Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 44(1), 2017, Pages 64–77
ISSN: 1223-6934

A Quasi-Uniformity On BCC-algebras

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Abstract. We introduce a quasi-uniformity $\mathcal{U}$ on a BCC-algebra X by a family of ideals of X. If $T(\mathcal{U})$ is the topology induced by $\mathcal{U}$, we study some conditions under which $(X, T(\mathcal{U}))$ becomes a (semi)topological BCC-algebra. Also, we show that bicompletion of the quasi-uniformity $\mathcal{U}$ can be considered a $T(\mathcal{U}^\star)$-topological BCC-algebra which contains X as a sub-dense space.

2010 Mathematics Subject Classification. 06F35, 22A26 .
Key words and phrases. BCC-algebra, (semi)topological BCC-algebra, filter, Quasi-uniforme space, Bicompletion.

1. Introduction

In 1966, Y. Imai and K. Iséki in [13] introduced a class of algebras of type $(2, 0)$ called BCK-algebras which generalizes on one hand the notion of algebra of sets whit the set subtraction as the only fundamental non-nullary operation, on the other hand the notion of impliction algebra. K. Iséki posed an interesting problem whether the class of BCK-algebras form a variety. In connection with this problem Y. Komori in [14] introduced a notion of BCC-algebras which is a generalization of notion BCK-algebras and proved that class of all BCC-algebras is not a variety. W.A. Dudek in [9] redefined the notion of BCC-algebras by using a dual form of the ordinary definition. Further study of BCC-algebras was continued [3, 6, 7, 8]. In 1937, André Weil in [17] introuduced the concept of a uniform space as a generalization of the concept of a metric space in which many non-topological invariants can be defined. The study of quasi-uniformities started in 1948 with Nachbin's investigations on uniform preordered spaces. In 1960, Á. Csaszar introduced quasi-uniform spaces and showed that every topological space is quasi-uniformizable. This result established an interesting analogy between metrizable spaces and topological spaces. quasi-uniform structures were also studied in algebraic structures. See for example [15]. In this paper, in section 3, we use of ideals of a BCC-algebra X to define a quasi-uniformity $\mathcal{U}$ on X. We show that $(X, \mathcal{U})$ is precompact but it is not T_1 and T_2. We prove that for each cardinal number α there is a T_0 quasi-uniform BCC-algebra. In section 4, by using of regular ideals we make the uniformity $\mathcal{U}^*$ on X and show that $(X, T(\mathcal{U}^*))$ is compact semi topological BCC-algebra, where $T(\mathcal{U}^*)$ is induced topology by $\mathcal{U}^*$ on X. Finally, we obtain $\mathcal{U}^*$- Cauchy filters and then construct a bicompletion BCC-algebra $(\widetilde{X}, \widetilde{\mathcal{U}})$ of $(X, \mathcal{U})$ and prove that $(\widetilde{X}, T(\widetilde{\mathcal{U}}))$ is a topological BCC-algebra which has X as a sub-dense-BCC-algebra.

Received June 4, 2015.

2. Preliminary

2.1. Topological Space. Recall that a set A with a family $\mathcal{T}$ of its subsets is called a *topological space*, denoted by $(A, \mathcal{T})$, if $\mathcal{T}$ is closed under finite intersections and arbitrary unions. The members of $\mathcal{U}$ are called *open sets* of A and the complement of $A \in \mathcal{U}$, that is $A \setminus U$, is said to be a *closed set*. If B is a subset of A, the smallest closed set containing B is called the *closure* of B and denoted by $\overline{B}$ (or $cl_u B$). A subfamily $\{U_\alpha : \alpha \in I\}$ of $\mathcal{T}$ is said to be a *base* of $\mathcal{T}$ if for each $x \in U \in \mathcal{T}$ there exists an $\alpha \in I$ such that $x \in U_\alpha \subseteq U$, or equivalently, each U in $\mathcal{T}$ is the union of members of $\{U_\alpha\}$. A subset P of A is said to be a *neighborhood* of $x \in A$, if there exists an open set U such that $x \in U \subseteq P$. Let $\mathcal{U}_x$ denote the totality of all neighborhoods of x in A. Then a subfamily $\mathcal{V}_x$ of $\mathcal{U}_x$ is said to form a *fundamental system* of neighborhoods of x, if for each U_x in $\mathcal{U}_x$, there exists a V_x in $\mathcal{V}_x$ such that $V_x \subseteq U_x$. Topological space $(A, \mathcal{T})$ is said to be *compact*, if each open covering of A is reducible to a finite open covering, *locally compact*, if for each $x \in A$ there exist an open neighborhood U of x and a compact subset K such that $x \in U \subseteq K$. Also $(A, \mathcal{T})$ is said to be *disconnected* if there are two nonempty, disjoint, open subsets $U, V \subseteq A$ such that $A = U \cup V$, and connected otherwise, *totally disconnected* if each nonempty connected subset of A has one point only, *locally connected* if each open neighborhood of every point x contains a connected open neighborhood of x. The maximal connected subset containing a point of A is called the *component* of that point [2].

2.2. Quasi-Uniform Space. Let A be a non-empty set and $\emptyset \neq \mathcal{F} \subseteq P(A)$. Then $\mathcal{F}$ is called a *filter* on $P(A)$, if for each $F_1, F_2 \in \mathcal{F}$:
(i) $F_1 \in \mathcal{F}$ and $F_1 \subseteq F$ imply $F \in \mathcal{F}$,
(ii) $F_1 \cap F_2 \in \mathcal{F}$,
(iii) $\emptyset \notin \mathcal{F}$.
A subset $\mathcal{B}$ of a filter $\mathcal{F}$ on A is a *base* of $\mathcal{F}$ iff, every set of $\mathcal{F}$ contains a set of $\mathcal{B}$. If $\mathcal{F}$ is a family of nonempty subsets of A, then we denote generated filter by $\mathcal{F}$ with $fil(\mathcal{F})$.

A *quasi-uniformity* on a set A is a filter Q on $P(X \times X)$ such that
(i) $\triangle = \{(x, x) \in A \times A : x \in A\} \subseteq q$, for each $q \in Q$,
(ii) For each $q \in Q$, there is a $p \in Q$ such that $p \circ p \subseteq q$ where

$$p \circ p = \{(x, y) \in A \times A : \exists z \in A \ s.t \ (x, z), (z, y) \in p\}.$$

The pair (A, Q) is called a *quasi-uniform space*. If Q is a quasi-uniformity on a set A, then $q^{-1} = \{q^{-1} : q \in Q\}$ is also a quasi-uniformity on A called the *conjugate* of Q. It is well-known that if a quasi-uniformity satisfies condition: $q \in Q$ implies $q^{-1} \in Q$, then Q is a *uniformity*. Also Q is a uniformity on A provided

$$\forall q \in Q \ \exists p \in Q \ s.t \ p^{-1} \circ p \subseteq q.$$

Furthermore, $Q^* = Q \vee Q^{-1}$ is a uniformity on A. A subfamily $\mathcal{C}$ of quasi-uniformity Q is said to be a base for Q iff, each $q \in Q$ contains some member of $\mathcal{C}$. The topology $T(Q) = \{G \subseteq X : \forall x \in G \ \exists q \in Q \ s.t \ q(x) \subseteq G\}$ is called the topology induced by the quasi-uniformity Q [11].

Proposition 2.1. [11] *Let $\mathcal{C}$ be a family of subset of $X \times X$ such that*
(i) *$\triangle \subseteq B$, for each $B \in \mathcal{C}$;*

(ii) for $B_1, B_2 \in \mathcal{C}$, there is a $B_3 \in \mathcal{C}$ such that $B_3 \subseteq B_1 \cap B_2$;
(iii) for each $B \in \mathcal{C}$, there is a $C \in \mathcal{C}$ such that $C \circ C \subseteq B$.
Then there is the unique quasi-uniformity $\mathcal{U} = \{U \subseteq X \times X : \exists B \in \mathcal{C} : \ B \subseteq U\}$ on X for which $\mathcal{C}$ is a base.

Definition 2.1. [11] (i) A filter $\mathcal{G}$ on quasi-uniform space (A, Q) is called $Q^\star$-*Cauchy* filter if for each $U \in Q$, there is a $G \in \mathcal{G}$ such that $G \times G \subseteq U$.
(ii) A quasi-uniform space (A, Q) is called *bicomplete* if each $Q^\star$-Cauchy filter converges with respect to the topology $T(Q^\star)$.
(iii) A *bicompletion* of a quasi-uniform space (A, Q) is a bicomplete quasi-uniform space $(Y, \mathcal{V})$ that has a $T(\mathcal{V}^\star)$-dense subspace quasi-unimorphic to (A, Q).
(iv) A $Q^\star$-Cauchy filter on a quasi-uniform space (A, Q) is *minimal* provided that it contains no $Q^\star$-Cauchy filter other than itself.

Lemma 2.2. [11] *Let $\mathcal{G}$ be a $Q^\star$-Cauchy filter on a quasi-uniform space (A, Q). Then, there is exactly one minimal $Q^\star$-Cauchy filter coarser than $\mathcal{G}$. Furthermore, if $\mathcal{B}$ is a base for $\mathcal{G}$, then $\{q(B) : B \in \mathcal{B}$ and q is a symetric member of $Q^\star\}$ is a base for the minimal $Q^\star$-Cauchy filter coarser than $\mathcal{G}$.*

Lemma 2.3. [11] *Let (A, Q) be a T_0 quasi-uniform space and $\widetilde{A}$ be the set of all minimal Q^*-Cauchy filters on it. For each $q \in Q$, let*

$$\widetilde{q} = \{(\mathcal{G}, \mathcal{H}) \in \widetilde{A} \times \widetilde{A} : \exists G \in \mathcal{G} \text{ and } H \in \mathcal{H} \text{ s.t } G \times H \subseteq q\},$$

and $\widetilde{Q} = fil\{\widetilde{q} : q \in Q\}$. Then the following statements hold:
(i) $(\widetilde{A}, \widetilde{Q})$ is a T_0 bicomplete quasi-uniform space and (A, Q) is a quasi-uniformly embedded as a $T(\widetilde{(Q^\star)})$-dense subspace of $(\widetilde{A}, \widetilde{Q})$ by the map $i : X \to \widetilde{A}$ such that, for each $x \in A$, $i(x)$ is the $T(Q^\star)$-neighborhood filter at x. Furthermore, the uniformities $(\widetilde{Q})^\star$ and $\widetilde{(Q^\star)}$ coincide.
(ii) Any T_0 bicomplete of (A, Q) is a quasi-unimorphic to $(\widetilde{A}, \widetilde{Q})$.

In Lemma 2.3, (A, Q) is T_0 if $(x, y) \in \bigcap_{B \in \mathcal{C}} B$ and $(y, x) \in \bigcap_{B \in \mathcal{C}} B$ imply $x = y$, for each $x, y \in A$. Also (A, Q) is T_0 quasi-uniform space if and only if $(A, T(Q))$ is a T_0 topological space.

2.3. BCC- Algebra. A BCC-algebra is a non empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms, for all $x, y, z \in X$:
(1) $((x * y) * (z * y)) * (x * z) = 0$,
(2) $0 * x = 0$,
(3) $x * 0 = x$,
(4)$x * y = 0$ and $y * x = 0$ imply $x = y$.

A non empty subset S of BCC-algebra X is called subalgebra of X if it is closed under BCC-operation. For a BCC-algebra X, we denote $x \wedge y = y * (y * x)$ for all $x, y \in X$. On any BCC-algebra X one can define the natural order $\leq$ putting

$$x \leq y \Leftrightarrow x * y = 0$$

it is not difficult to verify that this order is partial and 0 is its smallest element.
In BCC-algebra X, following hold: for any $x, y, z \in X$
(5) $(x * y) * (z * y) \leq x * z$,
(6) $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$,

(7) $x \wedge y \leq x, y$
(8) $x * y \leq x$
(9) $(x * y) * z \leq x * (y * z)$
(10) $x * x = 0$,
(11) $(x * y) * x = 0$. [8]

Definition 2.2. [4] Let X be a BCC-algebra and $\emptyset \neq I \subseteq X$. I is called an ideal of X if it satisfies the following conditions:
(12) $0 \in I$,
(13) $x * y \in I$ and $y \in I$ imply $x \in I$.

If I is an ideal in BCC-algebra of X, then I is a subalgebra. Moreover, if $x \in I$ and $y \leq x$, then $y \in I$. An ideal I is said to be *regular ideal* if the relation

$$x \equiv^I y \Longleftrightarrow x * y, y * x \in I$$

is a congruence relation. In this case we denote $x/I = \{y : x \equiv^I y\}$ and $X/I = \{x/I : x \in X\}$. X/I is a BCC-algebra by $x/I * y/I = (x * y)/I$.

3. A quasi-uniformity in BCC-algebras

In this section we let X be a BCC-algebra and η be an arbitrary family of ideals of X which is closed under intersection.

Definition 3.1. Let $\mathcal{T}$ be a topology on a BCC-algebra X. Then:
(i) $*$ is continuous in (first)second variable if $x * y \in U \in \mathcal{T}$, then there is a (V) $W \in \mathcal{T}$ such that $(x \in V)$ $y \in W$ and $(V * x \subseteq U)$ $x * W \subseteq U$. In this case, we also say $(X, *, \mathcal{T})$ is (right) left topological BCC-algebra.
(ii) $(X, *, \mathcal{T})$ is semitopological BCC-algebra if it is left and right topological BCC-algebra, i.e. if $x * y \in U \in \mathcal{T}$, then there are $V, W \in \mathcal{T}$ such that $x \in V, y \in W$ and $x * W \subseteq U$ and $V * y \subseteq U$.
(iii) $(X, *, \mathcal{T})$ is topological BCC-algebra if $*$ is continuous , i.e. if $x * y \subseteq U \in \mathcal{T}$, then there are two neighborhoods V, W of x, y, respectively, such that $V * W \subseteq U$.

Definition 3.2. A *quasi-uniform BCC-algebra* is a BCC-algebra endowed with a quasi-uniformity.

Theorem 3.1. *Let X be a BCC-algebra. The set $\mathcal{C} = \{I_L : I \in \eta\}$ is a base for a quasi-uniformity $\mathcal{U}$ on X, where $I_L = \{(x, y) \in X \times X : y * x \in I\}$.*

Proof. Let $I \in \eta$. Then $\triangle \subseteq I$, because for any $x \in X$, $x * x = 0 \in I$. Now we prove that $I_L \circ I_L \subseteq I_L$. Let $(x, y) \in I_L \circ I_L$. Then there exists $z \in X$ such that $(x, z) \in I_L$ and $(z, y) \in I_L$. Hence $z * x$ and $y * z$ are in I. Since $((y*x)*(z*x))*(y*z) = 0 \in I$ and $y*z \in I$, $(y*x)*(z*x) \in I$. Again since $z*x \in I$, we get that $y*x \in I$. This implies that $(x, y) \in I_L$ and so $I_L \circ I_L \subseteq I_L$. Since η is closed under intersection for each $I, J \in \eta$, $I_L \cap J_L = (I \cap J)_L \in \mathcal{C}$. Thus, $\mathcal{C}$ satisfies in conditions $(i), (ii), (iii)$ from Proposition 2.1. Hence $\mathcal{C}$ is a base for the quasi-uniformity $\{U \in X \times X : \exists I \in \eta \ s.t \ I_L \subseteq U\}$. □

Notation. From now on, $\mathcal{U}$ is the unifomity in Theorem 3.1 and $T(\mathcal{U}) = \{G \subseteq X : \forall x \in G \ \exists I \in \eta \ s.t \ I_L(x) \subseteq G\}$ is induced topology by it.

Example 3.1. Let $X = \{0, 1, 2, 3\}$ be a BCC-algebra with the following table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	1
3	3	3	3	0

Then obviously $I_1 = \{0\}$,$I_2 = \{0, 1, 2\}$ and $I_3 = X$ are ideals of X. Clearly,

$$(I_1)_L = \triangle \cup \{(1, 0), (2, 0), (3, 0), (2, 1)\},$$

$$(I_2)_L = \triangle \cup \{(1, 0), (2, 0), (3, 0), (2, 1), (0, 1), (0, 2)\}$$

and $(I_3)_L = X \times X$. Therefore, by Theorem 3.1, $\mathcal{B} = \{(I_i)_L : i = 1, 2, 3\}$ is a base of the quasi-uniformity $\mathcal{U} = \{U \subseteq X \times X : \exists i \in \{1, 2, 3\}\ s.t\ (I_i)_L \subseteq U\}$ on X. Moreover $(I_1)_L(0) = \{0\}$, $(I_1)_L(1) = \{0, 1\}$ and $(I_1)_L(3) = (I_2)_L(3) = \{0, 3\}$. Also,

$$(I_2)_L(0) = (I_2)_L(1) = (I_1)_L(2) = (I_2)_L(2) = \{0, 1, 2\},$$

$$(I_3)_L(0) = (I_3)_L(1) = (I_3)_L(2) = (I_3)_L(3) = X,$$

Therefore $T(\mathcal{U}) = \{U \subseteq X \times X : \forall x \in U\ \exists i \in \{1, 2, 3\}\ s.t\ (I_i)_L(x) \subseteq U\}$.

Recall subset I of BCC-algera X is called BCC-ideal if $0 \in I$ and $(x*y)*z \in I$,$y \in I$ imply $x * z \in I$. In a BCC-algebra any BCC-ideal is an ideal. [7]

Lemma 3.2. *For any $I \in \eta$ and $x \in X$, define $I_L(x) = \{y \in X : y * x \in I\}$. Then following holds:*
(i) $0 \in I_L(x)$,
(ii) if $x \le y$, then $I_L(x) \subseteq I_L(y)$,
(iii) if $y \in I_L(x)$, then $I_L(y) \subseteq I_L(x)$,
(iv) if $x \in I$, then $I_L(x) = I$,
(v) if $y \in I$, then $I_L(x * y) \subseteq I_L(x)$ for each $x \in X$,
(vi) if I is a BCC-ideal and $x \in I$, then for any $y \in X$, $I_L(x * y) \subseteq I_L(y)$.

Proof. *(i)* Since $0 = 0 * x \in I$, $0 \in I_L(x)$.
(ii) Let $z \in I_L(x)$. Then $z * x \in I$. Since $x \le y$, by (2), $z * y \le z * x$. Hence $z * y \in I$, which implies that $z \in I_L(y)$.
(iii) Let $z \in I_L(y)$. Then $z * y \in I$. Since $y \in I_L(x)$, $y * x \in I$. Now from $((z * x) * (y * x)) * (z * y) = 0$ we conclude that $z * x \in I$ and so $z \in I_L(x)$.
(iv) Since $x \in I$,

$$y \in I_L(x) \Leftrightarrow (x, y) \in I_L \Leftrightarrow y * x \in I \Leftrightarrow y \in I.$$

(v) Let $z \in I_L(x * y)$. Then $z * (x * y) \in I$. By (9), $(z * x) * y \le z * (x * y)$. Therefore $(z * x) * y \in I$. Since $y \in I$, $z * x \in I$. Hence $z \in I_L(x)$.
(vi) Let $z \in I_L(x*y)$. Then $(z*x)*y \in I$. Since $x \in I$ and I is a BCC-ideal, $z*y \in I$. Hence $z \in I_L(y)$. □

Theorem 3.3. *$T(\mathcal{U})$ is the smallest topology on X which includes η and $(X, *, T(\mathcal{U}))$ is a right topological BCC-algebra.*

Proof. By Lemma 3.2 *(iii)*, it is easy to prove that $I_L(x) \in T(\mathcal{U})$, for each $x \in X$ and $I \in \eta$. Now let $x, y \in X$ and $x * y \in G \in T(\mathcal{U})$. Then there exists $I \in \eta$ such that $I_L(x * y) \subseteq G$. Let $z \in I_L(x)$. Since $z * x \in I$ and $((z * y) * (x * y)) * (z * x) = 0 \in I$, $(z*y)*(x*y)$ is in I and so $z*y \in I_L(x*y)$. Hence $I_L(x)*y \subseteq I_L(x*y)$. This implies that $*$ is contiuous in first variable. Now suppose $\mathcal{T}$ is a topology on X such that $*$ is continuous in first variable and $\eta \subseteq \mathcal{T}$. We show that $T(\mathcal{U}) \subseteq \mathcal{T}$. For this, given $x \in G \in T(\mathcal{U}^{\star})$. Then there exists $I \in \eta$ such that $I_L(x) \subseteq G$. Since $x*x = 0 \in I \in \mathcal{T}$, there exists $V \in \mathcal{T}$ such that $x \in V$ and $V * x \subseteq I$. If $z \in V$, then $z * x \in I$ and so $z \in I_L(x)$. Hence $x \in V \subseteq I_L(x) \subseteq G$. Thus $T(\mathcal{U}) \subseteq \mathcal{T}$. □

Recall a non zero element $a \in X$ is called an *atom* of a BCC-algebra if $x \leq a$ implies $x = 0$ or $x = a$. It is easy to see if $a \neq b$ are atoms, then $a * b = a$. [6]

Proposition 3.4. *If all non zero elements of BCC-algebra X are atoms, then:*
(i) for each $I \in \eta$ and $x \in X$, $I_L(x) = I$,
(ii) $(X, *, T(\mathcal{U}))$ is a topological BCC-algebra,
(iii) $(X, \mathcal{U})$ is a uniform space,

Proof. (i) The proof is obvious.
(ii) Let $x, y \in X$ and $x * y \in G \in T(\mathcal{U})$. Then there exists $I \in \eta$ such that $I_L(x * y) = I \subseteq G$. Now

$$x * y \in I_L(x) * I_L(y) = I * I \subseteq I \subseteq G.$$

(iii) Let $U \in \mathcal{U}$. Then there exists, $I \in \eta$ such that $I_L \subseteq U$. We claim that $I_L^{-1} \circ I_L \subseteq U$. Let $(x, y) \in I_L^{-1} \circ I_L$. For some a $z \in X$ we have $(x, z) \in I_L^{-1}$ and $(z, y) \in I_L$. Hence $x * z \in I$ and $y * z \in I$. Since x, y are atoms, $x, y \in I$. Therefore, $(x, y) \in I_L \subseteq U$. □

Recall that a quasi-uniform space (A, Q) is said to be *precompact* if for each $q \in Q$ there exist $x_1, x_2, ..., x_n \in A$ such that $A = \cup_{i=1}^{n} q(x_i)$. [11]

Proposition 3.5. *Let X be a BCC-algebra. The following conditions are equivalent:*
(i) the topological space $(X, T(\mathcal{U}))$ is compact,
(ii) the quasi-uniform space $(X, \mathcal{U})$ is precompact,
(iii) there exists $S = \{x_1, x_2, ..., x_n\} \subseteq X$ such that for all $a \in X$ and $I \in \eta$, $a*x_i \in I$, for some $x_i \in S$.

Proof. $(i) \Rightarrow (ii)$ it is clear.
$(ii) \Rightarrow (iii)$ Let $I \in \eta$. Since $(X, \mathcal{U})$ is precompact, there exist $x_1, x_2, ..., x_n \in X$ such that $X = \cup_{i=1}^{n} I_L(x_i)$. If $a \in X$, then there exists x_i such that $a \in I_L(x_i)$. Therefore $a * x_i \in I$.
$(iii) \Rightarrow (i)$ Let $X = \cup_{\alpha \in \Omega} G_\alpha$, where each G_α is an open set of X. Then for any $x_i \in S$ there exists $\alpha_i \in \Omega$ such that $x_i \in G_{\alpha_i}$. Since G_{α_i} is an open set, there exists $I \in \eta$ such that $I_L(x_i) \subseteq G_{\alpha_i}$, For any $a \in X$ by hypothesis $a * x_i \in I$ for some $x_i \in S$. Hence $a \in I_L(x_i) \subseteq G_{\alpha_i}$. Therefore, $X = \cup_{i=1}^{n} I_L(x_i) \subseteq \cup_{i=1}^{n} G_{\alpha_i}$. So $(X, T(\mathcal{U}))$ is compact. □

Proposition 3.6. *Let $\eta = \{I\}$. Then:*
(i) if I^c is a finite set, then topological space $(X, T(\mathcal{U}))$ is compact,
(ii) the set I is compact in topological space $(X, T(\mathcal{U}))$,
(iii) for any $x \in X$, $I_L(x)$ is compact set in topological space $(X, T(\mathcal{U}))$.

Proof. (i) Let $\{G_\alpha : \alpha \in \Omega\}$ be an open cover of X and $I^c = \{x_1, x_2, ..., x_n\}$. Then there exist $\alpha_0, \alpha_1, ..., \alpha_n \in \Omega$ such that $0 \in G_{\alpha_0}, x_1 \in G_{\alpha_1}, x_2 \in G_{\alpha_2}, ...x_n \in G_{\alpha_n}$. By (3), $I = I_L(0) \subseteq G_{\alpha_0}$, so $X = I \cup I^c \subseteq G_{\alpha_0} \cup G_{\alpha_1} ... \cup G_{\alpha_n}$.
(ii) Let $I \subseteq \bigcup_{\alpha \in \Omega} G_\alpha$, where each G_α is an open set of X. Since $0 \in I$, there is $\alpha \in \Omega$

such that $0 \in G_\alpha$. Then $I = I_L(0) \subseteq G_\alpha$. Hence I is a compact set in topological space $(X, T(\mathcal{U}))$.
(iii) Suppose $x \in X$ and $\{G_\alpha : \alpha \in \Omega\}$ an open cover of $I_L(x)$. Since $x \in I_L(x)$, there exists $\alpha \in \Omega$ such that $x \in G_\alpha$. Hence $I_L(x) \subseteq G_\alpha$. □

Let (A, Q) be a quasi-uniform space and $\mathcal{C}$ be a base for it. Recall (A, Q) is said to be T_1 quasi-uniform space if $\triangle = \bigcap_{B \in \mathcal{C}} B$ and T_2 quasi-uniform space if $\triangle = \bigcap_{B \in \mathcal{C}} B^{-1} o B$. [11]

Proposition 3.7. *quasi-uniform space $(X, \mathcal{U})$ is T_0 space iff, $\{0\} \in \eta$. But it is not T_1 and T_2 space.*

Proof. Let $(x, y), (y, x) \in \bigcap_{I \in \eta} I_L$. Hence $x * y \in I, y * x \in I$, for all $I \in \eta$. Since $\{0\} \in \eta$, $x * y = y * x = 0$. By (4), $x = y$. Hence $(X, \mathcal{U})$ is T_0 space. Conversely, let $(X, \mathcal{U})$ be T_0. Let $x \in \bigcap_{I \in \eta} I$. Then for each $I \in \eta$, $x * 0 = x$ and $0 * x = 0$, both, are in I. So $(x, 0), (0, x) \in \bigcap_{I \in \eta} I_L$. Since $(X, \mathcal{U})$ is T_0, $x = 0$. Hence $\bigcap_{I \in \eta} I = \{0\}$. Since η is closed under intersection, $\{0\} \in \eta$.
For any $y \in X$, $(y, 0) \in \bigcap_{I \in \eta} I_L$. Hence $\bigcap_{U \in \mathcal{U}} U \neq \triangle$ which implies that $(X, \mathcal{U})$ is not T_1 and T_2. □

Proposition 3.8. *Let for any $a \in X$, $l_a : X \to X$ by $l_a(x) = a * x$ be an open map. Then $(X, T(\mathcal{U}))$ is a T_0 space.*

Proof. Let $x, y \in X$ and $x \neq y$. By (iv) of Lemma 3.2, I is in $T(\mathcal{U})$, so $x * I$ and $y * I$ are open neighborhoods of x, y, respectively. We claim that $y \notin x * I$ or $x \notin y * I$. If $y \in x * I$ and $x \in y * I$, then there exist $z_1, z_2 \in I$ such that $x = y * z_1$ and $y = x * z_2$. By (8), $x \leq y$ and $y \leq x$. So $x * y = y * x = 0$. By(4), $x = y$. This is a contradiction. □

Proposition 3.9. *The following conditions are equivalent:*
(i) $(X, T(\mathcal{U}))$ is a T_0 space,
(ii) for every $0 \neq x \in X$ there is $I \in \eta$ such that $x \notin I$,
(iii) for each $0 \neq x \in X$ there exists $U \in T(\mathcal{U})$ such that $x \notin U$.

Proof. $(i \Rightarrow ii)$ Let $0 \neq x \in X$. Since $(X, T(\mathcal{U}))$ is T_0, there is an open neighborhood G of 0 such that $x \notin G$. As $0 \in G$, there is $I \in \eta$ such that $0 \in I \subseteq G$. Clearly $x \notin I$.
$(ii \Rightarrow iii)$ Because for each $I \in \eta$, I belongs $T(\mathcal{U})$, the proof is obvious.
$(iii \Rightarrow i)$ Let $x, y \in X$ and $x \neq y$. Then $x * y \neq 0$ or $y * x \neq 0$. Without the lost of generality, suppose $x * y \neq 0$. By hypothesis there exists $G \in T(\mathcal{U})$ such that $x * y \notin G$. Since $0 \in G$, there exists $I \in \eta$ such that $I = I_L(0) \subseteq G$. Since $(X, *, T(\mathcal{U}))$ is right topological BCC-algebra and $0 * x = 0$, there is $J \in \eta$ such that $J_L(0) * x \subseteq I$. Let $K = I \cap J$. We claim that $x \notin K_L(y)$. If $x \in K_L(y)$, then $x * y \in K \subseteq I \subseteq G$. This is a contradiction. Hence $(X, T(\mathcal{U}))$ is a T_0 space. Conversely, Let $0 \neq x \in X$. Since $(X, T(\mathcal{U}))$ is a T_0 space and each open set in $(X, T(\mathcal{U}))$ contains 0, there exists $U \in T(\mathcal{U})$ such that $x \notin U$. □

Let (A, Q) and (A^*, R) be quasi-uniform spaces.The map $f : (A, Q) \to (A^*, R)$ is called quasi-uniform continuous if for each $r \in R$ there exists $q \in Q$ such that $(x, y) \in q$ implies $(f(x), f(y)) \in r$. [11],[16]

Proposition 3.10. *Let X be a BCC-algebra and $a \in X$. The mapping $r_a : (X, \mathcal{U}) \to (X, \mathcal{U})$ given by $r_a(x) = x * a$ for all $x \in X$ is quasi-uniform continuous.*

Proof. Let $U \in \mathcal{U}$. Then there exists $I \in \eta$ such that $I_L \subseteq U$. Let $(x, y) \in I_L$. Since $y * x \in I$ and $(y * a) * (x * a) \leq (y * x)$, we get that $(y * a) * (x * a) \in I$ and so

$$(r_a(x), r_a(y)) = ((x * a), (y * a)) \in I_L \subseteq U. \qquad \square$$

Theorem 3.11. *For each $n \geq 4$, there exists a quasi uniform BCC-algebra of order n.*

Proof. Let $(X, *, 0)$ be a BCC-algebra and η be a family of ideals in X which is closed under intersection. By Theorem 3.1, there is a uniformity $\mathcal{U}$ on X. Suppose $a \notin X$ and $X' = X \cup \{a\}$. Then X' is a BCC-algebra by

$$x \otimes y = \begin{cases} x * y & \text{if } x, y \in X \\ a & \text{if } x = a, y = 0 \\ 0 & \text{if } x = a, y \neq 0 \\ x & \text{if } x \in X, y = a \end{cases} \tag{1}$$

We prove that for all $I \in \eta$, $I' = I \cup \{a\}$ is an ideal of X'. Clearly, $0 \in I'$. Let $x \otimes y \in I'$ and $y \in I'$. If $x, y \neq a$, then $x * y \in I$. Since I is an ideal in X and $y \in I$, we get that $x \in I \subseteq I'$. If $x = a$, clearly $x \in I'$. If $x \in X$ and $y = a$, then $x = x \otimes y \in I'$. Thus $\eta' = \{I' : I \in \eta\}$ is a family of ideals in X' which is closed under intersection. By Theorem 3.1, there is a uniformity $\mathcal{U}'$ on X'.
By Example 3.1, there is a quasi-uniform BCC-algebra of order 4. If $(X, *, 0, \mathcal{U})$ is a quasi-uniform BCC-algebra of order n, then by the above paragraph there is a quasi-uniform BCC-algebra of order $n + 1$. $\square$

Corollary 3.12. *For each $n \geq 4$, there is a right topological BCC-algebra of order n.*

Proof. By Theorems 3.11 and 3.3, the proof is obvious. $\square$

Theorem 3.13. *For each $n \geq 4$, there is a T_0 quasi-uniform BCC-algebra of order n.*

Proof. Let $(X, *, 0)$ be a BCC-algebra and $a \notin X$. Then $X^{'} = X \cup \{a\}$ is a BCC-algebra by

$$x \otimes y = \begin{cases} x * y & \text{if } x, y \in X \\ 0 & \text{if } x \in X, y = a \\ a & \text{if } x = a, y \in X \end{cases} \tag{2}$$

First we show that every ideal in X is an ideal in X'. Let I be an ideal in X, $x \otimes y \in I$ and $y \in I$. $x \neq a$ because $a * y = a \notin I$. Since $x * y \in I$, $x, y \in X$ and I is an ideal in X, we get that $x \in I$. Hence if η is a family of ideals in X which is closed under intersection it is in X' so. By Theorem 3.1, there are quasi-uniformities $\mathcal{U}$, $\mathcal{U}^{'}$ on X, $X^{'}$, respectively. By Proposition 3.7, $(X, \mathcal{U})$ is a T_0 quasi-uniform space iff $\{0\} \in \eta$ iff $(X^{'}, \mathcal{U}^{'})$ is T_0 quasi-uniform space.
Now by Example 3.1, $(X, \mathcal{U})$ is a T_0 quasi-uniform BCC-algebra of order 4. Let $(X, *, 0, \mathcal{U})$ be a T_0 quasi-uniform BCC-algebra of order n. Then by the above paragraph, we can find a quasi-uniform BCC-algebra $(X^{'}, \mathcal{U}^{'})$ of order n+1. $\square$

Theorem 3.14. *Let α be an infinite cardinal number. Then there is a T_0 quasi-uniform BCC-algebra of order α.*

Proof. Let X be a set with cardinal number α. Consider $X_0 = \{x_0 = 0, x_1, x_2, ...\}$ a countable subset of X. Define

$$x_i * x_j = \begin{cases} 0 & \text{if } i = j \\ x_i & \text{if } i \neq j. \end{cases} \tag{3}$$

Then $(X_0, *, 0)$ is a BCC-algebra. Let η be a collection of ideals in X_0 which is closed under intersection and contains $\{0\}$. Then by Theorem 3.1 and Proposition 3.7, there is a quasi-uniformity $\mathcal{U}_0$ on X_0 such that $(X_0,\mathcal{U}_0)$ is a T_0 quasi-uniform BCC-algebra. Now, define the binary operation $\otimes$ on X by

$$x \otimes y = \begin{cases} x * y & \text{if } x, y \in X_0 \\ 0 & \text{if } x \in X_0, y \notin X_0 \\ x & \text{if } x \notin X_0, y \in X_0 \\ 0 & \text{if } x = y \notin X_0 \\ x & \text{if } x \neq y \;\; x, y \notin X_0. \end{cases} \tag{4}$$

It is routine to check that X is a BCC-algebra of order α. Let $I \in \eta$ and $x, y \in X$ such that $x \otimes y \in I$ and $y \in I$. Then $y \in X_0$. If $x \in X_0$, then since I is an ideal in X_0 and $x * y = x \otimes y \in I$, we get that $x \in I$. If $x \notin X_0$, then $x = x \otimes y \in I$. This proves that η is a collection of ideals in X which is closed under intersection and contains $\{0\}$. Hence by Theorem 3.1 and Proposition 3.7, there is a T_0 quasi-uniformity $\mathcal{U}$ on X. □

Corollary 3.15. *If α is a cardinal number, then there is a T_0 right topological BCC-algebra.*

4. The bicompletion of topological BCC-algebra

In this section, we let X be a BCC-algebra and η be an arbitrary family of regular ideals of X which is closed under intersection and prove that for T_0 quasi-uniform BCC-algebra $(X,\mathcal{U})$, the bicompletion $(\widetilde{X},\widetilde{\mathcal{U}})$ admits the structure of a topological BCC-algebra such that X is a $T(\widetilde{\mathcal{U}})^{\star}$-dense sub BCC-algebra of $\widetilde{X}$.

Proposition 4.1. *Let I be a regular ideal of BCC-algebra X. Define $I_L^{-1} = \{(x, y) \in X \times X : (y, x) \in I_L\}$ and $I_L^{\star} = I_L \cap I_L^{-1}$. Then following holds:*
(i) $I_L^{-1} = \{(x, y) \in X \times X : x * y \in I\}$,
(ii) $I_L^{-1}(x) = \{y \in X : x * y \in I\}$,
(iii) $I_L^{-1}(0) = X$,
(iv) $I_L^{\star} = \{(x, y) \in X \times X : x \equiv^I y\}$,
(v) $I_L^{\star}(x) = \{y \in X : x \equiv^I y\} = x/I$,
(vi) if $x \in I$, *then* $I_L^{\star}(x) = I$,
(vii) $I_L^{\star}(I_L^{\star}(0)) = I_L^{\star}(0)$,
(viii) $I_L^{\star}(G * H) = I_L^{\star}(G) * I_L^{\star}(H)$.

Proof. The proofs (i),(ii),(iv),(v) and $(viii)$ are easy. To prove (iii), let $x \in X$. Since $0 * x = 0 \in I$, by (ii) , $x \in I_L^{-1}(0)$. So $X \subseteq I_L^{-1}(0)$.
(vi)

$$z \in I_L^{\star}(x) \Leftrightarrow z \equiv^I x \Leftrightarrow x * z \in I, z * x \in I \Leftrightarrow z \in I.$$

(vii) By (iv) we have

$I_L^{\star}(I_L^{\star}(0)) = I_L^{\star}(I) = \{y \in X : \exists x \in I \; s.t \; y \equiv^I x\} = \{y \in X : y \in I\} = I = I_L^{\star}(0)$. □

Theorem 4.2. *There is a uniformity $\mathcal{U}^\star$ on X such that $(X, T(\mathcal{U}^\star))$ is a completely regular topological BCC-algebras, where $T(\mathcal{U}^\star)$ is the induced topology by $\mathcal{U}^\star$ on X.*

Proof. Let $\mathcal{B} = \{I_L^\star : I \in \eta\}$. As the proof of Theorem 3.1, we can show that $\mathcal{B}$ is a base for the quasi-uniformity $\mathcal{U}^\star = \{U \subseteq X \times X : \exists I \in \eta \ s.t \ I_L^\star \subseteq U\}$. We prove $\mathcal{U}^\star$ is a uniformity. For this we must show $U^{-1} \in \mathcal{U}^\star$, for all $U \in \mathcal{U}^\star$. Let $U \in \mathcal{U}^\star$. Then $I_L^\star \subseteq U$ for some $I \in \eta$. Since $I_L^\star = (I_L^\star)^{-1}$, $(I_L^\star)^{-1} \subseteq U$ and so $I_L^\star \subseteq U^{-1}$. This implies that $U^{-1} \in \mathcal{U}^\star$. Now suppose $T(\mathcal{U}^\star) = \{G \subseteq X : \forall x \in G \ \exists I \in \eta \ s.t \ I_L^\star(x) \subseteq U\}$ is the induced topology by $\mathcal{U}^\star$ on X. We will prove that $*$ is continuous. For this, suppose $x * y \in G \in T(\mathcal{U}^\star)$. Then there exists $I \in \eta$ such that $I_L^\star(x * y) \subseteq G$. Let $z \in I_L^\star(x) * I_L^\star(y)$. Then $z = \alpha * \beta$, for some $\alpha \in I_L^\star(x)$ and $\beta \in I_L^\star(y)$. Since $\alpha \equiv^I x$ and $\beta \equiv^I y$ and $\equiv^I$ is congruence relation, $x * y \equiv^I \alpha * \beta = z$. This implies that $z \in I_L^\star(x * y)$ and so $I_L^\star(x) * I_L^\star(y) \subseteq I_L^\star(x * y)$. Finally, since $T(\mathcal{U}^\star)$ is the induced topology by uniformity $\mathcal{U}^\star$, it is completely regular on X. □

Example 4.1. Let $(X, *, 0)$ be as BCC-algebra in example 3.1. It is easy to see that I_1, I_2 and I_3 are regular ideals of X. Hence $(I_1)_L^\star = \triangle$,

$$(I_2)_L^\star = \triangle \cup \{(0,1),(1,0),(0,2),(2,0),(1,2),(2,1)\}$$

and $(I_3)_L^\star = X \times X$. Therefore, $\mathcal{U}^\star = \{U \subseteq X \times X : \exists i \in \{1,2,3\} \ s.t \ (I_i)_L^\star \subseteq U\}$.

Example 4.2. Let $X = [0, \infty)$. Then X is a BCC-algebra with the following operation

$$x * y = \begin{cases} 0 & \text{if } x \leqslant y \\ x & \text{if } x > y. \end{cases} \tag{5}$$

Let $I_n = [0, n]$, for each $n \geq 1$. We show that I_n is a regular ideal. Let $(x * y) * z \in I_n$ and $y \in I_n$. If $y < x$, then $x * z = (x * y) * z \in I_n$. If $y \geq x$, then $x \in I_n$. Since $x * z$ is x or 0, we get that $x * z \in I_n$. Thus, I_n is a BCC-ideal and so is a regular ideal. Moreover,

$${I_n}_L^\star = \{(x,y) \in X \times X : x * y, y * x \leq n\} = \{(x,y) \in X \times X : x, y \in I_n\} = I_n \times I_n.$$

Now let $\eta = \{I_n : n \geq 1\}$. Then η is a family of regular ideals which is closed under intersection. By Theorem 4.2, $\mathcal{U}^\star = \{U \subseteq X \times X : \exists n \geq 1 \ s.t \ I_n \times I_n \subseteq U\}$.

A topological space A is *connected* if and only if it has only A and $\emptyset$ as closed and open subsets.

Proposition 4.3. *The space $(X, T(\mathcal{U}^\star))$ is connected if and only if $\eta = \{X\}$.*

Proof. Let $X \neq I \in \eta$ and $x \notin I$. It is clear that $I_L^\star(x) \in T(\mathcal{U}^\star)$. We show that $I_L^\star(x)$ is closed in this space. Let $y \in \overline{I_L^\star(x)}$. Then there is a $z \in I_L^\star(y) \cap I_L^\star(x)$. Hence $y \equiv^I z \equiv^I x$ which implies that $y \in I_L^\star(x)$. Obviously, $I_L^*(x)$ is nonempty. If $I_L^*(x) = X$, then 0 is in it and so $x \equiv^I 0$ which implies that $x \in I$, a contradiction. Thus, $I_L^*(x)$ is a nonempty proper subset of X which is closed and open. Hence this space is not connected. Conversely, let $\eta = \{X\}$. Then $T(\mathcal{U}^\star) = \{\emptyset, X\}$. Hence $(X, T(\mathcal{U}^\star))$ is connected. □

Recall quasi-uniform space (A, Q) is totally bounded, if for each $q \in Q$ there exist sets $S_1, S_2, ..., S_n$ such that $A = \bigcup_{i=1}^n S_i$ and for each $1 \leq i \leq n$, $S_i \times S_i \subseteq q$.[11],[16]

Proposition 4.4. *The following conditions are equivalent:*
(i) for each $I \in \eta$, X/I is finite,
(ii) $(X,\mathcal{U})$ is totally bounded,
(iii) $(X, T(\mathcal{U}^\star))$ is compact.

Proof. $(i \Rightarrow ii)$ Let for each $I \in \eta$, X/I be finite. We prove that $(X,\mathcal{U})$ is totally bounded. Let $I \in \eta$. Since X/I is finite, there are $x_1, x_2, ..., x_n \in X$ such that $X = \bigcup_{i=1}^{n} x_i/I$. For each $1 \leq i \leq n$, $x_i/I \times x_i/I \subseteq I_L$ because if $(x, y) \in x_i/I \times x_i/I$, then $x \equiv^I x_i \equiv^I y$ and so $(x, y) \in I_L$. This proves that $(X,\mathcal{U})$ is totally bounded.
$(ii \Rightarrow iii)$ Let $(X,\mathcal{U})$ be totally bounded and $I \in \eta$. There exist sets $S_1, S_2, ..., S_n$, such that $\bigcup_{i=1}^{n} S_i = X$ and for each $1 \leq i \leq n$, $S_i \times S_i \subseteq I_L$. Let $1 \leq i \leq n$ and $x, y \in S_i$. Since (x, y) and (y, x) are in I_L, we get $x \equiv^I y$. This proves that $S_i \subseteq x_i/I$, for some $x_i \in S_i$. Now to prove that $(X, T(\mathcal{U}^\star))$ is compact let $X = \bigcup_{\alpha\in\Omega} G_\alpha$, where each G_α is in $T(\mathcal{U}^\star)$. Then there are $G_{\alpha_1}, ..., G_{\alpha_n}$ such that $x_i \in G_{\alpha_i}$ for each $1 \leq i \leq n$. Now suppose $x \in X$, then $x \in x_i/I$, for some $1 \leq i \leq n$ and so $x \in I^\star_L(x_i) \subseteq G_{\alpha_i}$.Therefore $X \subseteq \bigcup_{i=1}^{n} G_{\alpha_i}$, which shows that $(X, T(\mathcal{U}^\star))$ is compact.
$(iii \Rightarrow i)$ Let $I \in \eta$. Since $\{I^\star_L(x) : x \in X\}$ is an open cover of X in $T(\mathcal{U}^\star)$, there are $x_1, x_2, ..., x_n \in X$ such that $X \subseteq \bigcup_{i=1}^{n} I^\star_L(x_i)$. Now it is easy to see that $X/I = \{x_1/I, ..., x_n/I\}$. □

Theorem 4.5. *Let $(X, *, \mathcal{T})$ be a semi topological BCC-algebra. If $\eta \subseteq \mathcal{T}$, then $T(\mathcal{U}^*) \subseteq \mathcal{T}$.*

Proof. Let $(X, *, \mathcal{T})$ be a semitopological BCC-algebra which includes η. Given $x \in G \in T(\mathcal{U}^\star)$. Then there exists $I \in \eta$ such that $I^\star_L(x) \subseteq G$. Since $x * x = 0 \in I \in \mathcal{T}$, there exists $U \in \mathcal{T}$ such that $x \in U$ and $x * U, U * x \subseteq I$. If $z \in U$, then $x * z, z * x \in I$ and so $z \in I^\star_L(x)$. Hence $x \in U \subseteq I^\star_L(x) \subseteq G$. Thus $T(\mathcal{U}^\star) \subseteq \mathcal{T}$. □

Lemma 4.6. *Let $\mathcal{B}$ be a base for $\mathcal{U}^\star$-Cauchy filter $\mathcal{G}$ on quasi-uniform BCC-algebra $(X,\mathcal{U})$. Then the set $\{I^\star_L(B) : I \in \eta\ ,\ B \in \mathcal{B}\}$ is a base for a uniqe minimal $\mathcal{U}^\star$-Cauchy filter coarser than $\mathcal{G}$.*

Proof. By Lemma 2.2, the set $\{U(B) : B \in \mathcal{B}\ ,\ U \in \mathcal{U}^\star\}$ is a base for the unique minimal $\mathcal{U}^\star$-Cauchy filter $\mathcal{G}_0$ coareser than $\mathcal{G}$. Let $U \in \mathcal{U}^\star$ and $B \in \mathcal{B}$. Then for some $I \in \eta$, $I^\star_L \subseteq U$. So $I^\star_L(B) \subseteq U(B)$. Now it is easy to prove that the set $\{I^\star_L(B) : I \in \eta\ ,\ B \in \mathcal{B}\}$ is a base for $\mathcal{G}_0$. □

Lemma 4.7. *η is a base for a minimal $\mathcal{U}^\star$-Cauchy filter $\mathcal{I}$ on quasi-uniform BCC-algebra $(X,\mathcal{U})$.*

Proof. Let $\mathcal{C} = \{S \subseteq X : \exists I \in \eta\ s.t\ I \subseteq S\}$. It is easy to prove that $\mathcal{C}$ is a filter with base η. To prove that $\mathcal{C}$ is a $\mathcal{U}^*$-Cauchy filter , let $U \in \mathcal{U}$. There is a $I \in \eta$ such that $I_L \subseteq U$. If $x, y \in I^\star_L(0)$, then $x \equiv^I y$ and so $(x, y) \in I^\star_L \subseteq I_L \subseteq U$. This proves that $I^\star_L(0) \times I^\star_L(0) \subseteq U$. By Proposition 4.1$(vi)$, $I \times I \subseteq U$. Hence $\mathcal{C}$ is a $\mathcal{U}^\star$-Cauchy filter. By Lemma 2.2, the set $\{I^\star_L(I^\star_L(0)) : I \in \eta\}$ is a base for the uniqe minimal $\mathcal{U}^\star$-Cauchy filter $\mathcal{I}$ coareser than $\mathcal{C}$. But by Proposition 4.1 (vii), $I^\star_L(I^\star_L(0)) = I^\star_L(0) = I$. Therefore, η is a base for $\mathcal{I} = \mathcal{C}$. □

Lemma 4.8. *Let $\mathcal{G}$ and $\mathcal{H}$ be $\mathcal{U}^\star$-Cauchy filters on X. Then $\mathcal{G} * \mathcal{H} = \{G * H : G \in \mathcal{G},\ H \in \mathcal{H}\}$ is a $\mathcal{U}^\star$-Cauchy filter base on X.*

Proof. Let $I \in \eta$. Since $\mathcal{G}$ and $\mathcal{H}$ are $\mathcal{U}^\star$-Cauchy filters, there are $G \in \mathcal{G}$ and $H \in \mathcal{H}$ such that $G \times G \subseteq I_L$ and $H \times H \subseteq I_L$. We show that $G * H \times G * H \subseteq I_L$. Let $g_1, g_2 \in G$ and $h_1, h_2 \in H$. Then, $(g_1, g_2), (g_2, g_1), (h_1, h_2), (h_2, h_1)$ are in I_L. So $g_1 \equiv^I g_2$ and $h_1 \equiv^I h_2$. Since $\equiv^I$ is congruence, $g_1 * h_1 \equiv^I g_2 * h_2$, which implies that $(g_1 * h_1, g_2 * h_2) \in I_L^\star$. □

Theorem 4.9. *There is a quasi-uniform space* $(\widetilde{X}, \widetilde{\mathcal{U}})$ *of minimal* $\mathcal{U}^\star$*-Cauchy filers of quasi-uniform BCC-algebra* $(X, \mathcal{U})$ *that admits a BCC-algebra structure.*

Proof. Let $\widetilde{X}$ be the family of all minimal $\mathcal{U}^\star$-Cauchy filters of quasi-uniform BCC-algebra $(X, \mathcal{U})$. Let for each $U \in \mathcal{U}$,

$$\widetilde{U} = \{(\mathcal{G}, \mathcal{H}) \in \widetilde{X} \times \widetilde{X} : \exists G \in \mathcal{G} \ , \ H \in \mathcal{H} \ s.t \ G \times H \subseteq U\}.$$

If $\widetilde{\mathcal{U}} = fil\{\widetilde{U} : U \in \mathcal{U}\}$, then $(\widetilde{X}, \widetilde{\mathcal{U}})$ is a quasi-uniform space of minimal $\mathcal{U}^\star$-Cauchy filters of $(X, \mathcal{U})$. Let $\mathcal{G}, \mathcal{H} \in \widetilde{X}$. Since $\mathcal{G}, \mathcal{H}$ are minimal $\mathcal{U}^\star$-Cauchy filters on X, then by Lemma 4.8, $\mathcal{G} * \mathcal{H}$ is $\mathcal{U}^\star$-Cauchy filter base on X. We define $\mathcal{G} \tilde{*} \mathcal{H}$ as the minimal $\mathcal{U}^\star$-Cauchy filter contained $\mathcal{G} * \mathcal{H}$. By Lemma 2.2, the set $\{I_L^*(G * H) : G \in \mathcal{G}, \ H \in \mathcal{H}, \ I \in \eta\}$ is a base of $\mathcal{G} \tilde{*} \mathcal{H}$. But by Proposition 4.1 $(viii)$, $I_L^*(G * H) = I_L^*(G) * I_L^*(H)$, so the set $\{I_L^*(G) * I_L^*(H) : G \in \mathcal{G}, \ H \in \mathcal{H}, \ I \in \eta\}$ is a base of it. Now we will prove that $(\widetilde{X}, \tilde{*})$ is a BCC-algebra. For this, we have to prove that:

(i) $((\mathcal{G} \tilde{*} \mathcal{H}) \tilde{*} (\mathcal{K} \tilde{*} \mathcal{H})) \tilde{*} (\mathcal{G} \tilde{*} \mathcal{K}) = \mathcal{I}$

(ii) $I \tilde{*} \mathcal{G} = \mathcal{I}$

(iii) $\mathcal{G} \tilde{*} \mathcal{I} = \mathcal{G}$

(iv) $\mathcal{G} \tilde{*} \mathcal{H} = \mathcal{H} \tilde{*} \mathcal{G} = \mathcal{I} \Rightarrow \mathcal{G} = \mathcal{H}$

where $\mathcal{G}, \mathcal{H}, \mathcal{K} \in \widetilde{X}$, and $\mathcal{I}$ is minimal $\mathcal{U}^\star$-Cauchy filter in Lemma 4.7.

(i) Let $\mathcal{G}, \mathcal{H}, \mathcal{K} \in \widetilde{X}$. By Lemma 4.6, the set S_1 defined by

$$\{I_{1L}^\star(I_{2L}^\star(I_{3L}^\star(G_1 * H_1) * I_{4L}^\star(K_1 * H_2)) * I_{5L}^\star(G_2 * K_2)) : I_i \in \eta, \ G_i \in \mathcal{G}, \ H_i \in \mathcal{H}, \ K_i \in \mathcal{K}\}$$

is the base of minimal $\mathcal{U}^\star$-Cauchy filter $((\mathcal{G} \tilde{*} \mathcal{H}) \tilde{*} (\mathcal{K} \tilde{*} \mathcal{H})) \tilde{*} (\mathcal{G} \tilde{*} \mathcal{K})$ and by Lemma 4.7, η is the base of minimal $\mathcal{U}^\star$-Cauchy filter $\mathcal{I}$. Let $I_{1L}^\star(I_{2L}^\star(I_{3L}^\star(G_1 * H_1) * I_{4L}^\star(K_1 * H_2)) * I_{5L}^\star(G_2 * K_2)) \in S_1$. Put $I = \bigcap_{j=1}^4 I_{jL}$, $G = G_1 \cap G_2$, $H = H_1 \cap H_2$ and $K = K_1 \cap K_2$. Then

$$I_L^\star(I_L^\star(I_L^\star(G * H) * I_L^\star(K * H)) * I_L^\star(G_* K))$$

is a subset of

$$I_{1L}^\star(I_{2L}^\star(I_{3L}^\star(G_1 * H_1) * I_{4L}^\star(K_1 * H_2)) * I_{5L}^\star(G_2 * K_2)) \in S_1.$$

Now since $((g * h) * (k * h)) * (g * k) = 0$, for each $g \in G, \ h \in H$ and $k \in K$, it is easy to prove that

$$I_L^\star(0) \subseteq I_L^\star(I_L^\star(I_L^\star(G * H) * I_L^\star(K * H)) * I_L^\star(G_* K)).$$

Hence $\mathcal{I} \subseteq ((\mathcal{G} \tilde{*} \mathcal{H}) \tilde{*} (\mathcal{K} \tilde{*} \mathcal{H})) \tilde{*} (\mathcal{G} \tilde{*} \mathcal{K})$. Minimality $((\mathcal{G} \tilde{*} \mathcal{H}) \tilde{*} (\mathcal{K} \tilde{*} \mathcal{H})) \tilde{*} (\mathcal{G} \tilde{*} \mathcal{K})$ implies that

$$\mathcal{I} = ((\mathcal{G} \tilde{*} \mathcal{H}) \tilde{*} (\mathcal{K} \tilde{*} \mathcal{H})) \tilde{*} (\mathcal{G} \tilde{*} \mathcal{K}).$$

(ii) The sets $S_1 = \{I_L^\star(I_L^\star(0) * G) : I \in \eta \ , G \in \mathcal{G}\}$ and $\eta = \{I_L^\star(0) : I \in \eta\}$ are bases of minimal $\mathcal{U}^\star$-Cauchy filters $\mathcal{I} \tilde{*} \mathcal{G}$ and $\mathcal{I}$, respectively. But for each $I \in \eta$ and $G \in \mathcal{G}$, by Proposition 4.1 $(viii)$,

$$I_L^\star(I_L^\star(0) * G) = I_L^\star(I_L^\star(0)) * I_L^\star(G) = I_L^\star(0) * I_L^\star(G) = I_L^\star(0 * G) = I_L^\star(0).$$

So $S_1 = \eta$ and $\mathcal{I} = \mathcal{I}\tilde{*}\mathcal{G}$.
(iii) The sets $\{I^\star_L(G * I^\star_L(0)) : G \in \mathcal{G},\ I \in \eta\}$ and $\{I^\star_L(G) : G \in \mathcal{G}\}$ are the bases of $\mathcal{G}\tilde{*}\mathcal{I}$ and $\mathcal{G}$. For each $I \in \eta$ and $G \in \mathcal{G}$, by Proposition 4.1 $(viii)$,

$$I^\star_L(G * I^\star_L(0)) = I^\star_L(G) * I^\star_L(I^\star_L(0)) = I^\star_L(G) * I^\star_L(0) = I^\star_L(G * 0) = I^\star_L(G).$$

So $S_1 = S_2$ and hence $\mathcal{G} = \mathcal{G}\tilde{*}\mathcal{I}$.
(iv) The sets $S_1 = \{I^\star_L(G) : I \in \eta\ ,\ G \in \mathcal{G}\}$, $S_2 = \{I^\star_L(H) : I \in \eta\ ,\ H \in \mathcal{H}\}$, $S_3 = \{I^\star_L(G*H) : I \in \eta\ ,\ G \in \mathcal{G},\ H \in \mathcal{H}\}$, $S_4 = \{I^\star_L(H*G) : I \in \eta\ ,\ G \in \mathcal{G},\ H \in \mathcal{H}\}$ and $\eta = \{I^\star_L(0) : I \in \eta\}$ are the bases of $\mathcal{G}$, $\mathcal{H}$, $\mathcal{G}\tilde{*}\mathcal{H}$, $\mathcal{H}\tilde{*}\mathcal{G}$ and $\mathcal{I}$ respectively. Let $I^\star_L(G') \in\ S_1$. Since $\mathcal{G}\tilde{*}\mathcal{H} = \mathcal{H}\tilde{*}\mathcal{G} = \mathcal{I}$, $J^\star_L(G_0 * H_0) = K^\star_L(H_1 * G_1) = I^\star_L(0) = I$ for some $J, K \in \eta$. Let $G = G' \cap G_0 \cap G_1$ and $H = H_0 \cap H_1$. Now for each $g \in G$ and $h \in H$,

$$g * h \in J^\star_L(g) * J^\star_L(h) = J^\star_L(g * h) \subseteq J^\star_L(G * H) \subseteq J^\star_L(G_0 * H_0) = I.$$

Hence $g * h \in I$. With the similar argument we have $h * g \in I$. So $I^\star_L(g) = I^\star_L(h)$. Therefore, $I^\star_L(H) = I^\star_L(G) \subseteq I^\star_L(G')$. Hence $I^\star_L(G') \in \mathcal{H}$. So $\mathcal{G} \subseteq \mathcal{H}$. By minimality, $\mathcal{H} = \mathcal{G}$. □

Theorem 4.10. *If quasi-uniform BCC-algebra* $(X,\mathcal{U})$ *is a* T_0*,Then*
(i) $(\widetilde{X},\widetilde{\mathcal{U}})$ *is the bicompletion of* $(X,\mathcal{U})$.
(ii) X *is a sub BCC-algebra of* $\widetilde{X}$.
(iii) $(\widetilde{X}, T(\widetilde{\mathcal{U}^\star}))$ *is a topological BCC-algebra.*

Proof. (i) By Lemma 2.2 and Lemma 2.3 , $(\widetilde{X},\widetilde{\mathcal{U}})$ is the unique T_0 bicompletion quasi-uniform of $(X,\mathcal{U})$ and the mapping $i : X \to \widetilde{X}$ defined by

$$i(x) = \{W \subseteq X : W\ is\ a\ T(\widetilde{\mathcal{U}^\star}) - neighborhood\ of\ x\}$$

is a quasi-uniform embedded and $cl_{T(\widetilde{\mathcal{U}^\star})} i(X) = \widetilde{X}$.
(ii) Let $x, y \in X$. We shall prove that $i(x)\tilde{*}i(y) = i(x * y)$. By Lemma 2.3, the set

$$S = \{I^\star_L(W_x * W_y) : I \in \eta\ ,\ W_x\ W_y\ are\ T(\widetilde{\mathcal{U}^\star}) - neighborhoods\ x, y\}$$

is base for $i(x)\tilde{*}i(y)$. Since $I^\star_L(x * y) \subseteq I^\star_L(W_x\tilde{*}W_y)$ and $I^\star_L(x * y) \in i(x * y)$, we deduced that filter $i(x)\tilde{*}i(y)$ is contained in the filter $i(x * y)$. Since they are minimal $\mathcal{U}^\star$-Cauchy filters, $i(x)\tilde{*}i(y) = i(x * y)$. Hence X is a sub-BCC-algebra of $\widetilde{X}$.
(iii) By Lemma 2.3, $(\widetilde{\mathcal{U}})^\star = \widetilde{\mathcal{U}^\star}$. Hence

$$T(\widetilde{\mathcal{U}^\star}) = \{S \subseteq \widetilde{X} : \forall \mathcal{G} \in S\ \exists I \in \eta\ s.t\ \widetilde{I^\star_L}(\mathcal{G}) \subseteq S\}.$$

We prove that $(\widetilde{X}, T(\widetilde{\mathcal{U}^\star}))$ is a topological BCC-algebra. Let $\mathcal{G}\tilde{*}\mathcal{H} \in \widetilde{I^\star_L}(\mathcal{G}\tilde{*}\mathcal{H})$. We show that $\widetilde{I^\star_L}(\mathcal{G})\tilde{*}\widetilde{I^\star_L}(\mathcal{H}) \subseteq \widetilde{I^\star_L}(\mathcal{G}\tilde{*}\mathcal{H})$. Let $\mathcal{G}_1 \in \widetilde{I^\star_L}(\mathcal{G})$ and $\mathcal{H}_1 \in \widetilde{I^\star_L}(\mathcal{H})$. Then, there are $G \in \mathcal{G}, G_1 \in \mathcal{G}_1, H \in \mathcal{H}$ and $H_1 \in \mathcal{H}_1$ such that $G \times G_1 \subseteq I^\star_L$ and $H \times H_1 \subseteq I^\star_L$. By Lemma 2.3, $I^\star_L(G * H) \in \mathcal{G}\tilde{*}\mathcal{H}$ and $I^\star_L(G_1 * H_1) \in \mathcal{G}_1\tilde{*}\mathcal{H}_1$. We have to prove that $\mathcal{G}_1\tilde{*}\mathcal{H}_1 \in \widetilde{I^\star_L}(\mathcal{G}\tilde{*}\mathcal{H})$. For this, it is enough to show that $I^\star_L(G * H) \times I^\star_L(G_1 * H_1) \subseteq I^\star_L$. Let $y \in I^\star_L(G * H)$ and $y_1 \in I^\star_L(G_1 * H_1)$. Then, $y \equiv^I g * h$ and $y_1 \equiv^I g_1 * h_1$ for some $g \in G, g_1 \in G_1, h \in H, h_1 \in H_1$. Since $(g, g_1), (h, h_1)$ are in $I^\star_L$, we get $g * h \equiv^I g_1 * h_1$. Hence $(y, y_1) \in I^\star_L$. □

5. Conclusion

In this paper on a BCC-algebra of X we introduced the quasi-uniformity $\mathcal{U}$ induced by a family η of BCC-ideals of X. We studied some properties of topological space $(X, T(\mathcal{U}))$. Next researches can study the following assertions:
(1) separation axioms on $(X, T(\mathcal{U}))$ and $(X, T(\mathcal{U}^*))$,
(2) quasi-uniform continuouty of the operation of X in quasi-uniform space $(X,\mathcal{U})$,
(3) quasi-uniform continuous homomorphisms on $(X,\mathcal{U})$,
(4) quasi-uniform quotient BCC-algebras.

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