \int_{\Omega_s} \big(a(x, z_n, \nabla z_n) - a(x, z_n, \nabla z\chi_s)\big)(\nabla z_n - \nabla z\chi_s)dx \\ &\le \int_{\Omega} \big(a(x, z_n, \nabla z_n) - a(x, z_n, \nabla z\chi_s)\big)(\nabla z_n - \nabla z\chi_s)dx. \end{aligned}$$

By *iii)*, we obtain

$$\lim_{n\longrightarrow\infty} \int_{\Omega_r} \big(a(x, z_n, \nabla z_n) - a(x, z_n, \nabla z)\big)(\nabla z_n - \nabla z)dx = 0.$$

So as in [22], we have

$$\nabla z_n \longrightarrow \nabla z \quad \text{a.e. in } \Omega. \tag{22}$$

On the other hand, we have

$$\begin{aligned} \int_\Omega a(x, z_n, \nabla z_n)\nabla z_n dx &= \int_\Omega \Big(a(x, z_n, \nabla z_n) - a(x, z_n, \nabla z\chi_s)\Big)(\nabla z_n - \nabla z\chi_s)dx \\ &+ \int_\Omega a(x, z_n, \nabla z\chi_s)(\nabla z_n - \nabla z\chi_s)dx + \int_\Omega a(x, z_n, \nabla z_n)\nabla z\chi_s dx. \end{aligned} \tag{23}$$

Since $(a(\cdot, z_n, \nabla z_n))_n$ is bounded in $(L_\psi(\Omega))^N$ and using the almost every where convergence of the gradients we obtain

$$a(x, z_n, \nabla z_n) \rightharpoonup a(x, z, \nabla z) \text{ weakly in } (L_\psi(\Omega))^N \text{ for } \sigma(\Pi L_\psi, \Pi E_\varphi).$$

Which implies that

$$\int_\Omega a(x, z_n, \nabla z_n)\nabla z\chi_s dx \longrightarrow \int_\Omega a(x, z, \nabla z)\nabla z\chi_s dx. \tag{24}$$

Letting $s \longrightarrow \infty$, we obtain

$$\int_\Omega a(x, z, \nabla z)\nabla z\chi_s dx \longrightarrow \int_\Omega a(x, z, \nabla z)\nabla z dx. \tag{25}$$

On the other hand, it is easy to see that second term of the right hand side of (23) tends to 0, as $n \longrightarrow \infty$, consequently, from *iii)*, (24) and (25), we have

$$\int_\Omega a(x, z_n, \nabla z_n)\nabla z_n dx \longrightarrow \int_\Omega a(x, z, \nabla z)\nabla z dx. \tag{26}$$

Using (20) and the convexity of φ, we have

$$\alpha\varphi\big(x, \frac{|\nabla z_n - \nabla z|}{2}\big) \leq \frac{1}{2}a(x, z_n, \nabla z_n) \cdot \nabla z_n + \frac{1}{2}a(x, z, \nabla z) \cdot \nabla z.$$

Then by (26) we get

$$\lim_{\text{meas}(E)\longrightarrow 0} \sup_{n\in\mathbb{N}} \int_E \varphi\Big(x, \frac{|\nabla z_n - \nabla z|}{2}\Big) dx = 0.$$

Then by using Vitali's theorem one has

$$z_n \longrightarrow z \text{ for the modular convergence in } W_0^1 L_\varphi(\Omega).$$

□

We define

$$\mathcal{T}_0^{1,\varphi}(\Omega) = \Big\{u \text{ measurable} \quad \text{ such that } \quad T_k(u) \in W_0^1 L_\varphi(\Omega), \ \forall k > 0\Big\}.$$

As in [14], we define the following notion of renormalized solution, which gives a meaning to a possible solution of $(\mathcal{P})$.

Definition 3.1. Assume that (17)-(20), (21) hold true. A function u is a renormalized solution of the problem $(\mathcal{P})$ if

$$\begin{cases} u \in \mathcal{T}_0^{1,\varphi}(\Omega), \ g(., u, \nabla u) \in L^1(\Omega), \ g(., u, \nabla u)u \in L^1(\Omega) \\ \displaystyle\int_\Omega a(x, u, \nabla u)h(u)\nabla v dx + \int_\Omega a(x, u, \nabla u)h'(u)\nabla u v dx + \int_\Omega g(x, u, \nabla u)h(u)v dx \\ \displaystyle\qquad = \int_\Omega f h(u) v dx \\ \text{for all } h \in W^{1,\infty}(\mathbb{R}) \text{ such that } h' \text{ has a compact support in } \mathbb{R} \\ \text{and for all } v \in W_0^1 L_\varphi(\Omega) \cap L^\infty(\Omega). \end{cases} \tag{27}$$

The weaker problem (27) is obtained by using the test function $h(u)v$ where $h \in W^{1,\infty}(\mathbb{R})$. and $v \in W_0^1 L_\varphi(\Omega) \cap L^\infty(\Omega)$ in $(\mathcal{P})$.

Remark 3.1. Let us note that in (27) every term is meaningful in the distributional sense.

Theorem 3.2. *Under assumptions (17)-(20), (21), there exists at least a renormalized solution u (in the sense of definition 3.1) of problem $(\mathcal{P})$.*

Proof. We devide the proof into seven steps.

Step 1: Approximate problem. Let us define, for each $k > 0$, the truncation

$$T_k(s) = \begin{cases} s & \text{if } |s| \leq k \\ k\dfrac{s}{|s|} & \text{if } |s| > k \end{cases}$$

and, for each $n \in \mathbb{N}$ the approximation

$$g_n(x, s, \xi) = T_n(g(x, s, \xi)).$$

Consider the nonlinear boundary elliptic problem

$$\begin{cases} u_n \in W_0^1 L_\varphi(\Omega) \\ -\text{div}\Big(a(\cdot, u_n, \nabla u_n)\Big) + g_n(\cdot, u_n, \nabla u_n) = f_n \quad \text{in } D'(\Omega). \end{cases} \tag{28}$$

where f_n be a sequence of regular functions which strongly converge to f in $L^1(\Omega)$ such that $\|f_n\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}$.

From Benkirane and Ould Mohameden Vall in [10], the problem (28), have at least one solution u_n.

Step 2: A priori estimates. Let $B(s) = \dfrac{1}{\alpha}\displaystyle\int_0^s b(|\tau|)d\tau$, $0 \leq B(s) \leq B(+\infty) = \dfrac{1}{\alpha}\displaystyle\int_0^s b(|\tau|)d\tau < \infty$ (b is the function in (21)).

Using $\sigma = T_k(u_n)e^{B(|u_n|)}$ as test function in (P_n), we obtain

$$\begin{aligned}\int_\Omega a(x,u_n,\nabla u_n)\nabla\Big(T_k(u_n)e^{B(|u_n|)}\Big)dx \quad &+ \quad \int_\Omega g_n(x,u_n,\nabla u_n)T_k(u_n)e^{B(|u_n|)}dx \\ &= \quad \int_\Omega f_n T_k(u_n)e^{B(|u_n|)}dx.\end{aligned}$$

Then by using (21) and the fact that

$$\begin{aligned}\nabla\Big(T_k(u_n)e^{B(|u_n|)}\Big) &= \nabla T_k(u_n)e^{B(|u_n|)} + \frac{1}{\alpha}T_k(u_n)\mathrm{sign}(u_n)b(|u_n|)\nabla u_n e^{B(|u_n|)} \\ &= \nabla T_k(u_n)e^{B(|u_n|)} + \frac{1}{\alpha}|T_k(u_n)|b(|u_n|)\nabla u_n e^{B(|u_n|)},\end{aligned}$$

we have

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n)e^{B(|u_n|)}dx + \frac{1}{\alpha}\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n b(|u_n|)|T_k(u_n)|e^{B(|u_n|)}dx \\ &\qquad \leq \int_\Omega \Big(|f_n| + |c(x)|\Big)|T_k(u_n)|e^{B(|u_n|)}dx + \int_\Omega b(|u_n|)|T_k(u_n)|\varphi(x,|\nabla u_n|)e^{B(|u_n|)}dx.\end{aligned}$$

By using (20) in the second integral, we deduce that

$$\begin{aligned}&\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)e^{B(|u_n|)}dx + \int_\Omega \varphi(x,|\nabla u_n|)b(|u_n|)|T_k(u_n)|e^{B(|u_n|)}dx \\ &\qquad \leq ke^{B(+\infty)}\int_\Omega \Big(|f_n| + |c(x)|\Big)dx + \int_\Omega \varphi(x,|\nabla u_n|)b(|u_n|)|T_k(u_n)|e^{B(|u_n|)}dx.\end{aligned}$$

Hence

$$\begin{aligned}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)e^{B(|u_n|)}dx \quad &\leq \quad ke^{B(+\infty)}\int_\Omega \Big(|f_n| + |c(x)|\Big)dx \\ &\leq \quad ck. \end{aligned} \tag{29}$$

Using again the condition (20)

$$\int_\Omega \varphi\Big(x,|\nabla T_k(u_n)|\Big)dx \leq c_1 k. \tag{30}$$

By using the Lemma 2.5, we have

$$\int_\Omega \varphi\Big(x,\frac{|T_k(u_n)|}{c}\Big)dx \leq \int_\Omega \varphi\Big(x,|\nabla T_k(u_n)|\Big)dx \leq c_2 k. \tag{31}$$

Then $(T_k(u_n))_n$ and $(\nabla T_k(u_n))_n$ are bounded in $L_\varphi(\Omega)$, hence $(T_k(u_n))_n$ is bounded in $W_0^1L_\varphi(\Omega)$, there exists some $v_k \in W_0^1L_\varphi(\Omega)$ such that

$$\begin{cases} T_k(u_n) \rightharpoonup v_k \quad \text{weakly in } W_0^1L_\varphi(\Omega) \text{ for } \sigma(\Pi L_\varphi, \Pi E_\psi) \\ T_k(u_n) \longrightarrow v_k \quad \text{strongly in } E_\psi(\Omega). \end{cases} \tag{32}$$

Step 3: Convegence in measure of $(u_n)_n$. Assume that exists a function M satisfies $\lim\limits_{t\longrightarrow\infty}\dfrac{M(t)}{t}=\infty$ and $M(t)\leq \operatorname{ess}\inf\limits_{x\in\Omega}\varphi(x,t)$.
Let $k>0$ large enough, by using (31), we have

$$\begin{aligned} M(k)\text{meas}\{|u_n|>k\} &= \int_{\{|u_n|>k\}} M(|T_k(u_n)|)dx \\ &\leq \int_{\{|u_n|>k\}} \varphi\big(x,|T_k(u_n)|\big)dx \leq \int_\Omega \varphi\big(x,|T_k(u_n)|\big)dx \\ &\leq c_3 k. \end{aligned}$$

Hence

$$\text{meas}\{|u_n|>k\}\leq \frac{c_3k}{M(k)}\longrightarrow 0 \text{ as } k\longrightarrow\infty.$$

For every $\lambda>0$, we have

$$\begin{aligned} \text{meas}\{|u_n-u_m|>\lambda\} &\leq \text{meas}\{|u_n|>k\}+\text{ meas}\{|u_m|>k\} \\ &\quad + \text{meas}\{|T_k(u_n)-T_k(u_m)|>\lambda\}. \end{aligned} \tag{33}$$

Consequently, by (31) we can assume that $(T_k(u_n))_n$ is a Cauchy sequence in measure in Ω.
Let $\varepsilon>0$, then by (33) there exists some $k=k(\varepsilon)>0$ such that

$$\text{meas}\{|u_n-u_m|>\lambda\}<\varepsilon,\qquad \text{for all } n,m\geq h_0(k(\varepsilon),\lambda).$$

This prove that $(u_n)_n$ is a Cauchy sequence in measure in Ω, thus converge almost every where to some measurables functions u. Then

$$\begin{cases} T_k(u_n)\rightharpoonup T_k(u) \quad \text{weakly in } W_0^1L_\varphi(\Omega) \text{ for } \sigma(\Pi L_\varphi,\Pi E_\psi) \\ \qquad T_k(u_n)\longrightarrow T_k(u) \quad \text{strongly in } E_\psi(\Omega). \end{cases} \tag{34}$$

Step 4: Boundness of $(a(x,T_k(u_n),\nabla T_k(u_n)))_n$ in $(L_\psi(\Omega))^N$. Let $w\in(E_\varphi(\Omega)^N$ be arbitrary such that $\|w\|_{\varphi,\Omega}\leq 1$, by (19), one has

$$\Big(a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\frac{w}{\nu})\Big)\Big(\nabla T_k(u_n)-\frac{w}{\nu}\Big)>0.$$

hence

$$\begin{aligned} \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\frac{w}{\nu}dx &\leq \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)dx \\ &\quad - \int_\Omega a(x,T_k(u_n),\frac{w}{\nu})(\nabla T_k(u_n)-\frac{w}{\nu})dx. \end{aligned} \tag{35}$$

Thanks to (29), we have

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)dx\leq ck.$$

On the other hand, for λ large enough ($\lambda>\beta$), we have by using (17).

$$\int_\Omega \psi_x\Big(\frac{a(x,T_k(u_n),\frac{w}{\nu})}{3\lambda}\Big)dx$$

$$\begin{aligned}
&\le \int_\Omega \psi_x\Big(\frac{\beta\big(d(x)+\psi_x^{-1}(\gamma(x,\nu|T_k(u_n)|))+\psi_x^{-1}(\varphi(x,|w|))\big)}{3\lambda}\Big)dx\\
&\le \frac{\beta}{\lambda}\int_\Omega \psi_x\Big(\frac{d(x)+\psi_x^{-1}(\gamma(x,\nu|T_k(u_n)|))+\psi_x^{-1}(\varphi(x,|w|))}{3}\Big)dx\\
&\le \frac{\beta}{3\lambda}\Big(\int_\Omega \psi_x(d(x))dx+\int_\Omega \gamma(x,\nu|T_k(u_n)|)dx+\int_\Omega \varphi(x,|w|)dx\Big)\\
&\le \frac{\beta}{3\lambda}\Big(\int_\Omega \psi_x(d(x))dx+\int_\Omega \gamma(x,\nu k)dx+\int_\Omega \varphi(x,|w|)dx\Big).
\end{aligned}$$

Now, since γ grows essentially less rapidly than φ near infinity ad by using the Remark 2.1, there exists $r(k)>0$ such that $\gamma(x,\nu k)\le r(k)\varphi(x,1)$ and so we have
$\int_\Omega \psi_x\Big(\frac{a(x,T_k(u_n),\frac{w}{\nu})}{3\lambda}\Big)dx$

$$\le \frac{\beta}{3\lambda}\Big(\int_\Omega \psi_x(d(x))dx+r(k)\int_\Omega \varphi(x,1)dx+\int_\Omega \varphi(x,|w|)dx\Big).$$

hence $a(x,T_k(u_n),\dfrac{w}{\nu})$ is bounded in $(L_\psi(\Omega))^N$.
Which implies that second term of the right hand side of (35) is bounded, consequently we obtain

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))w\,dx\le c_4(k),\qquad \text{for all } w\in (L^\varphi(\Omega))^N \text{ with } \|w\|_{\varphi,\Omega}\le 1.$$

Hence by the theorem of Banach Steinhous the sequence $(a(x,T_k(u_n),\nabla T_k(u_n)))_n$ remains bounded in $(L_\psi(\Omega))^N$.
Which implies that, for all $k>0$ there exists a function $h_k\in (L_\psi(\Omega))^N$ such that

$$a(x,T_k(u_n),\nabla T_k(u_n))\rightharpoonup h_k \text{ weakly in } (L_\psi(\Omega))^N \text{ for } \sigma(\Pi L_\psi,\Pi E_\varphi). \tag{36}$$

Step 5: Almost everywhere convergence of gradients
For $h>2k>0$, we set $b_k=\sup\{b(s):|s|\le k\}$ and

$$\begin{aligned}
w^j_{n,h} &= T_{2k}(u_n-T_h(u_n)+T_k(u_n)-T_k(v_j)),\\
w^j_h &= T_{2k}(u-T_h(u)+T_k(u)-T_k(v_j)),\\
w^j &= T_{2k}(T_k(u)-T_k(v_j)),\\
w_h &= T_{2k}(u-T_h(u)).
\end{aligned}$$

Let $v_j\in\mathcal{D}(\Omega)$ be a sequence such that $v_j\longrightarrow u$ in $W^1_0L_\varphi(\Omega)$ for the modular convergence.
Let $\theta_k(s)=se^{\delta s^2}$, with $\delta>\Big(\dfrac{b_k}{2\alpha}\Big)^2$, it is clear to see that

$$\theta_k'(s)-\frac{b_k}{\alpha}|\theta_k(s)|\ge\frac{1}{2},\quad \forall s\in\mathbb{R}. \tag{37}$$

Using $\sigma = \theta_k(w^j_{n,h})e^{B(|u_n|)}$ as test function in $(\mathcal{P}_n)$, we have

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)\nabla w^j_{n,h}\theta'_k(w^j_{n,h})e^{B(|u_n|)}dx + \int_\Omega g_n(x,u_n,\nabla u_n)\theta_k(w^j_{n,h})e^{B(|u_n|)}dx\\
&\quad + \int_\Omega a(x,u_n,\nabla u_n)\nabla u_n\theta_k(w^j_{n,h})\frac{b(|u_n|)}{\alpha}sign(u_n)e^{B(|u_n|)}dx\\
&= \int_\Omega f_n\theta_k(w^j_{n,h})e^{B(|u_n|)}dx.
\end{aligned}$$

Note that $\theta_k(w^j_{n,h})$ have the same sign as u_n on the set $\{|u_n| > k\}$, then by using (21), we have

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)\nabla w^j_{n,h}\theta'_k(w^j_{n,h})e^{B(|u_n|)}dx\\
&\quad + \int_{\{|u_n|>k\}} a(x,u_n,\nabla u_n)\nabla u_n|\theta_k(w^j_{n,h})|\frac{b(|u_n|)}{\alpha}e^{B(|u_n|)}dx\\
&\quad - \int_{\{|u_n|\le k\}} a(x,u_n,\nabla u_n)\nabla u_n|\theta_k(w^j_{n,h})|\frac{b(|u_n|)}{\alpha}e^{B(|u_n|)}dx\\
&\le \int_\Omega (|f_n|+c(x))|\theta_k(w^j_{n,h})|e^{B(|u_n|)}dx + \int_\Omega b(|u_n|)\varphi(x,|\nabla u_n|)|\theta_k(w^j_{n,h})|e^{B(|u_n|)}dx.
\end{aligned}$$

Using (18) in the second integral of the first hand side of last inequality, we obtain

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)\nabla w^j_{n,h}\theta'_k(w^j_{n,h})e^{B(|u_n|)}dx - \int_{\{|u_n|\le k\}} a(x,u_n,\nabla u_n)\nabla u_n|\theta_k(w^j_{n,h})|\frac{b(|u_n|)}{\alpha}e^{B(|u_n|)}dx\\
&\quad + \int_{\{|u_n|>k\}} \varphi(x,|\nabla u_n|)|\theta_k(w^j_{n,h})b(|u_n|)e^{B(|u_n|)}dx\\
&\le \int_\Omega (|f_n|+c(x))|\theta_k(w^j_{n,h})|e^{B(|u_n|)}dx + \int_{\{|u_n|\le k\}} b(|u_n|)\varphi(x,|\nabla u_n|)|\theta_k(w^j_{n,h})|e^{B(|u_n|)}dx\\
&\quad + \int_{\{|u_n|>k\}} \varphi(x,|\nabla u_n|)|\theta_k(w^j_{n,h})b(|u_n|)e^{B(|u_n|)}dx.
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)\nabla w^j_{n,h}\theta'_k(w^j_{n,h})e^{B(|u_n|)}dx - \int_{\{|u_n|\le k\}} a(x,u_n,\nabla u_n)\nabla u_n|\theta_k(w^j_{n,h})|\frac{b(|u_n|)}{\alpha}e^{B(|u_n|)}dx\\
&\quad - \int_{\{|u_n|\le k\}} b(|u_n|)\varphi(x,|\nabla u_n|)|\theta_k(w^j_{n,h})|e^{B(|u_n|)}dx\\
&\le \int_\Omega (|f_n|+c(x))|\theta_k(w^j_{n,h})|e^{B(|u_n|)}dx.
\end{aligned}$$

Using (18) in the third integral of the right hand side, we obtain

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)\nabla w^j_{n,h}\theta'_k(w^j_{n,h})e^{B(|u_n|)}dx - \frac{2b_k}{\alpha}\int_{\{|u_n|\le k\}} a(x,u_n,\nabla u_n)\nabla u_n|\theta_k(w^j_{n,h})e^{B(|u_n|)}dx\\
&\le e^{\frac{\|b\|_{L^1(\mathbb{R})}}{\alpha}}\int_\Omega (|f_n|+c(x))|\theta_k(w^j_{n,h})|dx.
\end{aligned}$$

In the other hand we have, $\theta_k(w^j_{n,h}) \rightharpoonup \theta_k(w^j_h)$ weakly * in $L^\infty(\Omega)$ as $n \longrightarrow \infty$ and $\theta_k(w^j_h) \longrightarrow \theta_k(w_h)$ by the modular convergence of $(v_j)_j$ in $L_\varphi(\Omega)$ as $j \longrightarrow \infty$.
Then

$$\int_\Omega (|f_n| + c(x))|\theta_k(w^j_{n,h})|dx \longrightarrow \int_\Omega (|f| + c(x))|\theta_k(w_h)|dx, \quad \text{as } n, j \longrightarrow \infty.$$

So, we have

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla w^j_{n,h}\theta'_k(w^j_{n,h})e^{B(|u_n|)}dx - \frac{2b_k}{\alpha}\int_{\{|u_n|\le k\}} a(x, u_n, \nabla u_n)\nabla u_n|\theta_k(w^j_{n,h})e^{B(|u_n|)}dx$$
$$\le \varepsilon(n, j, h). \tag{38}$$

Splitting the first integral of the right hand side where $\{|u_n| \le k\}$ and $\{|u_n| > k\}$ and using the fact that $\nabla w^j_{n,j} = 0$ on the set $\{|u_n| > m := h + 4k\}$, we obtain

$$\int_\Omega a(x, T_m(u_n), \nabla T_m(u_n))\nabla w^j_{n,h}\theta'_k(w^j_{n,h})e^{B(|u_n|)}dx =$$
$$\int_{\{|u_n|\le k\}} a(x, T_m(u_n), \nabla T_m(u_n))\Big(\nabla T_k(u_n) - \nabla T_k(v_j)\Big)\theta'_k(w^j_{n,h})e^{B(|u_n|)}dx.$$
$$- \int_{\{|u_n|>k\}} a(x, T_m(u_n), \nabla T_m(u_n))\nabla w^j_{n,h}\theta'_k(w^j_{n,h})e^{B(|u_n|)}dx. \tag{39}$$

The first term of the right hand side of last equality can write as

$$\int_{\{|u_n|\le k\}} a(x, T_m(u_n), \nabla T_m(u_n))\nabla w^j_{n,h}\theta'_k(w^j_{n,h})e^{B(|u_n|)}dx \ge$$
$$\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))\Big(\nabla T_k(u_n) - \nabla T_k(v_j)\Big)\theta'_k(w^j_{n,h})e^{B(|u_n|)}dx.$$
$$-e^{\frac{\|b\|_{L^1(\mathbb{R})}}{\alpha}}\theta'_k(2k)\int_{\{|u_n|>k\}} |a(x, T_k(u_n), 0)||\nabla T_k(v_j)|dx. \tag{40}$$

Recalling that $|a(x, T_k(u_n), 0)|\chi_{\{|u_n|>k\}}$ converge to $|a(x, T_k(u), 0)|\chi_{\{|u|>k\}}$ strongly in $L_\varphi(\Omega)$, moreover, since $|\nabla T_k(v_j)|$ converge by the modular convergence to $\nabla T_k(u)$, then

$$-e^{\frac{\|b\|_{L^1(\mathbb{R})}}{\alpha}}\theta'_k(2k)\int_{\{|u_n|>k\}} |a(x, T_k(u_n), 0)||\nabla T_k(v_j)|dx = \varepsilon(n, j).$$

For the second term of the right hand side of (39), we can write

$$\int_{\{|u_n|>k\}} a(x, T_m(u_n), \nabla T_m(u_n))\nabla w^j_{n,h}\theta'_k(w^j_{n,h})e^{B(|u_n|)}dx \ge$$
$$-e^{-\frac{\|b\|_{L^1(\mathbb{R})}}{\alpha}}\theta'_k(2k)\int_{\{|u_n>k|\}} |a(x, T_m(u_n), \nabla T_m(u_n))||\nabla T_k(v_j)|dx. \tag{41}$$

Since $|a(x, T_m(u_n), \nabla T_m(u_n))|$ is bounded in $L_\psi(\Omega)$ and since also $\nabla v_j\chi_{\{|u_n|>k\}}$ converge to $\nabla v_j\chi_{\{|u|>k\}}$ strongly in $E_\varphi(\Omega)$ as $n \longrightarrow \infty$, we obtain by using (36) that the integral

$$-\theta'_k(2k)\int_{\{|u_n|>k\}} |a(x, T_m(u_n), \nabla T_m(u_n))||\nabla T_k(v_j)|dx.$$

converge as $n \longrightarrow \infty$ to the quantity

$$-\theta_k'(2k)\int_{\{|u|>k\}} h_m|\nabla T_k(v_j)|dx.$$

Using now the modular convergence of $(v_j)_j$, we get

$$-\theta_k'(2k)\int_{\{|u|>k\}} h_m|\nabla T_k(v_j)|dx \longrightarrow -\theta_k'(2k)\int_{\{|u|>k\}} h_m|\nabla T_k(u)|dx = 0.$$

Finally, we have

$$-\theta_k'(2k)\int_{\{|u_n|>k\}} |a(x,T_m(u_n),\nabla T_m(u_n))||\nabla T_k(v_j)|dx = \varepsilon_h(n,j). \quad (42)$$

So, we deduce that

$$\begin{aligned}&\int_\Omega a(x,T_m(u_n),\nabla T_m(u_n))\nabla w_{n,h}^j\theta_k'(w_{n,h}^j)e^{B(|u_n|)}dx \geq \quad (43)\\ &\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla T_k(v_j))\theta_k'(w_{n,h}^j)e^{B(|u_n|)}dx + \varepsilon(n,j,h).\end{aligned}$$

Which implies by (43)

$$\begin{aligned}&\int_\Omega a(x,T_m(u_n),\nabla T_m(u_n))\nabla w_{n,h}^j\theta_k'(w_{n,h}^j)e^{B(|u_n|)}dx\\ &\quad\geq \int_\Omega \Big(a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j)\Big)\\ &\quad\quad\times(\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j)\theta_k'(w_{n,h}^j)e^{B(|u_n|)}dx\\ &\quad\quad+\int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j)(\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j)\theta_k'(w_{n,h}^j)e^{B(|u_n|)}dx\\ &\quad\quad-\int_{\Omega\setminus\Omega_s^j} a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(v_j)\theta_k'(w_{n,h}^j)e^{B(|u_n|)}dx + \varepsilon(n,j,h). \quad (44)\end{aligned}$$

where χ_s^j is the characteristic function of the set $\Omega_s^j = \{x\in\Omega : |\nabla T_k(v_j)| \leq s\}$.
By the fact that $\nabla T_k(v_j)\chi_{\Omega\setminus\Omega_s^j}\theta_k'(w_{n,h}^j) \longrightarrow \nabla T_k(v_j)\chi_{\Omega\setminus\Omega_s^j}\theta_k'(w_h^j)$ strongly in $(E_\varphi(\Omega))^N$, the third term of the right hand side of (44) tends as $n \longrightarrow \infty$ to

$$\int_\Omega h_k\nabla T_k(v_j)\chi_{\Omega\setminus\Omega_s^j}\theta_k'(w_h^j)e^{B(|u|)}dx.$$

Letting now $j \longrightarrow \infty$, by using the modular convergence of $(v_j)_j$, we have

$$\int_\Omega h_k\nabla T_k(v_j)\chi_{\Omega\setminus\Omega_s^j}\theta_k'(w_h^j)e^{B(|u|)}dx \longrightarrow \int_\Omega h_k\nabla T_k(u)\chi_{\Omega\setminus\Omega_s}\theta_k'(w_h)e^{B(|u|)}dx.$$

Finally, we have

$$\begin{aligned}&\int_{\Omega\setminus\Omega_s^j} a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(v_j)\theta_k'(w_h^j)e^{B(|u_n|)}dx\\ &\quad= \int_\Omega h_k\nabla T_k(u)\chi_{\Omega\setminus\Omega_s}\theta_k'(0)e^{B(|u|)}dx + \varepsilon(n,j,h). \quad (45)\end{aligned}$$

Concerning the second term of the right hand side of (44), we can write

$$\int_\Omega a(xT_k(u_n), \nabla T_k(v_j)\chi_s^j)\Big(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_s^j\Big)\theta'_k(w_{n,h}^j)e^{B(|u_n|)}dx$$
$$= \int_\Omega a(x, T_k(u_n), \nabla T_k(v_j)\chi_s^j)\nabla T_k(u_n)\theta'_k(T_k(u_n) - T_k(v_j))e^{B(|u_n|)}dx$$
$$- \int_\Omega a(x, T_k(u_n), \nabla T_k(v_j)\chi_s^j)\nabla T_k(v_j)\chi_s^j\theta'_k(w_{n,j}^j)e^{B(|u_n|)}dx. \quad (46)$$

The first term of the right hand side of (46) tends as $n \longrightarrow \infty$ to the quantity

$$\int_\Omega a(x, T_k(u), \nabla T_k(v_j)\chi_s^j)\nabla T_k(u)\theta'_k(T_k(u) - T_k(v_j))e^{B(|u|)}dx.$$

Since $a(x, T_k(u_n), \nabla T_k(v_j)\chi_s^j)\theta'_k(T_k(u_n) - T_k(v_j))$ converge strongly as $n \longrightarrow \infty$ in $(E_\psi(\Omega))^N$ to the quantity $a(x, T_k(u), \nabla T_k(v_j)\chi_s^j)\theta'_k(T_k(u) - T_k(v_j))$, by the Lemma 2.6 and that $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$ weakly by $\sigma(\Pi L_\varphi, \Pi E_\psi)$ in $L_\varphi(\Omega)$.
For the second term of the right hand side of (46), it is easy to see that

$$\int_\Omega a(x, T_k(u_n), \nabla T_k(v_j)\chi_s^j)\nabla T_k(v_j)\chi_s^j\theta'_k(w_{n,h}^j)e^{B(|u_n|)}dx$$
$$\longrightarrow \int_\Omega a(x, T_k(u), \nabla T_k(v_j)\chi_s^j)\nabla T_k(v_j)\chi_s^j\theta'_k(w_h^j)e^{B(|u|)}dx. \quad (47)$$

Consequently, we have

$$\int_\Omega a(x, T_k(u_n), \nabla T_k(v_j)\chi_s^j)\Big(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_s^j\Big)\theta'_k(w_{n,h}^j)e^{B(|u_n|)}dx$$
$$= \int_\Omega a(x, T_k(u), \nabla T_k(v_j)\chi_s^j)\Big(\nabla T_k(u) - \nabla T_k(v_j)\chi_s^j\Big)\theta'_k(w_h^j)e^{B(|u|)}dx + \varepsilon(n).$$

Since $\nabla T_k(v_j)\chi_s^j\theta'_k(w_h^j) \longrightarrow \nabla T_k(u)\chi_s\theta'_k(w_h)$ in $(E_\varphi(\Omega))^N$ by the modular convergence as $j \longrightarrow \infty$, it is easy to see that

$$\int_\Omega a(x, T_k(u), \nabla T_k(v_j)\chi_s^j)\Big(\nabla T_k(u) - \nabla T_k(v_j)\chi_s^j\Big)\theta'_k(w_h^j)e^{B(|u|)}dx$$
$$\longrightarrow \int_{\Omega\setminus\Omega_s} a(x, T_k(u), 0)\nabla T_k(u)\theta'_k(w_h)e^{B(|u|)}dx.$$

Thus

$$\int_\Omega a(x, T_k(u_n), \nabla T_k(v_j)\chi_s^j)\Big(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_s^j\Big)\theta'_k(w_{n,h}^j)e^{B(|u_n|)}dx$$
$$= \int_{\Omega\setminus\Omega_s} a(x, T_k(u), 0)\nabla T_k(u)\theta'_k(0)e^{B(|u|)}dx + \varepsilon(n, j, h). \quad (48)$$

Combining (44), (45) and (48), we get

$$\begin{aligned}
&\int_\Omega a(x,T_m(u_n),\nabla T_m(u_n))\nabla w^j_{n,h}\theta'_k(w^j_{n,h})e^{B(|u_n|)}dx\\
&\quad\geq \int_\Omega \Big[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi^j_s)\Big]\\
&\quad\times\Big(\nabla T_k(u_n)-\nabla T_k(v_j)\chi^j_s\Big)\theta'_k(w^j_{n,h})e^{B(|u_n|)}dx\\
&\quad-\int_{\Omega\setminus\Omega_s} h_k\nabla T_k(u)\theta'_k(0)e^{B(|u|)}dx\\
&\quad+\int_{\Omega\setminus\Omega_s} a(x,T_k(u),0)\nabla T_k(u)\theta'_k(0)e^{B(|u|)}dx+\varepsilon(n,j,h). \qquad (49)
\end{aligned}$$

Concerning the second term of the first hand side of (38), we can write

$$\begin{aligned}
&-\frac{2b_k}{\alpha}\int_{\{|u_n|\leq k\}} a(x,u_n,\nabla u_n)\nabla u_n|\theta_k(w^j_{n,h})|e^{B(|u_n|)}dx\\
&=-\frac{2b_k}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)|\theta_k(w^j_{n,h})|e^{B(|u_n|)}dx\\
&=-\frac{2b_k}{\alpha}\int_\Omega \Big(a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi^j_s)\Big)\\
&\times\Big(\nabla T_k(u_n)-\nabla T_k(v_j)\chi^j_s\Big)|\theta_k(w^j_{n,h})|e^{B(|u_n|)}dx\\
&-\frac{2b_k}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi^j_s)\Big(\nabla T_k(u_n)-\nabla T_k(v_j)\chi^j_s\Big)|\theta_k(w^j_{n,h})|e^{B(|u_n|)}dx\\
&-\frac{2b_k}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(v_j)\chi^j_s|\theta_k(w^j_{n,h})|e^{B(|u_n|)}dx.
\end{aligned}$$

As above it is easy to show that

$$-\frac{2b_k}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi^j_s)\Big(\nabla T_k(u_n)-\nabla T_k(v_j)\chi^j_s\Big)|\theta_k(w^j_{n,h})|e^{B(|u_n|)}dx=\varepsilon(n,j,h),$$

and

$$-\frac{2b_k}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(v_j)\chi^j_s|\theta_k(w^j_{n,h})|e^{B(|u_n|)}dx=\varepsilon(n,j,h).$$

Then

$$\begin{aligned}
&-\frac{2b_k}{\alpha}\int_{\{|u_n|\leq k\}} a(x,u_n,\nabla u_n)\nabla u_n|\theta_k(w^j_{n,h})|e^{B(|u_n|)}dx\\
&\quad= -\frac{2b_k}{\alpha}\int_\Omega \Big(a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi^j_s)\Big)\\
&\quad\times(\nabla T_k(u_n)-\nabla T_k(v_j)\chi^j_s)|\theta_k(w^j_{n,h})|e^{B(|u_n|)}dx\\
&\quad+\varepsilon(n,j,h). \qquad (50)
\end{aligned}$$

Combining (38), (49) and (50), we obtain

$$\int_\Omega \Big[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j)\Big]\Big(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_s^j\Big)$$

$$\times\Big(\theta_k'(w_{n,h}^j) - \frac{2b_k}{\alpha}|\theta_k'(w_{n,h}^j)|\Big)e^{B(|u_n|)}dx.$$

$$\begin{aligned}\leq\ & -\int_{\Omega\setminus\Omega_s} h_k \nabla T_k(u)\theta_k'(0)e^{B(|u|)}dx \\ & + \int_{\Omega\setminus\Omega_s} a(x,T_k(u),0)\nabla T_k(u)\theta_k'(0)e^{B(|u|)}dx + \varepsilon(n,j,h). \qquad (51)\end{aligned}$$

Which implies by using (37)

$$\begin{aligned}&\int_\Omega \Big[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j)\Big] \\ &\qquad \times\Big(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_s^j\Big)\theta_k'(w_{n,h}^j)e^{B(|u_n|)}dx \\ &\leq -2\int_{\Omega\setminus\Omega_s} h_k\nabla T_k(u)\theta_k'(0)e^{B(|u|)}dx + 2\int_{\Omega\setminus\Omega_s} a(x,T_k(u),0)\nabla T_k(u)\theta_k'(0)e^{B(|u|)}dx \\ &\qquad +\varepsilon(n,j,h). \qquad (52)\end{aligned}$$

Now, remark that

$$\begin{aligned}&\int_\Omega \Big[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi_s)\Big]\Big(\nabla T_k(u_n) - \nabla T_k(u)\chi_s\Big)dx \\ &= \int_\Omega\Big[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j)\Big]\Big(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_s^j\Big)dx \\ &\qquad + \int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j)\Big(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_s^j\Big)dx \\ &\qquad - \int_\Omega a(x,T_k(u_n),\nabla T_k(u)\chi_s)\Big(\nabla T_k(u_n) - \nabla T_k(u)\chi_s\Big)dx \\ &\qquad + \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\Big(\nabla T_k(v_j)\chi_s^j - \nabla T_k(u)\chi_s\Big)dx. \qquad (53)\end{aligned}$$

We shall pass to the limit as $n,j \longrightarrow \infty$ in the last three terms of the right hand side of the last inequality, we get

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j)\Big(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_s^j\Big)dx$$

$$= \int_{\Omega\setminus\Omega_s} a(x,T_k(u),0)\nabla T_k(u)dx + \varepsilon(n,j),$$

and

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u)\chi_s)\Big(\nabla T_k(u_n) - \nabla T_k(u)\chi_s\Big)dx$$

$$= \int_{\Omega\setminus\Omega_s} a(x,T_k(u),0)\nabla T_k(u)dx + \varepsilon(n),$$

similarly, we show that

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\Big(\nabla T_k(v_j)\chi_s^j-\nabla T_k(u)\chi_s\Big)dx=\varepsilon(n,j).$$

Which implies that

$$\begin{aligned}&\int_\Omega\Big[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)\chi_s)\Big]\Big(\nabla T_k(u_n)-\nabla T_k(u)\chi_s\Big)dx\\&\quad=\int_\Omega\Big[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j)\Big]\\&\quad\quad\times\Big(\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j\Big)dx+\varepsilon(n,j)\end{aligned}\tag{54}$$

Combining (52) and (54), we have

$$\begin{aligned}&\int_\Omega\Big[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)\chi_s)\Big]\Big(\nabla T_k(u_n)-\nabla T_k(u)\chi_s\Big)dx\\&\quad\leq\ -2\int_{\Omega\setminus\Omega_s}h_k\nabla T_k(u)\theta_k'(0)dx+2\int_{\Omega\setminus\Omega_s}a(x,T_k(u),0)\nabla T_k(u)\theta_k'(0)dx\\&\quad\quad+\varepsilon(n,j,h).\end{aligned}\tag{55}$$

By passing to the lim sup over n and letting $j,h,s\longrightarrow\infty$, w obtain

$$\begin{aligned}&\lim_{s\to\infty}\lim_{n\to\infty}\int_\Omega\Big[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)\chi_s)\Big]\\&\quad\quad\times\Big(\nabla T_k(u_n)-\nabla T_k(u)\chi_s\Big)dx=0.\end{aligned}\tag{56}$$

Thus implies by using Lemma 3.1

$$T_k(u_n)\longrightarrow T_k(u)\text{ for the modular convergence in }W_0^1L_\varphi(\Omega).\tag{57}$$

Then

$$\nabla u_n\longrightarrow\nabla u\quad\text{ a.e. in }\Omega.\tag{58}$$

Step 6: The equi-integrability of $g_n(x,u_n,\nabla u_n)$.

We shall show that

$$g_n(x,u_n,\nabla u_n)\to g(x,u,\nabla u)\quad\text{in }L^1(\Omega).\tag{59}$$

Thanks to Vitali's theorem, it suffices to prove that $g_n(x,u_n,\nabla u_n)$ is a uniformly equi-integrable.

We define the function $\overline{B}(s)=\dfrac{2}{\alpha}\displaystyle\int_0^s b(|r|)dr$ and we take $T_1(u_n-T_h(u_n))e^{\overline{B}(|u_n|)}$ as a test function in (28), we have

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla T_1(u_n-T_h(u_n))e^{\overline{B}(|u_n|)}\,dx\\&\quad+\frac{2}{\alpha}\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla u_n d(|u_n|)|T_1(u_n-T_h(u_n))|e^{\overline{B}(|u_n|)}\,dx\\&\quad+\int_\Omega g_n(x,u_n,\nabla u_n)T_1(u_n-T_h(u_n))e^{\overline{B}(|u_n|)}\,dx\\&=\int_\Omega f_nT_1(u_n-T_h(u_n))e^{\overline{B}(|u_n|)}\,dx.\end{aligned}$$

According to (18) and (21), we obtain

$$\begin{aligned}\alpha \int_{\{h<|u_n|\le h+1\}} & a(x,u_n,\nabla u_n)\nabla u_n e^{\overline{B}(|u_n|)}\,dx \\ &+ \int_{\{h<|u_n|\}} \varphi(x,|\nabla u_n|)b(|u_n|)|T_1(u_n-T_h(u_n))|e^{\overline{B}(|u_n|)}\,dx \\ &\le \int_{\{h<|u_n|\}} (|f_n|+|f_0|)e^{\overline{B}(|u_n|)}\,dx, \end{aligned} \tag{60}$$

it follows that

$$\int_{\{h+1<|u_n|\}} b(|u_n|)\varphi(x,|\nabla u_n|)\,dx \le e^{\overline{B}(\infty)} \int_{\{h<|u_n|\}} (|f|+|f_0|)\,dx.$$

Thus, for all $\eta>0$, there exists $h(\eta)\ge 1$ such that

$$\int_{\{h(\eta)<|u_n|\}} b(|u_n|)\varphi(x,|\nabla u_n|)\,dx \le \frac{\eta}{2}. \tag{61}$$

On the other hand, we set

$$b_{h(\eta)} := \max\{b(s) : |s|\le h(\eta)\},$$

for any measurable subset $E\subseteq\Omega$, we have

$$\begin{aligned}\int_E b(|u_n|)\varphi(x,|\nabla u_n|)\,dx \ \le\ & b_{h(\eta)}\int_E \varphi(x,|\nabla T_{h(\eta)}(u_n)|)\,dx \\ &+\int_{\{h(\eta)<|u_n|\}} b(|u_n|)\varphi(x,|\nabla u_n|)\,dx.\end{aligned} \tag{62}$$

From (57), there exists $\lambda(\eta)>0$ such that

$$\int_E b(|T_{h(\eta)}(u_n)|)\varphi(x,|\nabla T_{h(\eta)}(u_n)|)\,dx \le \frac{\eta}{2} \ \text{ for all } E \text{ such that meas(E)} \le \lambda(\eta). \tag{63}$$

Finally, by combining (61), (62) and (63), one easily has

$$\int_E b(|u_n|)\varphi(x,|\nabla u_n|)\,dx \le \eta \quad \text{for all} \quad \text{meas}(E)\le\beta(\eta), \tag{64}$$

using (21), we then deduce that $(g_n(x,u_n,\nabla u_n))_n$ are equi-integrable, and since

$$g_n(x,u_n,\nabla u_n)\to g(x,u,\nabla u) \quad \text{a.e. in} \quad \Omega.$$

In view of Vitali's theorem, we conclude (59).

Step 7: Passage to the limit.

Let $h(\cdot)\in W^{1,\infty}(\mathbb{R})$ be such that supp $h'(\cdot)\in[-M,M]$ for some $M>0$. For every $v\in\mathcal{D}(\Omega)$.

We have $h(T_M(u_n))v\in W_0^1L_\varphi(\Omega)$. Indeed, Since $T_M(u_n)$ is bounded in $W_0^1L_\varphi(\Omega)$ there exists two constant $c>0$ depends on M such that $\int_\Omega \varphi(x,|\nabla T_M(u_n)|)dx \le c$.

Let $c_1>0$ such that $\|h(T_M(u_n)\nabla v\|_\infty \le c_1$ and $\|h'(T_M(u_n))v\|_\infty\le c_1$.

Then, we have by using (11)

$$\begin{aligned}
&\int_\Omega \varphi\Big(x, \frac{h(T_M(u_n))\nabla v + h'(T_M(u_n))v|\nabla T_M(u_n)|}{2c_1}\Big)dx \\
&\leq \int_\Omega \varphi\Big(x, \frac{c_1 + c_1|\nabla T_M(u_n)|}{2c_1}\Big)dx \\
&\leq \frac{1}{2}\int_\Omega \varphi(x,1)dx + \frac{1}{2}\int_\Omega \varphi(x,|\nabla T_M(u_n)|)dx \\
&\leq c.
\end{aligned}$$

Taking $h(T_M(u_n))v$ as a test function in (28), we obtain

$$\begin{aligned}
&\int_\Omega a(x,T_M(u_n),\nabla T_M(u_n))\cdot(h'(T_M(u_n))v\nabla T_M(u_n) + h(T_M(u_n))\nabla v)dx \\
&\quad + \int_\Omega g_n(x,T_M(u_n),\nabla T_M(u_n))\ h(T_M(u_n))vdx \\
&= \int_\Omega f_n\ h(T_M(u_n))vdx. \qquad (65)
\end{aligned}$$

We start with the first integral in (65), we have

$$\begin{aligned}
&\int_\Omega a(x,T_M(u_n),\nabla T_M(u_n))\cdot(h'(T_M(u_n))v\nabla T_M(u_n) + h(T_M(u_n))\nabla v)dx \\
&= \int_\Omega a(x,T_M(u_n),\nabla T_M(u_n))\cdot h(T_M(u_n))\nabla v dx \\
&\quad + \int_\Omega \Big[a(x,T_M(u_n),\nabla T_M(u_n)) - a(x,T_M(u_n),\nabla T_M(u))\Big] \\
&\quad \times(\nabla T_M(u_n) - \nabla T_M(u))h'(T_M(u_n))vdx \\
&\quad + \int_\Omega a(x,T_M(u_n),\nabla T_M(u))(\nabla T_M(u_n) - \nabla T_M(u))h'(T_M(u_n))vdx \\
&\quad + \int_\Omega a(x,T_M(u_n),\nabla T_M(u_n))\nabla T_M(u)h'(T_M(u_n))vdx \qquad (66)
\end{aligned}$$

In the following we passe to the limit as $n \longrightarrow \infty$, in the each terms of (66), for the first term, we have $a(x,T_M(u_n),\nabla T_M(u_n)) \to a(x,T_M(u),\nabla T_M(u))$ a.e. in Ω and $a(x,T_M(u_n),\nabla T_M(u_n))$ is bounded in $(L_\psi(\Omega))^N$, using Lemma 2.4, we get

$$a(x,T_M(u_n),\nabla T_M(u_n)) \rightharpoonup a(x,T_M(u),\nabla T_M(u)) \quad \text{in} \quad (L_\psi(\Omega))^N,$$

and since

$$h(T_M(u_n))\nabla v \longrightarrow h(T_M(u))\nabla v \quad \text{strongly in} \quad (E_\varphi(\Omega))^N.$$

we deduce that

$$\begin{aligned}
&\lim_{n\to\infty}\int_\Omega a(x,T_M(u_n),\nabla T_M(u_n))\cdot h(T_M(u_n))\nabla v dx \\
&= \int_\Omega a(x,T_M(u),\nabla T_M(u))\cdot h(T_M(u))\nabla v dx \\
&= \int_\Omega a(x,u,\nabla u)\cdot h(u)\nabla v dx.. \qquad (67)
\end{aligned}$$

For the second term on the right hand side of (66), we have an argument as in (56), thinks to (19) and since

$$\Big|\int_\Omega \Big[a(x,T_M(u_n),\nabla T_M(u_n)) - a(x,T_M(u_n),\nabla T_M(u))\Big]$$
$$\times(\nabla T_M(u_n) - \nabla T_M(u))h'(T_M(u_n))vdx\Big|$$
$$\le \int_\Omega \Big[a(x,T_M(u_n),\nabla T_M(u_n)) - a(x,T_M(u_n),\nabla T_M(u))\Big]$$
$$\times(\nabla T_M(u_n) - \nabla T_M(u))\|h'(T_M(u_n))v\|_\infty dx$$
$$\le c_1\int_\Omega\Big[a(x,T_M(u_n),\nabla T_M(u_n)) - a(x,T_M(u_n),\nabla T_M(u))\Big](\nabla T_M(u_n) - \nabla T_M(u))dx.$$

Then

$$\lim_{n\longrightarrow\infty}\int_\Omega \Big[a(x,T_M(u_n),\nabla T_M(u_n)) - a(x,T_M(u_n),\nabla T_M(u))\Big]$$
$$\times(\nabla T_M(u_n) - \nabla T_M(u))h'(T_M(u_n))vdx = 0. \tag{68}$$

For the third term on the right hand side of (66), by using Lemma 2.6, we get

$$a(x,T_M(u_n),\nabla T_M(u)) \longrightarrow a(x,T_M(u),\nabla T_M(u))$$

as $n\longrightarrow\infty$ strongly in $(E_\psi(\Omega))^N$ and since $\nabla T_M(u_n) \rightharpoonup \nabla T_M(u)$ weakly in $L_\varphi(\Omega)$ and the fact that $T_M(u_n)\longrightarrow T_M(u)$ strongly in $E_\varphi(\Omega)$, we obtain

$$\lim_{n\longrightarrow\infty}\int_\Omega a(x,T_M(u_n),\nabla T_M(u))(\nabla T_M(u_n) - \nabla T_M(u))h'(T_M(u_n))vdx = 0. \tag{69}$$

For the third term on the right hand side of (66), as above we have

$$a(x,T_M(u_n),\nabla T_M(u_n)) \rightharpoonup a(x,T_M(u),\nabla T_M(u))$$

as $n\longrightarrow\infty$ weakly in $(L_\psi(\Omega))^N$ and since $T_M(u_n)\longrightarrow T_M(u)$ strongly in $E_\varphi(\Omega)$, we obtain

$$\lim_{n\longrightarrow\infty}\int_\Omega a(x,T_M(u_n),\nabla T_M(u_n))\nabla T_M(u)h'(T_M(u_n))vdx$$
$$= \int_\Omega a(x,T_M(u),\nabla T_M(u))\nabla T_M(u)h'(T_M(u))vdx$$
$$= \int_\Omega a(x,u,\nabla u)\nabla u h'(u)vdx. \tag{70}$$

Concerning the other terms, we have $h(T_M(u_n))\ v \rightharpoonup h(T_M(u))\ v$ weak$-*$ in $L^\infty(\Omega)$, then by using (59)

$$\lim_{n\longrightarrow\infty}\int_\Omega g_n(x,T_M(u_n),\nabla T_M(u_n))\ h(T_M(u_n))vdx$$
$$= \int_\Omega g(x,T_M(u),\nabla T_M(u))\ h(T_M(u))vdx$$
$$= \int_\Omega g(x,u,\nabla u)\ h(u)vdx \tag{71}$$

and

$$\int_\Omega f_n \ h(T_M(u_n)) \ v \, dx \longrightarrow \int_\Omega f \ h(T_M(u)) \ v \, dx. \tag{72}$$

By combining (65) – (72), we deduce that

$$\begin{aligned}\int_\Omega a(x,u,\nabla u)\cdot(h'(u)v\nabla u + h(u)\nabla v)\, dx \quad &+ \quad \int_\Omega g(x,u,\nabla u) \ h(u) \ v \, dx \\ &= \quad \int_\Omega f \ h(u) \ v \, dx.\end{aligned}$$

which is (27) in Definition 3.1. Therefore u is a renormalized solution to problem $(\mathcal{P})$. □

References

[1] M.L. Ahmed Oubeid, A. Benkirane, M. Ould Mohamedhen val, Nonlinear elliptic equations involiving measure data in Musielak-Orlicz-Sobolev spaces, *J.A. Diff. Eq and App.* **4** (2013), no. 1, 43–57.

[2] L. Aharouch, A. Benkirane, M. Rhoudaf, Existence results for some unilateral problems without sign condition with obstacle free in Orlicz spaces, *Nonlinear Analysis* **68** (2008) 2362–2380.

[3] M. Ait Khellou, A. Benkirane, S.M. Douiri, An inequality of type Poincaré in Musielak spaces and application to some non-linear elliptic problems with L^1 data, *Complex Variables and Elliptic Equations* **60** (2015), no 9, 1217–1242.

[4] M. Ait Khellou, A. Benkirane, S.M. Douiri, Existance of solutions for elliptic equations having naturel growth terms in Musielak Orlicz spaces, *J. Math. Comput. Sci.* **4** (2014), no. 4, 665–688.

[5] E. Azroul, M.B. Benboubker, M. Rhoudaf, Entropy solution for some $p(x)-$ quasilinear problem with right hand side measure, *African Diaspora, Journal of Mathematics* **13** (2012), No. 2, 23–44.

[6] M. Bendahmane, P. Wittbold, Renormalized solutions for nonlinear elliptic equations with variable exponents and L^1 data, *Nonlinear Anal.* **70** (2009), 567–583.

[7] A. Benkirane, F. Blali, M. Sidi El Vally (Ould Mohamedhen Val), An existence theorem for a strongly Nonlinear elliptique problem in Musielak-Orlicz spaces, *Applicationnes Mathematocae* **41** (2014), 175–184.

[8] A. Benkirane, A. Elmahi, An existence theorem for a strongly non-linear elliptic problem in Orlicz spaces. *Nonlinear Anal. Theory Meth. Appl.* **36** (1999), 11–24.

[9] A. Benkirane, A. Elmahi, Almost everywhere convergence of gradients of solutions to elliptic equations in Orlicz spaces and application, *Nonlinear Anal. Theory Meth. Appl.* **28** (1997), 1769–1784.

[10] A. Benkirane, M. Sidi El Vally (Ould Mohamedhen val), Variational inequalities in Musielak-Orlicz-Sobolev spaces, *Bull. Belg. Math. Soc. Simon Stevin* **21** (2014), 787–811.

[11] A. Benkirane, M. Sidi El Vally (Ould Mohamedhen Val), Some approximation properties in Musielak-Orlicz- Sobolev spaces, *Thai.J. Math.* **10** (2012), 371–381.

[12] D. Blanchard, F. Murat, Renormalised solutions of nonlinear parabolic problems with L1 data: Existence and uniqueness, *Proc. Roy. Soc. Edinburgh Sect. A* **127** (1997), no. 6, 1137–1152.

[13] D. Blanchard, F. Murat, H. Redwane, Existence and uniqueness of a renormalized solution for a fairly general class of nonlinear parabolic problems, *J. Differential Equations* **177** (2001), no. 2, 331–374.

[14] L. Boccardo, D. Giachetti, J.I. Diaz, F. Murat, Existence and regularity of renormalized solutions for some elliptic problems involving derivatives of nonlinear terms, *J. Differential Equations* **106** (1993), no. 2, 215–237.

[15] L. Boccardo and, T. Gallouet, Nonlinear elliptic equations with right-handside measures, *Commun. Partial Differential Equations* **17** (1992), 641–655.

[16] L. Boccardo and T. Gallouët, On some nonlinear elliptic and parabolic equations involving measure data, *J. Funct. Anal.* **87** (1989), 149–169.

[17] G. DalMaso, F. Murat, L. Orsina, A. Prignet, Renormalized solutions of elliptic equations with general measure data, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **12** (1999), no. 4, 741–808.
[18] L. Diening, P. Harjulehto, P. Hst, M. Rika, *Lebesgue and Sobolev spaces with variable exponents*, Springer-Verlag Berlin Heidelberg, 2011.
[19] M.S.B. Elemine vall, A. Ahemd, A. Touzani, A. Benkirane, Existence of entropy solutions for nonlinear elliptic equations in Musielak framework with L1 data, *Boletim da Soedade Paranaense de Matemtica* **36** (2018), no. 1, 127–152.
[20] A. Elmahi, D. Meskine, Existence of solutions for elliptic equations having natural growth terms in Orlicz spaces, *Abstr. Appl. Anal.* **2004** (2004), no. 12, 1031–1045.
[21] J.P. Gossez, A strongly non-linear elliptic problem in OrliczSobolev spaces. *Proc. Am. Math. Soc. Symp. Pure Math.* **45** (1986), 455–462.
[22] J.P. Gossez, Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients, *Trans. Amer. Math. Soc.* **190** (1974), 163–205.
[23] P. Gwiazda, P. Wittbold, A. Wroblewska, A. Zimmermann, Renormalized solutions of nonlinear elliptic problems in generalized Orlicz spaces, *J. Differential Equations* **253** (2012) 635–666.
[24] R. Landes, On the existence of weak solutions for quasilinear parabolic initial-boundary value problems, *Proc. Roy. Soc. Edinburgh Sect A* **89** (1981), 217–237.
[25] P.L. Lions, F. Murat, Solutions renormalisées d'équations elliptiques, in preparation.
[26] P.L. Lions, *Mathematical Topics in Fluid Mechanics, Vol. 1: Incompressible models*, Oxford Univ. Press, Oxford, 1996
[27] F. Murat, Equations elliptiques non linaires avec secondmembre L^1 ou mesure, In: *Comptes Rendus du 26me Congrs National d'Analyse Numrique, Les Karellis* (1994), 12–24.
[28] F. Murat, Soluciones renormalizadas de EDP elipticas non lineales, Cours á l'Université de Séville, Publication R93023, Laboratoire d'Analyse Numérique, Paris VI, 1993.
[29] J. Musielak, *Modular spaces and Orlicz spaces*, Lecture Notes in Math. 1034, Springer-Verlag Berlin Heidelberg, 1983.
[30] A. Porretta, Nonlinear equations with natural growth terms and measure data, *Electron. J. Differ. Equ. Conf.* **09** (2002) 181–202.
[31] J. M. Rakotoson, R. Temam, Relative rearrangement in quasilinear elliptic variational inequalities, *Indiana Univ. Math. J.* **36** (1987), no. 4, 757–810.

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