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Localization of semi-Heyting algebras

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Abstract. In this note, we introduce the notion of ideal on semi-Heyting algebras which allows us to consider a topology on them. Besides, we define the concept of $\mathfrak{F}-$multiplier, where $\mathfrak{F}$ is a topology on a semi-Heyting algebra L, which is used to construct the localization semi-Heyting algebra $L_{\mathfrak{F}}$. Furthermore, we prove that the semi-Heyting algebra of fractions L_S associated with an $\wedge-$closed system S of L is a semi-Heyting of localization. Finally, in the finite case we prove that L_S is isomorphic to a special subalgebra of L. Since Heyting algebras are a particular case of semi-Heyting algebras, all these results generalize those obtained in [11].

Key words and phrases. localization, $\mathfrak{F}-$multipliers, semi-Heyting algebras, $\wedge-$closed system.

1. Introduction

Starting from the example of the ring, J. Schmid introduces in [20], [21] the notion of maximal lattice of quotients for a distributive lattice. The central role in this construction is played by the concept of multipliers, defined by W. H. Cornish in [13], [14]. Using the model of localization ring, in [15] is defined for a bounded distributive lattice L the localization lattice $L_{\mathfrak{F}}$ of L with respect to a topology $\mathfrak{F}$ on L and is proved that the maximal lattice of quotients for a distributive lattice is a lattice of localization (relative to the topology of regular ideals). The same theory is also valid for the lattice of fractions of a distributive lattice with 0 and 1 relative to an $\wedge-$closed system. A theory of localization for Hilbert and Hertz algebras was developed in [5]; for the case of commutative bounded BCK-algebras see [17]; for the case of $n-$valued Łukasiewicz-Moisil algebras see [12]; for the case of Heyting algebras see [11], for the case of MV and pseudo MV$-$algebras see [6, 7], for the case of BL and pseudo BL-algebras see [8, 9, 10] and for the case of hoop-algebras see [16].

On the other hand, semi-Heyting algebras were introduced as a new equational class by H. P. Sankappanavar in [19] (see also [1, 2, 3]). These algebras represent a generalization of Heyting algebras. Nevertheless the behavior of semi-Heyting algebras is much more complicated than that of Heyting algebras.

An algebra $\mathcal{L} = \langle L, \vee, \wedge, \rightarrow, 0, 1\rangle$ is a semi-Heyting algebra if the following conditions hold:

(sH1) $\mathcal{L} = \langle L, \vee, \wedge, 0, 1\rangle$ is a lattice with 0 and 1.
(sH2) $x \wedge (x \rightarrow y) = x \wedge y$.
(sH3) $x \wedge (y \rightarrow z) = x \wedge [(x \wedge y) \rightarrow (x \wedge z)]$.

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(sH4) $x \to x = 1$.

We will denote by $\mathcal{SH}$ the variety of semi-Heyting algebras. The variety $\mathcal{H}$ of Heyting algebras is the subvariety of $\mathcal{SH}$ characterized by the equation $(x \wedge y) \to x = 1$.

These new algebras share with Heyting algebras some important properties (see [19]). For instance, they are pseudocomplemented, with the pseudocomplement given by $x^* = x \to 0$, congruences on them are determined by filters and the variety of semi-Heyting algebras is arithmetic. But, at the same time, semi-Heyting algebras present strong diferences with Heyting algebras. For example, the operation of implication on a semi-Heyting algebra $\mathcal{L}$ is not determined by the order of the underlying structure of lattice of $\mathcal{L}$: there are two non-isomorphic structures of semi-Heyting definable on the two-element lattice and ten on the three-element lattice. That is, we can have many operations of semi-Heyting implications on a given lattice. Among all these implications, the Heyting implication is the biggest one.

The aim of this paper is to generalize some of the results established in [11], using the model of bounded distributive lattices from [15] to semi-Heyting algebras. To this end, we introduce the notion of ideal on semi-Heyting algebras, dual to that of filter, which allows us to consider a topology on them. Besides, we define the concept of $\mathfrak{F}-$multiplier, where $\mathfrak{F}$ is a topology on an semi-Heyting algebra L, which is used to construct the localization semi-Heyting algebra $L_{\mathfrak{F}}$. Furthermore, we prove that the semi-Heyting algebra of fractions L_S associated with an $\wedge-$closed system S of L is an semi-Heyting algebra of localization. In the last part of this paper we give an explicit description of the semi-Heyting algebras $L_{\mathfrak{F}}$ and L_S in the finite case.

2. $\mathfrak{F}-$multipliers and localization of semi-Heyting algebras

Taking into account the notion of topology for bounded distributive lattices introduced in [15], we will consider this concept in the particular case of semi-Heyting algebras.

Definition 2.1. Let $\mathcal{L} = \langle L, \vee, \wedge, \to, 0, 1\rangle$ be a semi-Heyting algebra. A non-empty subset I of L is an ideal of L, if I is an ideal of the bounded lattice $\langle L, \vee, \wedge, 0, 1\rangle$.

It is worth noting that $\{0\}$ and L are ideals of L. We will denote by $I(L)$ the set of all ideals of L.

If X is a non-empty subset of L, we will denote by $\langle X\rangle$ the ideal generated by X. In particular, if $X = \{a\}$ we will write $\langle a\rangle$ instead of $\langle\{a\}\rangle$. We have that $\langle X\rangle = \{y \in L :$ there are $x_1, \cdots, x_n \in X$ such that $y \leq \bigvee\limits_{i=1}^{k} x_i\}$. Moreover, if $a \in L$ then $\langle a\rangle = \{x \in L : x \leq a\}$.

Definition 2.2. Let L be a semi-Heyting algebra. A nomempty set $\mathfrak{F}$ of elements $I \in I(L)$ will be called a topology on L if the following properties hold:
(1) If $I_1 \in \mathfrak{F}$, $I_2 \in I(L)$ and $I_1 \subseteq I_2$, then $I_2 \in \mathfrak{F}$ (hence $L \in \mathfrak{F}$),
(2) If $I_1, I_2 \in \mathfrak{F}$, then $I_1 \cap I_2 \in \mathfrak{F}$.

Clearly, if $\mathfrak{F}$ is a topology on L, then $(L, \mathfrak{F} \cup \{\emptyset\})$ is a topological space. Any intersection of topologies on L is a topology, hence the set $\mathcal{T}(L)$ of all topologies of L is a complete lattice with respecto to inclusion.

$\mathfrak{F}$ is a topology on L iff $\mathfrak{F}$ is a filter of the lattice of power set of L, for this reason a topology on L is usually called a Gabriel filter on $I(L)$.

Definition 2.3. Let $\mathfrak{F}$ be a topology on a semi-Heyting algebra L. Define the relation $\theta_{\mathfrak{F}}$ on L as follows: $(x, y) \in \theta_{\mathfrak{F}}$ iff there exists $I \in F$ such that $e \wedge x = e \wedge y$ for any $e \in I$ for all $x, y \in L$.

Theorem 2.1. *$\theta_{\mathfrak{F}}$ is a congruence on L.*

Proof. The proofs of reflexivity and the symmetry of $\theta_{\mathfrak{F}}$ are straightforward. We show that $\theta_{\mathfrak{F}}$ is transitive. Suppose that $(x, y), (y, z) \in \theta_{\mathfrak{F}}$. Then there exist $I_1, I_2 \in \mathfrak{F}$ such that $e \wedge x = e \wedge y$ for every $e \in I_1$, and $f \wedge y = f \wedge z$ for every $f \in I_2$. Put $I = I_1 \cap I_2 \in \mathfrak{F}$. Then for every $g \in I$, we have $g \wedge x = g \wedge z$. Hence $(x, z) \in \theta_{\mathfrak{F}}$.

Now suppose that $(x, y), (z, w) \in \theta_{\mathfrak{F}}$. Then there exist $I_1, I_2 \in \mathfrak{F}$ such that $e \wedge x = e \wedge y$ for every $e \in I_1$, and $f \wedge z = f \wedge w$ for every $f \in I_2$. Put $I = I_1 \cap I_2 \in \mathfrak{F}$. Then $g \wedge x = g \wedge y$ and $g \wedge z = g \wedge w$, for every $g \in I$. Hence, we get that $g \wedge (x \wedge z) = (g \wedge y) \wedge z = (g \wedge w) \wedge y = g \wedge (y \wedge w)$. Then $(x \wedge z, y \wedge w) \in \theta_{\mathfrak{F}}$. On the other hand, $g \wedge (x \vee z) = (g \wedge x) \vee (g \wedge z) = (g \wedge y) \vee (g \wedge w) = g \wedge (y \vee w)$. Then $(x \vee z, y \vee w) \in \theta_{\mathfrak{F}}$. By (sH3), we have $g \wedge (x \to z) = g \wedge [(g \wedge x) \to (g \wedge z)] = g \wedge [(g \wedge y) \to (g \wedge w)] = g \wedge (y \to w)$. Thus $(x \to z, y \to w) \in \theta_{\mathfrak{F}}$. Hence $\theta_{\mathfrak{F}}$ is a congruence on L. □

Remark 2.1. Theorem 2.1 proves that $L/\theta_{\mathfrak{F}}$ is a semi-Heyting algebra. We will denote by $[x]_{\theta_{\mathfrak{F}}}$ the congruence class of an element $x \in L$ and by $p_{\mathfrak{F}} : L \longrightarrow L/\theta_{\mathfrak{F}}$ the canonical morphism of semi-Heyting algebras.

Definition 2.4. Let $\mathfrak{F}$ be a topology on a semi-Heyting algebra L. An $\mathfrak{F}-$multiplier is a mapping $f : I \longrightarrow L/\theta_{\mathfrak{F}}$ where $I \in \mathfrak{F}$ and for every $x \in I$ and $e \in L : f(e \wedge x) = [e]_{\theta_{\mathfrak{F}}} \wedge f(x)$.

Example 2.1. (a) The map $1 : L \longrightarrow L/\theta_{\mathfrak{F}}$ defined by $1(x) = [x]_{\theta_{\mathfrak{F}}}$ for every $x \in L$ is an $\mathfrak{F}-$multiplier.
(b) The map $0 : L \longrightarrow L/\theta_{\mathfrak{F}}$ defined by $0(x) = [0]_{\theta_{\mathfrak{F}}}$ for every $x \in L$ is an $\mathfrak{F}-$multiplier.
(c) For $a \in L$ and $I \in \mathfrak{F}$, $f_a : I \longrightarrow L/\theta_{\mathfrak{F}}$ defined by $f_a(x) = [a]_{\theta_{\mathfrak{F}}} \wedge [x]_{\theta_{\mathfrak{F}}}$ for every $x \in I$, is an $\mathfrak{F}-$multiplier. If $dom(f_a) = L$, we denote f_a by $\overline{f_a}$.

Lemma 2.2. *For each $\mathfrak{F}-$multiplier $f : I \longrightarrow L/\theta_{\mathfrak{F}}$ the following properties hold:*
(a) $f(x) \leq [x]_{\theta_{\mathfrak{F}}}$, *for all* $x \in I$,
(b) $f(x \wedge y) = f(x) \wedge f(y)$,
(c) $f(x \vee y) = f(x) \vee f(y)$,
(d) $[x]_{\theta_{\mathfrak{F}}} = [y]_{\theta_{\mathfrak{F}}} \wedge f(x)$.

Proof. It is straightforward. □

We denote the set of all the $\mathfrak{F}-$multipliers having the domain $I \in \mathfrak{F}$ by $M(I, L/\theta_{\mathfrak{F}})$ and $M(L/\theta_{\mathfrak{F}}) = \bigcup\limits_{I \in \mathfrak{F}} M(I, L/\theta_{\mathfrak{F}})$.

If $I_1, I_2 \in \mathfrak{F}$ and $I_1 \subseteq I_2$, we have a canonical mapping $\varphi_{I_1,I_2} : M(I_2, L/\theta_{\mathfrak{F}}) \longrightarrow M(I_1, L/\theta_{\mathfrak{F}})$, defined by $\varphi_{I_1,I_2}(f) = f\,|_{I_1}$ for $f \in M(I_2, L/\theta_{\mathfrak{F}})$.

Consider the directed system of sets $\langle \{M(I, L/\theta_{\mathfrak{F}})\}_{I \in \mathfrak{F}}, \{\varphi_{I_1,I_2}\}_{I_1,I_2 \in \mathfrak{F}, I_1 \subseteq I_2} \rangle$ and denote the inductive limit (in the category of sets) by $L_{\mathfrak{F}}$:

$$L_{\mathfrak{F}} = \varinjlim_{I \in \mathfrak{F}} M(I, L/\theta_{\mathfrak{F}}).$$

For any $\mathfrak{F}-$multiplier $f : I \longrightarrow L/\theta_{\mathfrak{F}}$, we denote by $\widehat{(I, f)}$ the equivalence class of f in $L_{\mathfrak{F}}$.

Remark 2.2. If $f_i : I_i \longrightarrow L/\theta_{\mathfrak{F}}$, $i = 1, 2$ are $\mathfrak{F}-$multipliers, then $\widehat{(I_1, f_1)} = \widehat{(I_2, f_2)}$ (in $L_{\mathfrak{F}}$) iff there exists $I \in \mathfrak{F}$ and $I \subseteq I_1 \cap I_2$ such that $f_1 \mid_I = f_2 \mid_I$.

Let $f_1 \in M(I_1, L/\theta_{\theta_{\mathfrak{F}}})$, $f_2 \in M(I_2, L/\theta_{\mathfrak{F}})$ two $\mathfrak{F}-$multipliers and we consider the mappings $f_1 \vee f_2, f_1 \wedge f_2, f_1 \to f_2 : I_1 \cap I_2 \longrightarrow L/\theta_{\mathfrak{F}}$ defined by:

- $(f_1 \vee f_2)(x) = f_1(x) \vee f_2(x)$
- $(f_1 \wedge f_2)(x) = f_1(x) \wedge f_2(x)$
- $(f_1 \to f_2)(x) = [f_1(x) \to f_2(x)] \wedge p_{\mathfrak{F}}(x)$ for any $x \in I_1 \cap I_2$.

The fact that $f_1 \vee f_2$ and $f_1 \wedge f_2$ are $\mathfrak{F}-$multipliers is proved in [15].

Proposition 2.3. *$f_1 \to f_2$ defined as before is a $\mathfrak{F}-$multiplier.*

Proof. Let $x \in I_1 \cap I_2$ and $e \in L$. Then by (sH3) we have

$$\begin{aligned}(f_1 \to f_2)(x \wedge y) &= p_{\mathfrak{F}}(x \wedge y) \wedge [f_1(x \wedge y) \to f_2(x \wedge y)] \\ &= p_{\mathfrak{F}}(x) \wedge p_{\mathfrak{F}}(y) \wedge [(f_1(x) \wedge p_{\mathfrak{F}}(y)) \to (f_2(x) \wedge p_{\mathfrak{F}}(y))] \\ &= p_{\mathfrak{F}}(x) \wedge p_{\mathfrak{F}}(y) \wedge [f_1(x) \to f_2(x)] \\ &= (f_1 \to f_2)(x) \wedge p_{\mathfrak{F}}(y).\end{aligned}$$

Hence $f_1 \to f_2 \in M(I_1 \cap I_2, L/\theta_{\mathfrak{F}})$. □

We will define on $L_{\mathfrak{F}}$ the following operations:

- $\widehat{(I_1, f_1)} \wedge \widehat{(I_2, f_2)} = (I_1 \cap \widehat{I_2, f_1} \wedge f_2)$
- $\widehat{(I_1, f_1)} \vee \widehat{(I_2, f_2)} = (I_1 \cap \widehat{I_2, f_1} \vee f_2)$

and for $\widehat{(I_1, f_1)}$, $\widehat{(I_2, f_2)}$ we define the element

- $\widehat{(I_1, f_1)} \to \widehat{(I_2, f_2)} = (I_1 \cap \widehat{I_2, f_1} \to f_2)$.

We denote by $\mathbf{0}$ the equivalence class of $(L, 0)$ and by $\mathbf{1}$ the equivalence class of $(L, 1)$.

Lemma 2.4. *For each $I \in \mathfrak{F}$, $\langle M(I, L/\theta_{\mathfrak{F}}), \wedge, \vee, \to, 0, 1\rangle$ is a semi-Heyting algebra.*

Proof. It is easy to verify that $\langle M(I, L/\theta_{\mathfrak{F}}), \wedge, \vee, 0, 1\rangle$ is a bounded lattice. To prove that it is a semi-Heyting algebra, we have for all $f_1, f_2, f_3 \in M(I, L/\theta_{\mathfrak{F}})$ and $x \in I$,

(sH1)
$$\begin{aligned}(f_1 \wedge (f_1 \to f_2))(x) &= f_1(x) \wedge (f_1 \to f_2)(x) \\ &= f_1(x) \wedge (f_1(x) \to f_2(x)) \wedge [x]_{\theta_{\mathcal{F}}} \\ &= f_1(x) \wedge (f_1(x) \to f_2(x)) \\ &= f_1(x) \wedge f_2(x) \\ &= (f_1 \wedge f_2)(x)\end{aligned}$$

(sH2)
$$\begin{aligned}(f_1 \wedge [(f_1 \wedge f_2) \to (f_1 \wedge f_3)])(x) &= f_1(x) \wedge [(f_1 \wedge f_2) \to (f_1 \wedge f_3)](x) \\ &= f_1(x) \wedge [(f_1 \wedge f_2)(x) \to (f_1 \wedge f_3)(x)] \wedge [x]_{\theta_{\mathcal{F}}} \\ &= f_1(x) \wedge (f_2(x) \to f_3(x)) \wedge [x]_{\theta_{\mathcal{F}}} \\ &= f_1(x) \wedge (f_2 \to f_3)(x) \\ &= (f_1 \wedge (f_2 \to f_3))(x).\end{aligned}$$

(sH3)
$$\begin{aligned}(f_1 \to f_1)(x) &= (f_1(x) \to f_1(x)) \wedge [x]_{\theta_{\mathcal{F}}} \\ &= [1]_{\theta_{\mathcal{F}}} \wedge [x]_{\theta_{\mathcal{F}}} \\ &= [1 \wedge x]_{\theta_{\mathcal{F}}}\end{aligned}$$

$$= [x]_{\theta_{\mathcal{F}}}$$
$$= 1(x).$$

□

Theorem 2.5. $\langle L_{\mathfrak{F}}, \vee, \wedge, \rightarrow, \mathbf{0}, \mathbf{1}\rangle$ *is a semi-Heyting algebra.*

Proof. It follows as a special case of Corollary 2.1 in [18]. Indeed, condition (ii) in Definition 2.2 is stronger than the property of being dwon directed, the operations $\vee, \wedge, \rightarrow, 0$ and 1 of $M(I, L/\theta_{\mathfrak{F}})$ obviously satisfy conditions (2.1) and (2.2) in [18, Section 2.1] and $M(I, L/\theta_{\mathfrak{F}})$ is a semi-Heyting algebra by Lemma 2.4. □

The semi-Heyting algebra $L_{\mathfrak{F}}$ will be called the localization semi-Heyting algebra of L with respect to the topology $\mathfrak{F}$.

3. Semi-Heyting algebra of fractions relative to an $\wedge-$closed system

Definition 3.1. A subset $S \subseteq A$ is called $\wedge-$closed system if $1 \in S$ and if $x, y \in S$ implies $x \wedge y \in S$.

We will denote by $S(L)$ the set of all $\wedge-$closed system of L.

Lemma 3.1. *Let $S \in S(L)$. Then, the binary relation θ_S defined by $(x, y) \in \theta_S \Leftrightarrow$ there is $s \in S$ such that $x \wedge s = y \wedge s$ is a congruence on L.*

Proof. We will only prove that θ_S is compatible with $\rightarrow$. Suppose that $(x, y), (z, w) \in \theta_S$. Then, there exists $s, e \in S$ such that $x \wedge s = y \wedge s$ and $z \wedge e = w \wedge e$. From these statements and (sH2), we have that

$$\begin{aligned} s \wedge e \wedge (x \rightarrow z) &= s \wedge e \wedge [(s \wedge e \wedge x) \rightarrow (s \wedge e \wedge z)] \\ &= s \wedge e \wedge [(s \wedge e \wedge y) \rightarrow (s \wedge e \wedge w)] \\ &= s \wedge e \wedge (y \rightarrow w). \end{aligned}$$

Thus, $(x \rightarrow z, y \rightarrow w) \in \theta_S$. Hence θ_S is a congruence on L. □

For $x \in L$ we denote the equivalence class of x relative to θ_S by $[x]_S$ and by $L[S] = L/\theta_S$. We denote the canonical map $p_S : L \longrightarrow L[S]$ defined by $p_S(x) = [x]_S$ for every $x \in L$. Clearly, $L[S]$ become a semi-Heyting algebra.

Remark 3.1. Since for every $s \in S$, $s \wedge s = s \wedge 1$, we deduce that $[s]_S = [1]_S$, hence $p_S(S) = \{[1]_S\}$.

Theorem 3.2. *If L is a semi-Heyting algebra and $f : L \longrightarrow L'$ is an morphism of semi-Heyting algebras such that $f(S) = \{1\}$, then there is a unique morphism of semi-Heyting algebras $f' : L[S] \longrightarrow L'$ such that $f' \circ p_S = f$.*

Proof. If follows from [18, Theorem 4.1] and Remark 3.1. □

Theorem 3.2 allows us to call $L[S]$ the semi-Heyting algebra of fractions relative to the $\wedge-$closed system S.

Remark 3.2. From Theorem 3.2 we have that

(i) If $S = \{1\}$, then θ_S coincides with the identity congruence on L and so, $L[S] \simeq L$.

(ii) If S is an $\wedge-$closed system of L such that $0 \in S$ (for example $S = L$), then $\theta_S = L \times L$. Hence, $L[S] = \{[0]_S\}$.

Remark 3.3. Let L a semi-Heyting algebra and $S \in S(L)$. Then $\mathfrak{F}_S = \{I \in I(L) : I \cap S \neq \emptyset\}$ is a topology on L.

The topology $\mathfrak{F}_S$ will be called the topology associated with the $\wedge-$closed system S of L.

Lemma 3.3. *Let $\mathcal{F}_S$ be the topology associated with the $\wedge-$closed system S. Then $\theta_{\mathcal{F}_S} = \theta_S$.*

Proof. Let $(x, y) \in \theta_{\mathcal{F}_S}$. Then there is $I \in \mathfrak{F}_S$ such that $s \wedge x = s \wedge y$, for all $s \in I$. Since there exists $s_o \in I \cap S$ verifying $s_o \wedge x = s_o \wedge y$, we infer that $(x, y) \in \theta_S$. Conversely, let $(x, y) \in \theta_S$. Then there is $s_o \in S$ such that $x \wedge s_o = y \wedge s_o$. By considering $I = \langle s_o \rangle$ we conclude that $(x, y) \in \theta_{\mathfrak{F}_S}$. ☐

Remark 3.4. From Lemma 3.3, we have that $L/\theta_{\mathfrak{F}_S} = L[S]$. Then an $\mathfrak{F}_S-$multiplier can be consider as a map $f : I \longrightarrow L[S]$ where $I \in \mathfrak{F}_S$ and $f(e \wedge x) = [e]_S \wedge f(x)$ for all $x \in I$ and $e \in I$.

Lemma 3.4. *Let $\widehat{(I_1, f_1)}, \widehat{(I_2, f_2)} \in L_{\mathfrak{F}_S}$ be such that $\widehat{(I_1, f_1)} = \widehat{(I_2, f_2)}$. Then there exists $I \subseteq I_1 \cap I_2$ such that $f_1(s_o) = f_2(s_o)$ for all $s_o \in I \cap S$.*

Proof. From the hypothesis and Remark 2.2, we have that there exists $I \in \mathfrak{F}_S$, $I \subseteq I_1 \cap I_2$ such that $f_1 \mid_I = f_2 \mid_I$ and so, $f_1(s_o) = f_2(s_o)$ for each $s_o \in I \cap S$. ☐

Theorem 3.5. *Let L be a semi-Heyting algebra. If $\mathfrak{F}_S$ is the topology associated with the $\wedge-$closed system S, then $L_{\mathfrak{F}_S}$ is isomorphic to $L[S]$.*

Proof. Let $\alpha : L_{\mathfrak{F}_S} \longrightarrow L[S]$ be defined by $\alpha\widehat{(I, f)} = f(s)$ for all $s \in I \cap S$. From Lemma 3.4, we have that α is well-defined. Besides, α is onte-to-one. Indeed, suppose that $\alpha\widehat{(I_1, f_1)} = \alpha\widehat{(I_2, f_2)}$. Then there exist $s_1 \in I_1 \cap S$ and $s_2 \in I_2 \cap S$ such that $f_1(s_1) = f_2(s_2)$. Hence, by considering $f_1(s_1) = [x]_S$ and $f_2(s_2) = [y]_S$, we have that there is $s \in S$ verifying $x \wedge s = y \wedge s$. If $s' = s \wedge s_1 \wedge s_2$, then we infer that $f_1(s') = f_1(s' \wedge s_1) = [s']_S \wedge f_1(s_1) = [s']_S \wedge f_2(s_2) = f_2(s')$. Let $I = \langle s' \rangle$. So, $I \in \mathfrak{F}_S$, $I \subseteq I_1 \cap I_2$ and $f_1 \mid_I = f_2 \mid_I$. Remark 2.2 allows us to infer that $\widehat{(I_1, f_1)} = \widehat{(I_2, f_2)}$. In order to prove that α is surjective, let $[a]_S \in L[S]$ and $f_a : L \longrightarrow L[S]$ defined by $f_a(x) = [a \wedge x]_S$ for all $x \in L$. It is simple to verify that f_a is an $\mathfrak{F}_S-$multiplier. Moreover, from Remark 3.1, $\alpha\widehat{(L, f_a)} = f_a(s) = [a \wedge s]_S = [a]_S$, being $s \in S$. It is simple to verify that this map is an morphism of semi-Heyting algebras. ☐

4. Localization and fractions in finite semi-Heyting algebras

In this section, our attention is focus on considering the above results in the particular case of finite semi-Heyting algebras. More precisely, we will prove that for each finite semi-Heyting algebra L and $S \in S(L)$ the algebra $L[S]$ is isomorphic to a special subalgebra of L. In order to do this, the following propositions will be fundamental.

Proposition 4.1. *Let L be a finite semi-Heyting algebra and $I \subseteq L$. Then, the following conditions are equivalent:*

(i) $I \in I(A)$,

(ii) $I = \langle a \rangle$ *for some* $a \in L$.

Proof. It is straightforward. ☐

Proposition 4.2. *Let L be a finite semi-Heyting algebra and $S \in S(L)$. Then* $\mathfrak{F}_S = \{\langle a\rangle : \bigwedge\limits_{x\in S} x \leq a\}$.

Proof. Let us consider $\mathfrak{T} = \{\langle a\rangle : \bigwedge\limits_{x\in S} x \leq a\}$. Assume that $I \in \mathfrak{F}_S$. Then, by Proposition 4.1 we have that $I = \langle a\rangle$ for some $a \in L$. On the other hand, from Remark 3.3 there is $s \in S \cap \langle a\rangle$ which implies that $\bigwedge\limits_{x\in S} \leq s \leq a$. Therefore, $I \in \mathfrak{T}$. Conversely, suppose that $I \in \mathfrak{T}$. Hence, $\bigwedge\limits_{x\in S} x \in I \cap S$. Furthermore by Proposition 4.1 we have that $I \in I(L)$. From these last assertions and Remark 3.3 we conclude that $I \in \mathfrak{F}_S$. $\square$

Proposition 4.3. *Let L be a finite semi-Heyting algebra and $S \in S(L)$. Then, the following conditions are equivalent:*

(i) $(x, y) \in \theta_{\mathfrak{F}_S}$

(ii) $x \wedge b = y \wedge b$ *where* $b = \bigwedge\limits_{x\in S} x$.

Proof. It is routine. $\square$

Lemma 4.4. ([19, Lemma 6.2]) *Let L be a finite semi-Heyting algebra and $\langle a\rangle \in I(L)$. Then, $L_a = \langle\langle a\rangle, \vee, \wedge, \to^a, 0, a\rangle$ is a semi-Heyting algebra.*

Finally, we obtain our desired goal.

Theorem 4.5. *Let L be a finite semi-Heyting algebra and $S \in S(L)$. Then $L[S]$ is isomorphic to L_b where* $b = \bigwedge\limits_{x\in S} x$.

Proof. Let $\beta : L \longrightarrow L_b$ be the function defined by the prescription $\beta(x) = x \wedge b$ By [19, Lemma 6.2] β is a semi-Heyting morphism. It is easy to check that β is surjective. Therefore, β is an semi-Heyting epimorphism. Moreover, $x \in [1]_{\theta_S} \Leftrightarrow (x, 1) \in \theta_S \Leftrightarrow$ there is $s \in S$ such that $x \wedge s = s \Leftrightarrow x \wedge b = b \Leftrightarrow \beta(x) = b \Leftrightarrow x \in \ker(\beta)$. Therefore, taking into account a well-known result of universal algebra (see [4, p. 59]) we conclude that $L[S]$ is isomorphic to L_b. $\square$

Corollary 4.6. *Let L be a finite semi-Heyting algebra and $S \in S(L)$. Then, $L_{\mathfrak{F}_S}$ is isomorphic to L_b where $b = \bigwedge\limits_{x\in S} x$. More precisely,* $L_{\mathfrak{F}_S} = \{\widehat{(\langle b\rangle, f_x)} : x \in \langle b\rangle\}$.

Proof. It follows as a consequence of Theorem 3.5 and Theorem 4.5. $\square$

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