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Some weighted Grüss-type inequalities with not necessarily positive weights on time scales

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Abstract. The aim of this paper is to present weighted versions of some Grüss-type inequalities on time scales, using weights that are allowed to take some negative values, these are Hermite–Hadamard weights, the Steffensen–Popoviciu weights and the Grüss–Popoviciu weights. The notions of weighted integral mean, weighted variance and weighted covariance are expanded according to these weights. Some applications of these inequalities are presented.

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1. Introduction

In the last years, the theory of time scales has received new developments and the applications of dynamic derivatives on time scales emerged. This study provides an unification and an extension of traditional differential equations and, in the same time, could be considered an unification of the discrete theory with the continuous theory, from the scientific point of view. Also, it is a tool of the utmost importance in many computational and numerical applications.

A combined dynamic derivative, so called $\diamond_\alpha$ (diamond-α) dynamic derivative, was introduced as a linear combination of the well-known Δ (delta) and ∇ (nabla) dynamic derivatives on time scales (see [15]). Starting with delta an nabla derivatives, the notions of delta and nabla integrals were defined. Throughout this paper, it is assumed that the basic notions of time scales are well known and understood. For the basic rules of calculus on time scales, please refer to [1, 3, 8, 15, 24, 26].

The probability theory is the perfect field to grow such an amazing calculus, since random variables and distributions functions can be either discrete or continuous. Moreover, negative probabilities and their applications in physics and quantum mechanics received a lot of attention during the last years, especially because they helped us solving and understanding several problems and paradoxes (see [9, 12, 18]).

The classical Grüss inequality gives an estimation, from above, of a product of two functions compared to the product of the integrals of the two functions (see [14]).

Theorem 1.1. *Let f and g be two bounded functions on $[a, b]$ with $\gamma_1 \le f(x) \le \Gamma_1$ and $\gamma_2 \le g(x) \le \Gamma_2$, where $\gamma_1, \gamma_2, \Gamma_1, \Gamma_2$ are four real constants. Then*

$$\left| \frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \frac{1}{b-a}\int_a^b g(x)\mathrm{d}x \right| \le \frac{1}{4}(\Gamma_1-\gamma_1)(\Gamma_2-\gamma_2).$$

The constant $1/4$ is sharp, since it cannot be replaced by a smaller one.

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Many authors obtained a lot of integral Grüss-type inequalities for different classes of functions. They also used these inequalities to obtain some other sharp inequalities or to estimate the deviation of the values of the functions from its mean value. See [5, 6, 10, 11, 13, 16, 17, 19, 25].

In [20], Minculete and Ciurdariu presented several integral inequalities, including a generalized form of the Grüss-type inequality using some positive weights and they also rewrote several integral inequalities using the h-integral arithmetic mean for a Riemann-integrable functions. In [7], Dinu defined some classes of weights that are allowed to take some negative values, these are the Steffensen–Popoviciu and Hermite–Hadamard weights and then he proved a complete weighted version of the Hermite–Hadamard inequality for convex functions on time scales. The connection between these weights is also presented there.

The aim of this paper is to prove some analogues of many integral inequalities on time scales, including some Grüss-type inequalities by computing the mean value of a function using the diamond-α integrals and some not necessarily positive classes of weights. We shall obtain improved integral inequalities than in [20] by enlarging the class of weights, and also better discrete inequalities than those obtained by Popoviciu in [23], improved by Bhatia and Davis in [2].

In section 2 we present some notions of integral mean, variance and covariance from probability theory, adjusted with the classes of weights that are allowed to take some negative values, although they have to satisfy some end positivity conditions. Some of their properties, analogues to the classic notions are also included.

In section 3 we give our main results, regarding the weighted version of some inequalities on time scales, including the discrete and integral forms of the Grüss-type inequality.

2. Basic notions

Let $\mathbb{T}$ be a time scale and $a, b \in \mathbb{T}$ with $a \leq b$. Also, let $g : \mathbb{T} \to \mathbb{R}$ be a continuous function and $w : \mathbb{T} \to \mathbb{R}$ another continuous function, called an *α-weight* for g on $[a,b]_{\mathbb{T}}$ if $\int_a^b w(t)\diamond_\alpha t > 0$.

The *w-weighted integral arithmetic mean* of the function g on $[a,b]_{\mathbb{T}}$ is defined as

$$M_w[g] = \frac{\int_a^b g(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}. \tag{1}$$

The following property of the w-weighted integral arithmetic mean of the function g on $[a,b]_{\mathbb{T}}$ is straightforward:

$$M_w[g \pm k] = M_w[g] \pm k,$$

for every k constant real number.

The *w-weighted variance* of the function g on $[a,b]_{\mathbb{T}}$ is also defined as

$$\mathrm{Var}_w(g) = M_w\left[(g - M_w[g])^2\right], \tag{2}$$

but it can be easily expanded in the following manner:

$$\mathrm{Var}_w(g) = \frac{1}{\int_a^b w(t)\diamond_\alpha t} \int_a^b \left(g(t) - \frac{\int_a^b g(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} \right)^2 w(t)\diamond_\alpha t.$$

As in the usual case, the w-variance has another form, given by:

$$\mathrm{Var}_w(g) = M_w\left[g^2\right] - M_w^2\left[g\right],$$

and the following property is straightforward

$$\mathrm{Var}_w(g \pm k) = \mathrm{Var}_w(g),$$

for every constant real number k.

The w-*weighted covariance* of the function g and h on $[a,b]_{\mathbb{T}}$ is a “measure” of how much two functions change together and it is defined as

$$\mathrm{Cov}_w(g,h) = M_w\left[(g - M_w[g])\,(h - M_w[h])\right], \tag{3}$$

but it can be expanded thus:

$$\mathrm{Cov}_w(g,h) = M_w\left[gh\right] - M_w\left[g\right] M_w\left[h\right],$$

or

$$\mathrm{Cov}_w(g,h) = \frac{\int_a^b g(t)h(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} - \frac{\int_a^b g(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}\cdot\frac{\int_a^b h(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}.$$

Using the notions of covariance, the Grüss inequality can be restated as:

$$\mathrm{Cov}_w(g,h) \le \frac{1}{4}(\Gamma_1 - \gamma_1)(\Gamma_2 - \gamma_2), \tag{4}$$

where w is a constant positive function and $[a,b]_{\mathbb{T}} = [a,b]$.

Let $x_1, x_2, ..., x_n$ be real numbers, with $\gamma \le x_i \le \Gamma$ for all $i = \overline{1,n}$ and their mean $\overline{x} = \frac{1}{n}\sum_{i=1}^n x_i$. Popoviciu proved in [23] the following discrete inequality:

$$\frac{1}{n}\sum_{i=1}^n (x_i - \overline{x})^2 \le \frac{1}{4}(\Gamma - \gamma)^2, \tag{5}$$

improved by Bhatia and Davis in [2] thus:

$$\frac{1}{n}\sum_{i=1}^n (x_i - \overline{x})^2 \le (\Gamma - \overline{x})(\overline{x} - \gamma). \tag{6}$$

Using our notations, inequalities (5) and (6) have the following forms:

$$\mathrm{Var}_w(g) \le \frac{1}{4}(\Gamma - \gamma)^2, \tag{7}$$

and

$$\mathrm{Var}_w(g) \le (\Gamma - M_w\left[g\right])(M_w\left[g\right] - \gamma), \tag{8}$$

where $[a,b]_{\mathbb{T}} = \{1, 2, ..., n+1\}$, $\alpha = 1$ and w is a constant positive function.

We also recall here the notions of α-Steffensen–Popoviciu weight for g on $[a,b]_{\mathbb{T}}$ and α-Hermite–Hadamard weight for g on $[a,b]_{\mathbb{T}}$ from [7]. Thus, the continuous function $w : \mathbb{T} \to \mathbb{R}$ is an α-*Steffensen–Popoviciu weight* for g on $[a,b]_{\mathbb{T}}$ (abbreviated α-SP weight) if

$$\int_a^b w(t)\diamond_\alpha t > 0 \quad \textit{and} \quad \int_a^b f(g(t))^+ w(t)\diamond_\alpha t \ge 0, \tag{9}$$

while, the continuous function $w : \mathbb{T} \to \mathbb{R}$ is an α-*Hermite–Hadamard weight* for g on $[a,b]_{\mathbb{T}}$ (abbreviated α-HH weight) if

$$\int_a^b w(t)\diamond_\alpha t > 0 \quad \textit{and} \quad \frac{\int_a^b f(g(t))w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} \le \frac{\Gamma - M_w[g]}{\Gamma - \gamma}f(\gamma) + \frac{M_w[g] - \gamma}{\Gamma - \gamma}f(\Gamma). \tag{10}$$

for every $f : [\gamma, \Gamma] \to \mathbb{R}$ continuous convex function, where $\gamma = \inf_{t\in[a,b]_{\mathbb{T}}} g(t)$ and $\Gamma = \sup_{t\in[a,b]_{\mathbb{T}}} g(t)$.

The connection between these two classes of weights is given in [7, Theorem 3] together with a full characterization of each kind of weights. That is, every α-Steffensen–Popoviciu weight for g on $[a,b]_{\mathbb{T}}$ is an α-Hermite–Hadamard weight for g on $[a,b]_{\mathbb{T}}$, for every $\alpha \in [0,1]$. Obviously, all the positive weights are α-SP weights for any continuous function g and every $\alpha \in [0,1]$. But there are some α-SP weights that are allowed to take negative values. For instance, if $[a,b]_{\mathbb{T}} = [-1,1]$, then $w(t) = t^2 + a$ is an α-SP weight for $g(t) = t$ on $[-1,1]$ if $a > -1/3$.

3. Main results

We give the integral and discrete versions of the inequalities (4), (7) and (8) improved not only by allowing the weight w to take nonconstant values, but also to take some negative ones.

Lemma 3.1. *Let $g : \mathbb{T} \to \mathbb{R}$ be a continuous function with $\gamma \le g(t) \le \Gamma$ for all $t \in [a,b]_{\mathbb{T}}$ and $w : \mathbb{T} \to \mathbb{R}$ an α-Hermite–Hadamard weight for g on $[a,b]_{\mathbb{T}}$. Then, we have*

$$Var_w(g) \le \frac{1}{4}(\Gamma - \gamma)^2. \tag{11}$$

Proof. Taking into account the aforementioned properties of the w-variance of a function, we obtain

$$\begin{aligned}
\mathrm{Var}_w(g) &= \mathrm{Var}_w\left(g - \frac{\Gamma+\gamma}{2}\right) \\
&= M_w\left[\left(g - \frac{\Gamma+\gamma}{2}\right)^2\right] - M_w^2\left[g - \frac{\Gamma+\gamma}{2}\right] \\
&\le M_w\left[\left(g(t) - \frac{\Gamma+\gamma}{2}\right)^2\right] \\
&= \frac{1}{\int_a^b w(t)\diamond_\alpha t}\int_a^b \left(g(t) - \frac{\Gamma+\gamma}{2}\right)^2 w(t)\diamond_\alpha t \\
&\le \frac{(\Gamma-\gamma)^2}{4}.
\end{aligned}$$

The last inequality is true by considering the convex function $f : [\gamma,\Gamma] \to \mathbb{R}$, $f(x) = \left(x - \frac{\Gamma+\gamma}{2}\right)^2$ in the definition of the α-HH weight. □

Lemma 3.2. *Let $g : \mathbb{T} \to [\gamma,\Gamma]$ be a continuous function and $w : \mathbb{T} \to \mathbb{R}$ an α-Steffensen–Popoviciu weight for g on $[a,b]_{\mathbb{T}}$. Then, we have*

$$\mathrm{Var}_w(g) \ge 0. \tag{12}$$

Proof. We have

$$\mathrm{Var}_w(g) = \frac{1}{\int_a^b w(t)\diamond_\alpha t}\int_a^b \left(g(t) - \frac{\int_a^b g(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}\right)^2 w(t)\diamond_\alpha t \ge 0,$$

using the complete weighted Jensen inequality for the convex function $f : [\gamma,\Gamma] \to \mathbb{R}$, $f(x) = (x - M_w[g])^2$, (see [7, Theorem 2]). □

In order to obtain our next result, we need to derive a new class of α-weights from the class of α-SP weights of a function on a time scale.

Definition 3.1. Let $\mathbb{T}$ be a time scale and $g : \mathbb{T} \to \mathbb{R}$ a continuous function and $w : \mathbb{T} \to \mathbb{R}$ an α-Steffensen–Popoviciu weight for g on $[a, b]_{\mathbb{T}}$. The function w is an *α-Grüss–Popoviciu weight for g on* $[a, b]_{\mathbb{T}}$ (abbreviated α-*GP* weight) if

$$\int_a^b (\Gamma - g(t))(g(t) - \gamma)w(t)\diamond_\alpha t \geq 0. \tag{13}$$

Recalling the above example, we notice that if $[a, b]_{\mathbb{T}} = [-1, 1]$, then $w(t) = t^2 + a$ is an α-*GP* weight for $g(t) = t$ on $[-1, 1]$ if $a > -1/5$.

Now, we can state an important result of this paper.

Lemma 3.3. *Let $g : \mathbb{T} \to \mathbb{R}$ be a continuous function with $\gamma \leq g(t) \leq \Gamma$ for all $t \in [a, b]_{\mathbb{T}}$ and $w : \mathbb{T} \to \mathbb{R}$ an α-GP weight for g on $[a, b]_{\mathbb{T}}$. Then, we have*

$$\mathrm{Var}_w(g) \leq (\Gamma - M_w[g])(M_w[g] - \gamma). \tag{14}$$

Proof. As we noticed before,

$$\mathrm{Var}_w(g) = \frac{\int_a^b g^2(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} - \left(\frac{\int_a^b g^2(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}\right)^2$$

and adjusting the computational methods used in [10] and [20], we get

$$\begin{aligned}
&\frac{\int_a^b (\Gamma - g(t))(g(t) - \gamma)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} \\
&\quad = -\Gamma\gamma + (\Gamma + \gamma)\frac{\int_a^b g(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} - \frac{\int_a^b g^2(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} \\
&\quad = -\Gamma\gamma + (\Gamma + \gamma)\frac{\int_a^b g(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} - \mathrm{Var}_w(g) - \left[\frac{\int_a^b g(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}\right]^2 \\
&\quad = \left(\Gamma - \frac{\int_a^b g(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}\right)\left(\frac{\int_a^b g(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} - \gamma\right) - \mathrm{Var}_w(g).
\end{aligned}$$

But, since w is an α-*GP* weight for g on $[a, b]_{\mathbb{T}}$, then $\int_a^b (\Gamma - g(t))(g(t) - \gamma)w(t)\diamond_\alpha t \geq 0$ and $\int_a^b w(t)\diamond_\alpha t > 0$, and so, the proof is complete. □

We should also remember the equality founded above, in terms of random variables, that is:

$$\mathrm{Cov}_w(\Gamma - g, g - \gamma) = (\Gamma - M_w[g])(M_w[g] - \gamma) - \mathrm{Var}_w(g). \tag{15}$$

Now, adapting some ideas from [20], we improve some relations between w-variance and w-covariance.

Lemma 3.4. *Let $g, h : \mathbb{T} \to \mathbb{R}$ be continuous functions and $w : \mathbb{T} \to \mathbb{R}$ an α-weight on $[a, b]_{\mathbb{T}}$. Then, we have*

$$\mathrm{Var}_w(ag + bh) = a^2\mathrm{Var}_w(g) + b^2\mathrm{Var}_w(h) + 2ab\mathrm{Cov}_w(g, h), \tag{16}$$

where a, b are constant real numbers.

Proof. Using the properties of the variance, we obtain

$$\begin{aligned}\mathrm{Var}_w(ag+bh) &= M_w\left[(ag+bh)^2\right] - M_w^2\left[ag+bh\right]\\ &= M_w\left[a^2g^2+2abgh+b^2h^2\right] - \left(aM_w[g]+bM_w[h]\right)^2\\ &= a^2M_w\left[g^2\right]+2abM_w\left[gh\right]+b^2M_w\left[h^2\right]\\ &\quad - a^2M_w^2\left[g\right]-2abM_w\left[g\right]M_w\left[h\right]-b^2M_w^2\left[h\right]\\ &= a^2\mathrm{Var}_w(g)+b^2\mathrm{Var}_w(h)+2ab\mathrm{Cov}_w(g,h)\end{aligned}$$

and so, the statement is true. □

Taking $a=b=1$ and $a=1,\ b=-1$ in (16), we get

Corollary 3.5. *In the above conditions, we have*

$$\mathrm{Var}_w(g+h) = \mathrm{Var}_w(g)+\mathrm{Var}_w(h)+2\mathrm{Cov}_w(g,h), \tag{17}$$

and

$$\mathrm{Var}_w(g-h) = \mathrm{Var}_w(g)+\mathrm{Var}_w(h)-2\mathrm{Cov}_w(g,h). \tag{18}$$

Lemma 3.6. *Let $g,h,p,q:\mathbb{T}\to\mathbb{R}$ be continuous functions and $w:\mathbb{T}\to\mathbb{R}$ an α-weight on $[a,b]_{\mathbb{T}}$. Then, we have*

$$\begin{aligned}&\mathrm{Cov}_w(ag+bh,cp+dq)\\ &\qquad= ac\mathrm{Cov}_w(g,p)+ad\mathrm{Cov}_w(g,q)+bc\mathrm{Cov}_w(h,p)+bd\mathrm{Cov}_w(h,q),\end{aligned} \tag{19}$$

where a,b,c,d are constant real numbers.

Proof. It suffices to recall the above mentioned properties of covariance to get

$$\begin{aligned}&\mathrm{Cov}_w(ag+bh,cp+dq)\\ &\quad= M_w[(ag+bh)(cp+dq)]-M_w[ag+bh]M_w[cp+dq]\\ &\quad= M_w[acgp+adgq+bchp+bdhq]\\ &\qquad -\left(aM_w[g]+aM_w[h]\right)\left(cM_w[p]+dM_w[q]\right)\\ &\quad= ac\left(M_w[gp]-M_w[g]M_w[p]\right)+ad\left(M_w[gq]-M_w[g]M_w[q]\right)\\ &\qquad +bc\left(M_w[hp]-M_w[h]M_w[p]\right)+bd\left(M_w[hq]-M_w[h]M_w[q]\right)\\ &\quad= ac\mathrm{Cov}_w(g,p)+ad\mathrm{Cov}_w(g,q)+bc\mathrm{Cov}_w(h,p)+bd\mathrm{Cov}_w(h,q).\end{aligned}$$

□

The next statement is a version of the weighted inequality of Cauchy–Schwartz for random variables, but we improve it using α-SP weights.

Proposition 3.7. *Let $g,h:\mathbb{T}\to\mathbb{R}$ be continuous functions and $w:\mathbb{T}\to\mathbb{R}$ an α-SP weight for the function f, g and h, where $f:\mathbb{T}\to\mathbb{R}$, $f(t)=g(t)-h(t)\mathrm{Cov}_w(g,h)/\mathrm{Var}_w(h)$ on $[a,b]_{\mathbb{T}}$, if $\mathrm{Var}_w(h)\neq 0$. Then, we have*

$$\mathrm{Cov}_w^2(g,h)\le \mathrm{Var}_w(g)\mathrm{Var}_w(h). \tag{20}$$

Proof. If $\mathrm{Var}_w(g)=0$ or $\mathrm{Var}_w(h)=0$ then the relation (20) is valid. If $\mathrm{Var}_w(h)\neq 0$, then using Lemma (3.2) and the fact that w is an α-SP weight for the function f, where $f:\mathbb{T}\to\mathbb{R}$, $f(t)=g(t)-h(t)\mathrm{Cov}_w(g,h)/\mathrm{Var}_w(h)$ on $[a,b]_{\mathbb{T}}$, we get

$$\mathrm{Var}_w(f)=\mathrm{Var}_w(g)-\frac{\mathrm{Cov}_w^2(g,h)}{\mathrm{Var}_w(h)}\ge 0.$$

□

Now, we are able to give an improvement of the Grüss inequality, using not necessarily positive weights.

Theorem 3.8. *Let $g,h:[a,b]_{\mathbb{T}}\to\mathbb{R}$ be two continuous functions with $\gamma_1\le g(t)\le\Gamma_1$ and $\gamma_2\le h(t)\le\Gamma_2$, where $\gamma_1,\gamma_2,\Gamma_1,\Gamma_2$ are four constants and $w:[a,b]_{\mathbb{T}}\to\mathbb{R}$ an α-GP weight for g and h on $[a,b]_{\mathbb{T}}$ and an α-SP weight for the function f, where $f:[a,b]_{\mathbb{T}}\to\mathbb{R}$, $f(t)=g(t)-h(t)\mathrm{Cov}_w(g,h)/\mathrm{Var}_w(h)$ on $[a,b]_{\mathbb{T}}$, if $\mathrm{Var}_w(h)\neq 0$. Then*

$$\begin{aligned}|\mathrm{Cov}_w(g,h)| &\le \sqrt{(\Gamma_1-M_w[g])(M_w[g]-\gamma_1)(\Gamma_2-M_w[h])(M_w[h]-\gamma_2)}\\ &\le \frac{1}{4}(\Gamma_1-\gamma_1)(\Gamma_2-\gamma_2).\end{aligned}\tag{21}$$

The proof is straightforward by applying Proposition 3.7 and Lemma 3.3.

Taking some well-known time scales, ($\mathbb{T}=\mathbb{R}$, $\mathbb{T}=\mathbb{N}$ and $\mathbb{T}=q^{\mathbb{N}}$) we find interesting inequalities.

Corollary 3.9. (The continuous case). *Let $g,h:[a,b]\to\mathbb{R}$ be two integrable functions with $\gamma_1\le g(x)\le\Gamma_1$ and $\gamma_2\le h(x)\le\Gamma_2$, where $\gamma_1,\gamma_2,\Gamma_1,\Gamma_2$ are four constants and $w:[a,b]\to\mathbb{R}$ an GP weight for g and h on $[a,b]$ and an SP weight for the function f, where $f:[a,b]\to\mathbb{R}$, $f(t)=g(t)-h(t)\mathrm{Cov}_w(g,h)/\mathrm{Var}_w(h)$ on $[a,b]$, if $\mathrm{Var}_w(h)\neq 0$. Then*

$$\begin{aligned}&\left|\frac{\int_a^b g(t)h(t)w(t)\mathrm{d}t}{\int_a^b w(t)\mathrm{d}t}-\frac{\int_a^b g(t)w(t)\mathrm{d}t}{\int_a^b w(t)\mathrm{d}t}\cdot\frac{\int_a^b h(t)w(t)\mathrm{d}t}{\int_a^b w(t)\mathrm{d}t}\right|\\ &\le\sqrt{\left(\Gamma_1-\frac{\int_a^b g(t)w(t)\mathrm{d}t}{\int_a^b w(t)\mathrm{d}t}\right)\left(\frac{\int_a^b g(t)w(t)\mathrm{d}t}{\int_a^b w(t)\mathrm{d}t}-\gamma_1\right)}\\ &\quad\cdot\sqrt{\left(\Gamma_2-\frac{\int_a^b h(t)w(t)\mathrm{d}t}{\int_a^b w(t)\mathrm{d}t}\right)\left(\frac{\int_a^b h(t)w(t)\mathrm{d}t}{\int_a^b w(t)\mathrm{d}t}-\gamma_2\right)}\\ &\le\frac{1}{4}(\Gamma_1-\gamma_1)(\Gamma_2-\gamma_2).\end{aligned}\tag{22}$$

Corollary 3.10. (The discrete case). *Let $\mathbb{T}=\mathbb{Z}$, $n\in\mathbb{N}$, $\alpha=1$, $a=1$, $b=n+1$, $g,h:\{1,...,n+1\}\to\mathbb{R}$, $g(i)=x_i$, $h(i)=y_i$ with $\gamma_1\le x_i\le\Gamma_1$ and $\gamma_2\le y_i\le\Gamma_2$, where $\gamma_1,\gamma_2,\Gamma_1,\Gamma_2$ are four constants and $w:\{1,...,n+1\}\to\mathbb{R}$, $w(i)=w_i$, such that w_i are GP weights for x_i and y_i , but also an SP weight for $f_i=x_i-y_i\mathrm{Cov}_w(g,h)/\mathrm{Var}_w(h)$, if $\mathrm{Var}_w(h)\neq 0$, for every $i\in\{1,...,n+1\}$. Then*

$$\begin{aligned}&\left|\frac{\sum_{i=1}^n x_iy_iw_i}{\sum_{i=1}^n w_i}-\frac{\sum_{i=1}^n x_iw_i}{\sum_{i=1}^n w_i}\cdot\frac{\sum_{i=1}^n y_iw_i}{\sum_{i=1}^n w_i}\right|\\ &\le\sqrt{\left(\Gamma_1-\frac{\sum_{i=1}^n x_iw_i}{\sum_{i=1}^n w_i}\right)\left(\frac{\sum_{i=1}^n x_iw_i}{\sum_{i=1}^n w_i}-\gamma_1\right)}\\ &\quad\cdot\sqrt{\left(\Gamma_2-\frac{\sum_{i=1}^n y_iw_i}{\sum_{i=1}^n w_i}\right)\left(\frac{\sum_{i=1}^n y_iw_i}{\sum_{i=1}^n w_i}-\gamma_2\right)}\\ &\le\frac{1}{4}(\Gamma_1-\gamma_1)(\Gamma_2-\gamma_2).\end{aligned}\tag{23}$$

Corollary 3.11. (The quantum calculus case). *Let $\mathbb{T}=q^{\mathbb{Z}}$, $q>1$, $l,n\in\mathbb{N}$, with $l<n$, $\alpha\in[0,1]$, $a=q^l$, $b=q^n$, $g:\{l,...,n\}\to\mathbb{R}$, $g(i)=x_i$, $h(i)=y_i$ with $\gamma_1\le x_i\le\Gamma_1$ and $\gamma_2\le y_i\le\Gamma_2$, where $\gamma_1,\gamma_2,\Gamma_1,\Gamma_2$ are four constants and $w:\{l,...,n\}\to\mathbb{R}$,*

$w(i)=w_i$, such that w_i are α-GP weights for x_i and y_i, but also an α-SP weight for $f_i=x_i-y_i\mathrm{Cov}_w(g,h)/\mathrm{Var}_w(h)$, if $\mathrm{Var}_w(h)\neq 0$, for every $i\in\{l,...,n\}$. Then

$$\begin{aligned}
&\left|\frac{\alpha\sum_{i=l}^{n-1}q^ix_iy_iw_i+(1-\alpha)\sum_{i=l+1}^{n}q^ix_iy_iw_i}{\alpha\sum_{i=l}^{n-1}q^iw_i+(1-\alpha)\sum_{i=l+1}^{n}q^iw_i}\right.\\
&\qquad-\frac{\alpha\sum_{i=l}^{n-1}q^ix_iw_i+(1-\alpha)\sum_{i=l+1}^{n}q^ix_iw_i}{\alpha\sum_{i=l}^{n-1}q^iw_i+(1-\alpha)\sum_{i=l+1}^{n}q^iw_i}\\
&\qquad\left.\cdot\frac{\alpha\sum_{i=l}^{n-1}q^iy_iw_i+(1-\alpha)\sum_{i=l+1}^{n}q^iy_iw_i}{\alpha\sum_{i=l}^{n-1}q^iw_i+(1-\alpha)\sum_{i=l+1}^{n}q^iw_i}\right|\\
&\leq\sqrt{\left(\Gamma_1-\frac{\alpha\sum_{i=l}^{n-1}q^ix_iw_i+(1-\alpha)\sum_{i=l+1}^{n}q^ix_iw_i}{\alpha\sum_{i=l}^{n-1}q^iw_i+(1-\alpha)\sum_{i=l+1}^{n}q^iw_i}\right)}\\
&\qquad\cdot\sqrt{\left(\frac{\alpha\sum_{i=l}^{n-1}q^ix_iw_i+(1-\alpha)\sum_{i=l+1}^{n}q^ix_iw_i}{\alpha\sum_{i=l}^{n-1}q^iw_i+(1-\alpha)\sum_{i=l+1}^{n}q^iw_i}-\gamma_1\right)}\\
&\qquad\cdot\sqrt{\left(\Gamma_2-\frac{\alpha\sum_{i=l}^{n-1}q^iy_iw_i+(1-\alpha)\sum_{i=l+1}^{n}q^iy_iw_i}{\alpha\sum_{i=l}^{n-1}q^iw_i+(1-\alpha)\sum_{i=l+1}^{n}q^iw_i}\right)}\\
&\qquad\cdot\sqrt{\left(\frac{\alpha\sum_{i=l}^{n-1}q^iy_iw_i+(1-\alpha)\sum_{i=l+1}^{n}q^iy_iw_i}{\alpha\sum_{i=l}^{n-1}q^iw_i+(1-\alpha)\sum_{i=l+1}^{n}q^iw_i}-\gamma_2\right)}\\
&\leq\frac{1}{4}(\Gamma_1-\gamma_1)(\Gamma_2-\gamma_2).
\end{aligned}\tag{24}$$

Our next result is a weighted version of Cauchy inequality for time scales.

Lemma 3.12. *Let $g,h:\mathbb{T}\to\mathbb{R}$ be continuous functions and $w:\mathbb{T}\to\mathbb{R}$ an α-SP-weight for the functions g and $\lambda g-h$ on $[a,b]_{\mathbb{T}}$, for all $\lambda\in\mathbb{R}$. Then, we have*

$$M_w[g^2]M_w[h^2]\geq M_w^2[gh].\tag{25}$$

Proof. Since w is an α-SP-weight for g on $[a,b]_{\mathbb{T}}$, then $M_w[g^2]\geq 0$. If $M_w[g^2]=0$ then $M_w^2[gh]=0$ and the inequality (25) is valid. Now, we suppose that $M_w[g^2]>0$. Taking into account that w is an α-SP-weight for $\lambda g-h$ on $[a,b]_{\mathbb{T}}$ we get

$$M_w[(\lambda g-h)^2]\geq 0,\qquad\text{for all}\quad\lambda\in\mathbb{R}.$$

But

$$M_w[(\lambda g-h)^2]=\lambda^2M_w[g^2]-2\lambda M_w[gh]+M_w[h^2],$$

which implies

$$\lambda^2M_w[g^2]-2\lambda M_w[gh]+M_w[h^2]\geq 0,\qquad\text{for all}\quad\lambda\in\mathbb{R}.$$

And so, the discriminant of the above quadratic function in λ is negative and we get the conclusion. □

Furthermore, we can compare the estimations given by the weighted Cauchy–Schwartz inequality (20) and the weighted Cauchy inequality (25).

Theorem 3.13. *Let $g,h:\mathbb{T}\to\mathbb{R}$ be continuous functions and $w:\mathbb{T}\to\mathbb{R}$ an α-SP-weight for the functions g and h on $[a,b]_{\mathbb{T}}$. Then, we have*

$$0\leq\mathrm{Var}_w(g)\mathrm{Var}_w(h)-\mathrm{Cov}_w^2(g,h)\leq M_w[g^2]M_w[h^2]-M_w^2[gh].\tag{26}$$

Proof. The left side of the inequality is obtained from (20). For the right side, we evaluate the difference

$$M_w[g^2]M_w[h^2] - M_w^2[gh] - \left[\mathrm{Var}_w(g)\mathrm{Var}_w(h) - \mathrm{Cov}_w^2(g,h)\right]$$
$$= M_w[g^2]M_w^2[h] - 2M_w[gh]M_w[g]M_w[h] + M_w[h^2]M_w^2[g].$$

Since w is an α-SP-weight for g and h on $[a,b]_{\mathbb{T}}$, then $M_w[g], M_w[h] \geq 0$ and we can apply the classic AM-GM inequality to obtain

$$M_w[g^2]M_w^2[h] + M_w[h^2]M_w^2[g] \geq 2\sqrt{M_w[g^2]M_w[h^2]}M_w[g]M_w[h].$$
$$\geq 2M_w[gh]M_w[g]M_w[h],$$

that is

$$M_w[g^2]M_w^2[h] - 2M_w[gh]M_w[g]M_w[h] + M_w[h^2]M_w^2[g] \geq 0$$

and that ends the proof. □

4. Conclusions

Most of the well known properties for weighted variance and weighted covariance with positive weights can be "recovered" for weights that are not "entirely" positive, using powerful inequalities like the weighted Hermite–Hadamard inequality for α-Steffensen–Popoviciu weights and α-Hermite–Hadamard weights or the complete weighted Jensen inequality for α-Steffensen–Popoviciu weights (see [7]). The weights that verify most of the Grüss-type inequalities, called the α-Grüss–Popoviciu weights are derived from the α-Steffensen–Popoviciu ones, but they are not the same. Although, they are a little more restrictive, they take negative values too.

These weights can be applied in both continuous or discrete time scales, for continuous or discrete random variables, but the theory beyond both types is an unified one. Their applications are very important in many fields, such as quantum mechanics where they are associated with negative energies and negative probabilities. In probability theory, notions like negative probabilities and probabilities above unit are used more and more in calculations. In convolution quotients of nonnegative definite functions and algebraic probability theory, random variables are already in use with signed or quasi distributions where some of the probabilities are negative.

More recently, negative probabilities were applied to mathematical finance. Here, these are not real probabilities, they are called pseudo-probabilities and they are used in a series of assumptions to simplify calculations. A more rigorous mathematical theory of negative probabilities among with their properties were presented in [4], where the authors use these notions to financial option pricing.

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