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Properties of stratified languages

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Abstract. A new mechanism to generate formal languages is intuitively described in [7] and [8]. Two families of formal languages were defined: stratified languages of first type and stratified languages of second type. In this paper we present several algebraic properties of these families. Several open problems are relieved in the last section.

Key words and phrases. Peano algebra, morphism, stratified graph, formal languages.

1. Introduction

The concept of *labeled stratified graph* (*LSG*) was introduced in [9] related by the concept of knowledge base with output. The existence of this structure was proved in [10]. Since then many results and applications have been obtained. We can enumerate some applications: semantics of communication ([13]), geometrical image generation ([13]), reconstruction of geometrical image by extracting the semantics of a linguistic spatial description given in a natural language ([14]) obtained by a remote connection, application to attribute graphs in order to find the paths satisfying several restrictions ([12]), problem solving ([9]), a model of cooperation between two or more companies ([11], [12]), the use of stratified graphs in optimal planning ([16]). A recent research line refers to a new mechanism to generate formal languages (named *stratified languages*) given by stratified graphs ([7], [8]) and natural languages ([15]), thus the stratified graphs belong to the set of mechanisms that generate formal languages: automata, formal grammars, Lindenmayer systems, basic and recursive transition networks.

As it is shown in [7] and [8], there are two kinds of stratified languages. These are named *stratified languages of first type* and *stratified languages of second type*. In this paper we present several basic properties of these families of formal languages.

The paper is organized as follows: Section 2 presents the basic concepts and results to understand the next sections; Section 3 defines the erasing morphism and establishes several algebraic properties of this operator; Section 4 treats the family of the stratified languages of first type; Section 5 is devoted to the description of the stratified languages of second type; Section 6 lists several open problems.

2. Preliminary results and concepts

Let M be an arbitrary nonempty set and consider the Peano σ-algebra $\overline{M}$ generated by M ([1], [2], [5]). In order to obtain this structure we consider an operator symbol

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σ of arity 2 and the set B given by

$$B = \bigcup_{n\geq 0} B_n, \tag{1}$$

where

$$\begin{cases} B_0 = M, \\ B_{n+1} = B_n \cup \{\sigma(x_1, x_2) \mid (x_1, x_2) \in B_n \times B_n\},\ n \geq 0, \end{cases} \tag{2}$$

We suppose that $\sigma(x_1, x_2)$ is the word $\sigma x_1 x_2$ over the alphabet $\{\sigma\} \cup M$. The pair (B, σ) becomes a *Peano σ-algebra* over M ([1], [2]). It is often said that B is the Peano σ- algebra *generated* by M and this fact is specified by writing $B = \overline{M}$.

A *labeled graph* is a tuple $G = (S, L_0, T_0, f_0)$ where S is the set of nodes, L_0 is a finite set of labels, $T_0 \subseteq 2^{S\times S}$ is a set of binary relations on S and $f_0 : L_0 \longrightarrow T_0$ is a surjective mapping.

We recall that if $\rho_1, \rho_2 \in 2^{S\times S}$ then

$$\rho_1 \circ \rho_2 = \{(x, y) \in S \times S \mid \exists z \in S : (x, z) \in S \times S, (z, y) \in S \times S\}.$$

We define the mapping $prod_S : dom(prod_S) \longrightarrow 2^{S\times S}$ as follows:

- $dom(prod_S) = \{(\rho_1, \rho_2) \in 2^{S\times S} \times 2^{S\times S} \mid \rho_1 \circ \rho_2 \neq \emptyset\}$
- $prod_S(\rho_1, \rho_2) = \rho_1 \circ \rho_2$ for $(\rho_1, \rho_2) \in dom(prod_S)$

We denote $R(prod_S) = \{u \mid u \preceq prod_S\}$, where $u \preceq prod_S$ designates the property "u is a restriction of $prod_S$". If $u \in R(prod_S)$ then we denote by $Cl_u(T_0)$ the closure of T_0 in the partial algebra $(2^{S\times S}, u)$ ([17]).

A *stratified graph* $\mathcal{G}$ over G ([9], [10]) is a tuple (G, L, T, u, f) where

- $G = (S, L_0, T_0, f_0)$ is a labeled graph;
- $L \in Initial(L_0)$, that is $L_0 \subseteq L \subseteq \overline{L_0}$ and if $\sigma(\alpha, \beta) \in L$ for some arbitrary elements $\alpha, \beta \in \overline{L_0}$ then $\alpha \in L$ and $\beta \in L$;
- $u \in R(prod_S)$ and $T = Cl_u(T_0)$;
- $f : (L, \sigma_L) \longrightarrow (2^{S\times S}, u)$ is a morphism of partial algebras such that $f_0 \preceq f$, $f(L) = T$ and if $(f(x), f(y)) \in dom(u)$ then $(x, y) \in dom(\sigma_L)$, where $dom(\sigma_L) = \{(\alpha, \beta) \in \overline{L_0} \times \overline{L_0} \mid \sigma(\alpha, \beta) \in L\}$ and $\sigma_L(\alpha, \beta) = \sigma(\alpha, \beta)$ for every $(\alpha, \beta) \in dom(\sigma_L)$.

To build $\mathcal{G}$ we follow this steps:

- Take $\{B_n\}_{n\geq 0}$ as in (2) for $M = L_0$.
- Take $D_0 = L_0$ and, $dom(f_0) = D_0$.
- Define for every natural number $n \geq 0$:

$$D_{n+1} = \{\sigma(p, q) \in B_{n+1} \setminus B_n \mid p, q \in dom(f_n), (f_n(p), f_n(q)) \in dom(u)\}, \tag{3}$$

$$dom(f_{n+1}) = dom(f_n) \cup D_{n+1}, \tag{4}$$

$$f_{n+1}(x) = \begin{cases} f_n(x) & \text{if} \quad x \in dom(f_n), \\ \\ u(f_n(p), f_n(q)) & \text{if} \quad x = \sigma(p, q) \in D_{n+1}. \end{cases} \tag{5}$$

- Define the mapping $f : dom(f) \longrightarrow T$ as follows:

$$dom(f) = \bigcup_{n\geq 0} dom(f_n) = \bigcup_{k\geq 0} D_k, \tag{6}$$

$$f(x) = f_k(x) \text{ if } x \in D_k,\ k \geq 0. \tag{7}$$

- Take the set L defined as follows:
$$L = dom(f). \tag{8}$$

3. The erasing morphism

We define the operator $\epsilon_\sigma : \overline{M} \longrightarrow M^*$, where M^* is the set of all words over M, as follows:

$$\begin{cases} \epsilon_\sigma(a) = a \text{ for } a \in M, \\ \epsilon_\sigma(\sigma(\alpha,\beta)) = \epsilon_\sigma(\alpha)\epsilon_\sigma(\beta) \text{ if } \sigma(\alpha,\beta) \in \overline{M}, \alpha \in \overline{M}, \beta \in \overline{M}. \end{cases}$$

For example,

$$\begin{aligned} \epsilon_\sigma(\sigma(\sigma(\sigma(a,b),b),\sigma(b,a))) &= \epsilon_\sigma(\sigma(\sigma(a,b),b)\epsilon_\sigma(\sigma(b,a)) = \epsilon_\sigma(\sigma(a,b))\epsilon_\sigma(b)\epsilon_\sigma(b)\epsilon_\sigma(a) \\ &= \epsilon_\sigma(a)\epsilon_\sigma(b)\epsilon_\sigma(b)\epsilon_\sigma(b)\epsilon_\sigma(a) = abbba. \end{aligned}$$

For every $X \subseteq \overline{M}$ we denote

$$\epsilon_\sigma(X) = \{w \in M^* \mid \exists x \in X : w = \epsilon_\sigma(x)\}. \tag{9}$$

Thus the mapping ϵ_σ becomes an application of the form $\epsilon_\sigma : 2^{\overline{M}} \longrightarrow 2^{M^*}$. The next property uses the concept of *morphism of universal algebras* ([3], [4], [6]). First we remark that the pairs $(\overline{M}, \sigma)$ and $(M^*, \bullet)$, where $\bullet$ is the concatenation operation, are universal algebras.

Proposition 3.1. *The mapping $\epsilon_\sigma : (\overline{M}, \sigma) \longrightarrow (M^*, \bullet)$ is a morphism of universal algebras.*

Proof. This means that if we consider the graphical representation from Figure 1 then we have the following property: if we go along the path $(\overline{M} \times \overline{M}, \overline{M}, M^*)$ and along the path $(\overline{M} \times \overline{M}, M^* \times M^*, M^*)$ then we obtain the same result. Really, suppose that $(\alpha, \beta) \in \overline{M} \times \overline{M}$. We have $\sigma(\alpha,\beta) \in \overline{M}$ and $\epsilon_\sigma(\sigma(\alpha,\beta)) = \epsilon_\sigma(\alpha)\epsilon_\sigma(\beta)$. But $\epsilon_\sigma \times \epsilon_\sigma(\alpha,\beta) = (\epsilon_\sigma(\alpha), \epsilon_\sigma(\beta))$ and $\bullet(\epsilon_\sigma(\alpha), \epsilon_\sigma(\beta)) = \epsilon_\sigma(\alpha)\epsilon_\sigma(\beta)$. Thus

$$\epsilon_\sigma(\sigma(\alpha,\beta)) = \bullet(\epsilon_\sigma(\alpha), \epsilon_\sigma(\beta))$$

and the proposition is proved. □

$$\begin{tikzcd} \overline{M} \times \overline{M} \arrow[r, "\sigma"] \arrow[d, "\epsilon_\sigma \times \epsilon_\sigma"'] & \overline{M} \arrow[d, "\epsilon_\sigma"] \\ M^* \times M^* \arrow[r, "\bullet"'] & M^* \end{tikzcd}$$

FIGURE 1. The morphism condition.

Remark 3.1. Based on the previous property the mapping ϵ_σ can be named the **σ-erasing morphism**.

Proposition 3.2. *The mapping $\epsilon_\sigma : (2^{\overline{M}}, \subseteq) \longrightarrow (2^{M^*}, \subseteq)$ is an isotone application. In other words, if $A \subseteq \overline{M}$, $B \subseteq \overline{M}$ and $A \subseteq B$ then $\epsilon_\sigma(A) \subseteq \epsilon_\sigma(B)$.*

Proof. Immediate from (9). □

Proposition 3.3. *The mapping* $\epsilon_\sigma : (2^{\overline{M}}, \cup) \longrightarrow (2^{M^*}, \cup)$ *is a morphism of universal algebras.*

Proof. We prove that for every $A \subseteq \overline{M}$ and $B \subseteq \overline{M}$ we have

$$\epsilon_\sigma(A \cup B) = \epsilon_\sigma(A) \cup \epsilon_\sigma(B).$$

Take $w \in \epsilon_\sigma(A \cup B)$. There is $x \in A \cup B$ such that $w = \epsilon_\sigma(x)$. If $x \in A$ then $w \in \epsilon_\sigma(A)$. If $x \in B$ then $w \in \epsilon_\sigma(B)$. Thus

$$\epsilon_\sigma(A \cup B) \subseteq \epsilon_\sigma(A) \cup \epsilon_\sigma(B).$$

By Proposition 3.2 we have $\epsilon_\sigma(A) \subseteq \epsilon_\sigma(A \cup B)$ and $\epsilon_\sigma(B) \subseteq \epsilon_\sigma(A \cup B)$. Thus

$$\epsilon_\sigma(A) \cup \epsilon_\sigma(B) \subseteq \epsilon_\sigma(A \cup B).$$

□

Remark 3.2. The previous property shows that the diagram from Figure 2 is commutative.

$$\begin{tikzcd}
2^{\overline{M}} \times 2^{\overline{M}} \arrow[r, "\cup"] \arrow[d, "\epsilon_\sigma \times \epsilon_\sigma"'] & 2^{\overline{M}} \arrow[d, "\epsilon_\sigma"] \\
2^{M^*} \times 2^{M^*} \arrow[r, "\cup"'] & M^*
\end{tikzcd}$$

FIGURE 2. Commutative diagram.

Proposition 3.4. *For every* $A \subseteq \overline{M}$ *and* $B \subseteq \overline{M}$ *we have*

$$\epsilon_\sigma(A \cap B) \subseteq \epsilon_\sigma(A) \cap \epsilon_\sigma(B).$$

Proof. From $A \cap B \subseteq A$ and $A \cap B \subseteq B$ and Proposition 3.2 we obtain $\epsilon_\sigma(A \cap B) \subseteq \epsilon_\sigma(A) \cap \epsilon_\sigma(B)$. □

It is not difficult to show that the reverse inclusion is not true. Really, consider $A = \{a, b, \sigma(a, \sigma(a, b))\}$ and $B = \{a, b, \sigma(\sigma(a, a), b)\}$. We have $\epsilon_\sigma(A \cap B) = \{a, b\}$, $\epsilon_\sigma(A) = \{a, b, aab\}$, $\epsilon_\sigma(B) = \{a, b, aab\}$. It follows that $\epsilon_\sigma(A) \cap \epsilon_\sigma(B) \supset \epsilon_\sigma(A \cap B)$.

4. Stratified languages of first type

In this section we show a method to generate formal languages using stratified graphs. These languages are named *stratified language of first type*. In what follows we present several properties of these languages. Another method to generate formal languages by stratified graphs is given in the next section. The corresponding languages are named *stratified language of second type*.

Definition 4.1. Let us suppose that $\mathcal{G} = (G, L, T, u, f)$ is a stratified graph and L_0 is the label set of G. The language $X \subseteq L_0^*$ is **represented** or **recognized** by $\mathcal{G}$ if $X = \epsilon_\sigma(L)$. If this is the case then we denote $X = \mathcal{R}(\mathcal{G})$. The set L_0 is an **alphabet** of X.

Definition 4.2. The set X is a **stratified language of first type** if there is a stratified graph $\mathcal{G}$ such that $X = \mathcal{R}(\mathcal{G})$.

Remark 4.1. The name of stratified language comes from the following fact. Let $\mathcal{G} = (G, L, T, u, f)$ be a stratified graph, where $G = (S, L_0, T_0, f_0)$. We define

$$\begin{cases} Layer(L, 0) = L_0, \\ Layer(L, n+1) = L \cap (B_{n+1} \setminus B_n), \quad n \geq 0, \end{cases} \tag{10}$$

where $B_0 = L_0$ and $B = B_0 \cup \bigcup_{i \geq 0}(B_{i+1} \setminus B_i)$ is the Peano σ-algebra generated by L_0. The set $Layer(L, n)$ is called the n^{th} layer of L. The set L of labels is divided into several layers; the first layer is given by L_0; each element of the layer i is obtained by means of two elements, one of them belonging to the layer $i-1$ and the other being in the union of the layers $0, 1, \ldots, i-1$. We obtain a *stratified* structure of the set L of labels. This structure induces a stratified structure of the language X generated by $\mathcal{G}$.

Proposition 4.1. *Let us suppose that $\mathcal{G} = (G, L, T, u, f)$ is a stratified graph and L_0 is the label set of G. If $X = \mathcal{R}(\mathcal{G})$ then $X \supseteq L_0$.*

Proof. We have $X = \epsilon_\sigma(L)$. But $L \supseteq L_0$, therefore $X = \epsilon_\sigma(L) \supseteq \epsilon_\sigma(L_0) = L_0$. □

Proposition 4.2. *A stratified language of first type can be recognized by distinct stratified graphs. More precisely, there are $\mathcal{G}_1 = (G, L_1, T_1, u_1, f^1) \in \mathcal{L}_{lsg}$ and $\mathcal{G}_2 = (G, L_2, T_2, u_2, f^2) \in \mathcal{L}_{lsg}$ such that $L_1 \neq L_2$ and $X = \epsilon_\sigma(L_1) = \epsilon_\sigma(L_2)$.*

Proof. We show that the language $\{a, b, ab, bb, abb\}$ is recognized by two distinct stratified graphs. Consider the labeled graph $G = (S, L_0, T_0, f_0)$, where $S = \{x_1, x_2, x_3, x_4\}$, $L_0 = \{a, b\}$, $T_0 = \{\rho_1, \rho_2\}$, $\rho_1 = \{(x_1, x_2)\}$, $\rho_2 = \{(x_2, x_3), (x_3, x_4)\}$ and $f_0(a) = \rho_1$, $f_0(b) = \rho_2$. This graph is represented in Figure 3.

$$\boxed{x_1} \xrightarrow{a} \boxed{x_2} \xrightarrow{b} \boxed{x_3} \xrightarrow{b} \boxed{x_4}$$

FIGURE 3. The labeled graph G.

We consider $u_1 \in R(prod_S)$ defined by $u_1(\rho_2, \rho_2) = \rho_3$, $u_1(\rho_1, \rho_2) = \rho_4$ and $u_1(\rho_1, \rho_3) = \rho_5$, where $\rho_3 = \{(x_2, x_4)\}$, $\rho_4 = \{(x_1, x_3)\}$, $\rho_5 = \{(x_1, x_4)\}$. We build the stratified graph over G generated by u_1:

- $L_0 = \{a, b\}$, $f_0(a) = \rho_1$, $f_0(b) = \rho_2$;
- $T_1 = Cl_{u_1}(T_0) = \{\rho_1, \rho_2, \rho_3, \rho_4, \rho_5\}$;
- $D_0 = L_0$;
- $D_1 = \{\sigma(b, b), \sigma(a, b)\}$; $f^1(\sigma(b, b)) = u_1(f_0(b), f_0(b)) = u_1(\rho_2, \rho_2) = \rho_3$; $f^1(\sigma(a, b)) = u_1(f_0(a), f_0(b)) = u_1(\rho_1, \rho_2) = \rho_4$;
- $D_2 = \{\sigma(a, \sigma(b, b))\}$, $f^1(\sigma(a, \sigma(b, b))) = u_1(f^1(a), f^1(\sigma(b, b))) = u_1(\rho_1, \rho_3) = \rho_5$;
- $D_3 = \emptyset$.

We obtain the stratified graph $\mathcal{G}_1 = (G, L_1, T_1, u_1, f^1)$, where $L_1 = \{a, b, \sigma(b, b), \sigma(a, b), \sigma(a, \sigma(b, b))\}$, therefore $\epsilon_\sigma(L_1) = \{a, b, bb, ab, abb\}$.
We show that we can build another labeled graph over G representing the same language. We consider $\rho_1 = \{(x_1, x_2)\}$, $\rho_2 = \{(x_2, x_3), (x_3, x_4)\}$, $\rho_3 = \{(x_2, x_4)\}$ and $\rho_4 = \{(x_1, x_3)\}$. We denote $\omega_1 = \rho_4 \circ \rho_2 = \{(x_1, x_4)\}$. Take $u_2(\rho_1, \rho_2) = \rho_4$, $u_2(\rho_2, \rho_2) = \rho_3$, $u_2(\rho_4, \rho_2) = \omega_1$. In this context we obtain the following computations:

- $L_0 = \{a, b\}$, $f_0(a) = \rho_1$, $f_0(b) = \rho_2$;
- $T = Cl_{u_2}(T_0) = \{\rho_1, \rho_2, \rho_3, \rho_4, \omega_1\}$;
- $D_0 = L_0$, $f_0(a) = \rho_1$, $f_0(b) = \rho_2$;
- $D_1 = \{\sigma(a, b), \sigma(b, b)\}$; $f^2(\sigma(a, b)) = u_2(f_0(a), f_0(b)) = u_2(\rho_1, \rho_2) = \rho_4$; $f^2(\sigma(b, b)) = u_2(f_0(b),\ f_0(b)) = u_2(\rho_2, \rho_2) = \rho_3$;
- $D_2 = \{\sigma(\sigma(a, b), b)\}$, $f^2(\sigma(\sigma(a, b), b)) = u_2(f^2(\sigma(a, b)), f^2(b))) = u_2(\rho_4,\ \rho_2) = \omega_1$;
- $D_3 = \emptyset$.

We obtain the stratified graph $\mathcal{G}_2 = (G, L_2, T_2, u_2, f^2)$, where $L_2 = \{a, b,\ \sigma(a, b), \sigma(b, b),\ \sigma(\sigma(a, b), b)\}$, therefore $\epsilon_\sigma(L_2) = \{a, b, bb, ab, abb\}$. Thus $X = \epsilon_\sigma(L_1) = \epsilon_\sigma(L_2)$. □

In what follows we denote by λ the empty word.

Proposition 4.3. *If X is a stratified language of first type and $w \in X$ then $w \neq \lambda$.*

Proof. There is a stratified graph $\mathcal{G} = (G, L, T, u, f)$ such that $X = \epsilon_\sigma(L)$. If $w \in X$ then there is $u \in L$ such that $w = \epsilon_\sigma(u)$. If $u \in L_0$ then $w \in M$, therefore $w \neq \lambda$. If $u \in L \setminus L_0$ then the length of w is greater than 1, therefore $w \neq \lambda$. □

Proposition 4.4. *If X is a stratified language of first type then the following property is satisfied: for every $w \in X \setminus L_0$, there are $w_1 \in X$ and $w_2 \in X$ such that $w = w_1 w_2$.*

Proof. There is a stratified graph $\mathcal{G} = (G, L, T, u, f)$ such that $X = \epsilon_\sigma(L)$. Take $w \in X \setminus L_0$. There is $\omega \in L$ such that $w = \epsilon_\sigma(\omega)$. We have two cases:

(1) Suppose $\omega \in L_0$.
 This case is not possible because $w = \epsilon_\sigma(\omega) = \omega \in L_0$. But $w \in X \setminus L_0$.

(2) Suppose $\omega \in L \setminus L_0$. There are $\alpha, \beta \in L$ such that $\omega = \sigma(\alpha, \beta)$. We have $w = \epsilon_\sigma(\sigma(\alpha, \beta)) = \epsilon_\sigma(\alpha)\epsilon_\sigma(\beta)$. Take $w_1 = \epsilon_\sigma(\alpha)$ and $w_2 = \epsilon_\sigma(\beta)$. We have $w_1 \in X$, $w_2 \in X$ and $w = w_1 w_2$.

□

We remark that if L is a language over the alphabet A then $L \subseteq A^*$. If $A \subseteq B$ then B is also an alphabet for L because $L \subseteq B^*$. Next we define the alphabet of a stratified language.

Definition 4.3. If $X = \mathcal{R}(\mathcal{G})$ and $\mathcal{G}$ is a stratified graph over the labeled graph $G = (S, L_0, T_0, f_0)$, then L_0 is named an **alphabet** of X.

Proposition 4.5. *If X is a stratified language of first type then the alphabet of X is uniquely determined.*

Proof. Suppose that $G_1 = (S_1, L_{01}, T_{01}, f_{01})$ and $G_2 = (S_2, L_{02}, T_{02}, f_{02})$ are labeled graphs. Consider two stratified graphs $\mathcal{G}_1 = (G_1, L_1, T_1, u_1, f_1)$ and $\mathcal{G}_2 = (G_2, L_2, T_2, u_2, f_2)$ such that $X = \mathcal{R}(\mathcal{G}_1) = \mathcal{R}(\mathcal{G}_2)$. We prove that $L_{01} = L_{02}$.
Take the entities defined in (1), (2), (3), (4), (5), (6) and (8). We can write:

- Take $\{B_n\}_{n \geq 0}$ as in (2) for $M = L_{01}$;
- Take $T_1 = Cl_{u_1}(T_{01})$;
- Take $D_0^1 = L_{01}$ and $f_0^1 = f_{01}$, $dom(f_0^1) = D_0^1$;

- Define for every natural number $n \geq 0$:

$$D^1_{n+1} = \{\sigma(p,q) \in B_{n+1} \setminus B_n \mid p, q \in dom(f^1_n), (f^1_n(p), f^1_n(q)) \in dom(u)\}, \tag{11}$$

$$dom(f^1_{n+1}) = dom(f^1_n) \cup D^1_{n+1}, \tag{12}$$

$$f^1_{n+1}(x) = \begin{cases} f^1_n(x) & \text{if} \quad x \in dom(f^1_n), \\ \\ u(f^1_n(p), f^1_n(q)) & \text{if} \quad x = \sigma(p,q) \in D^1_{n+1}; \end{cases} \tag{13}$$

- Define the mapping $f_1 : dom(f_1) \longrightarrow T_1$ as follows:

$$dom(f_1) = \bigcup_{n \geq 0} dom(f^1_n) = \bigcup_{k \geq 0} D^1_k, \tag{14}$$

$$f_1(x) = f^1_k(x) \text{ if } x \in D^1_k,\ k \geq 0; \tag{15}$$

- Take the set L_1 defined as follows:

$$L_1 = dom(f_1). \tag{16}$$

Similarly we proceed for $\mathcal{G}_2$, and obtain L_2, T_2 and f_2. We obtain

$$L_1 = \bigcup_{k \geq 0} D^1_k = L_{01} \cup D^1_1 \cup \dots$$

$$L_2 = \bigcup_{k \geq 0} D^2_k = L_{02} \cup D^2_1 \cup \dots$$

Applying Proposition 3.3 we obtain

$$X = \epsilon_\sigma(L_1) = \epsilon_\sigma(L_{01}) \cup \epsilon_\sigma(\bigcup_{k \geq 1} D^1_k),$$

$$X = \epsilon_\sigma(L_2) = \epsilon_\sigma(L_{02}) \cup \epsilon_\sigma(\bigcup_{k \geq 1} D^2_k).$$

But $\epsilon_\sigma(L_{01}) = L_{01}$ and $\epsilon_\sigma(L_{02}) = L_{02}$ therefore

$$X = \epsilon_\sigma(L_1) = L_{01} \cup \epsilon_\sigma(\bigcup_{k \geq 1} D^1_k),$$

$$X = \epsilon_\sigma(L_2) = L_{02} \cup \epsilon_\sigma(\bigcup_{k \geq 1} D^2_k).$$

Suppose that $w \in L_{01}$. It follows that $w \in X$, therefore $w \in L_{02} \cup \epsilon_\sigma(\bigcup_{k \geq 1} D^2_k)$. But $L_{01} = L_{01} \cap X = L_{01} \cap (L_{02} \cup \epsilon_\sigma(\bigcup_{k \geq 1} D^2_k)) = (L_{01} \cap L_{02}) \cup (L_{01} \cap \bigcup_{k \geq 1} D^2_k) = L_{01} \cap L_{02}$ because $L_{01} \cap \bigcup_{k \geq 1} D^2_k = \emptyset$. Similarly we have $L_{02} = L_{01} \cap L_{02}$, therefore $L_{01} = L_{02}$. □

Proposition 4.6. *If L_0 is the alphabet of the stratified language of first type X then the complement $L_0^* \setminus X$ is not a stratified language of first type.*

Proof. If X is a stratified language over L_0 then the empty word λ is not a member of X. It follows that $\lambda \in L_0^* \setminus X$. Applying Proposition 4.3 we deduce that $L_0^* \setminus X$ is not a stratified language. □

Remark 4.2. If $\mathcal{G} = (G, L, T, u, f)$ is a stratified graph over G then we denote $label(\mathcal{G}) = L$. This notation is explained by the fact that each element of L is a label of a binary relation of $\mathcal{G}$. If $G = (S, L_0, T_0, f_0)$ then L_0 is named the **alphabet** of $\mathcal{G}$.

Proposition 4.7. *Suppose that L_0 is a finite set. There is a stratified graph $\mathcal{G}$ of alphabet L_0 such that label$(\mathcal{G}) = \overline{L_0}$.*

Proof. We consider the labeled graph $G = (S, L_0, T_0, f_0)$ defined as follows:

- $S = \{x_1\}$;
- $T_0 = \{\rho\}$, where $\rho = \{(x_1, x_1)\}$;
- $f_0(a) = \rho$ for every $a \in L_0$;

We consider the stratified graph $\mathcal{G} = (G, L, T, u, f)$, where $u(\rho, \rho) = \rho$. Take the entities defined in (1), (2), (11), (12), (13), (14) and (16). We obtain $T = Cl_u(T_0) = T_0$ because if we compute the closure of T_0 we have

$$T_1 = T_0 \cup \{\rho_1 \in 2^{S \times S} \mid \exists \rho_2 \in T_0 : \rho_1 = u(\rho_2, \rho_2)\} = T_0.$$

We verify by induction on $n \geq 0$ the following properties:

$$D_{n+1} = B_{n+1} \setminus B_n, \tag{17}$$

$$f_{n+1}(x) = \rho, \text{ for } x \in \bigcup_{k=0}^{n+1} D_k. \tag{18}$$

For $n = 0$ we obtain:

$$D_1 = \{\sigma(p, q) \in B_1 \setminus B_0 \mid p, q \in dom(f_0), (f_0(p), f_0(q)) \in dom(u)\} = B_1 \setminus B_0,$$

because $dom(f_0) = D_0 = L_0$, $f_0(p) = \rho$ for every $p \in L_0$ and $(\rho, \rho) \in dom(u)$. We have also

$$f_1(x) = \begin{cases} f_0(x) & \text{if} \quad x \in dom(f_0) = L_0, \\ \\ u(f_0(p), f_0(q)) & \text{if} \quad x = \sigma(p, q) \in D_1. \end{cases}$$

But $D_0 = L_0$ and $u(\rho, \rho) = \rho$ therefore (18) is true for $n = 0$.
Suppose that (17) and (18) are true for n. We have

$$D_{n+2} = \{\sigma(p, q) \in B_{n+2} \setminus B_{n+1} \mid p, q \in dom(f_{n+1}),$$
$$(f_{n+1}(p), f_{n+1}(q)) \in dom(u)\} = B_{n+2} \setminus B_{n+1},$$

because $f_{n+1}(p) = f_{n+1}(q) = \rho$ for every $p, q \in dom(f_{n+1})$ and $(\rho, \rho) \in dom(u)$.
From (5) we have

$$f_{n+2}(x) = \begin{cases} f_{n+1}(x) & \text{if} \quad x \in dom(f_{n+1}), \\ \\ u(f_{n+1}(p), f_{n+1}(q)) & \text{if} \quad x = \sigma(p, q) \in D_{n+2}, \end{cases}$$

therefore $f_{n+2}(x) = \rho$ for every $x \in \bigcup_{k=0}^{n+2} D_k$. Thus (18) is true for $n + 1$.
From (6) and (8) we obtain now $L = \bigcup_{k \geq 0} D_k = D_0 \cup \bigcup_{k \geq 0} (B_{k+1} \setminus B_k) = \overline{L_0}$. The proposition is proved. □

Proposition 4.8. *If $\overline{L_0}$ is the Peano algebra generated by L_0 then $\epsilon_\sigma(\overline{L_0}) = L_0^+$.*

Proof. If we take $\overline{L_0} = \bigcup_{n \geq 0} L_n$, where $L_{n+1} = L_n \cup \{\sigma(x_1, x_2) \mid (x_1, x_2) \in L_n \times L_n\}$, for $n \geq 0$ then

$$\epsilon_\sigma(\overline{L_0}) = L_0 \cup \bigcup_{n \geq 1} \epsilon_\sigma(L_n). \tag{19}$$

Let us prove by induction on $n \geq 1$ the property

$$\epsilon_\sigma(L_n) = \bigcup_{k=1}^{2^n} L_0^k. \tag{20}$$

We have $L_1 = L_0 \cup \{\sigma(x_1, x_2) \mid (x_1, x_2) \in L_0 \times L_0\}$, therefore

$$\begin{aligned}\epsilon_\sigma(L_1) &= L_0 \cup \{\epsilon_\sigma(x_1)\epsilon_\sigma(x_2) \mid (x_1, x_2) \in L_0 \times L_0\} \\ &= L_0 \cup \{x_1 x_2 \mid (x_1, x_2) \in L_0 \times L_0\} = L_0 \cup L_0^2.\end{aligned}$$

It follows that (19) is true for $n = 1$. Suppose that (20) is true for $n = m$ and we verify this property for $n = m + 1$. We have

$$L_{m+1} = L_m \cup \{\sigma(x_1, x_2) \mid (x_1, x_2) \in L_m \times L_m\}$$

therefore

$$\begin{aligned}\epsilon_\sigma(L_{m+1}) &= \epsilon_\sigma(L_m) \cup \{\epsilon_\sigma(x_1)\epsilon_\sigma(x_2) \mid (x_1, x_2) \in L_m \times L_m\} \\ &= \epsilon_\sigma(L_m) \cup \epsilon_\sigma(L_m)\epsilon_\sigma(L_m) = \bigcup_{k=1}^{2^m} L_0^k \cup (\bigcup_{k=1}^{2^m} L_0^k)(\bigcup_{p=1}^{2^m} L_0^p) \\ &= \bigcup_{k=1}^{2^m} L_0^k \cup \bigcup_{k,p=1}^{2^m} L_0^{k+p} = \bigcup_{k=1}^{2^m} L_0^k \cup \bigcup_{k=2}^{2^{m+1}} L_0^k = \bigcup_{k=1}^{2^{m+1}} L_0^k\end{aligned}$$

and thus (20) is true for $n = m + 1$.
From (19) and (20) we obtain $\epsilon_\sigma(\overline{L_0}) = L_0 \cup \bigcup_{n \geq 1} \bigcup_{k=1}^{2^n} L_0^k = L_0^+$. □

Proposition 4.9. *For every finite set L_0 the language $L_0^+ = \bigcup_{k \geq 1} L_0^k$ is a stratified language of first type of alphabet L_0.*

Proof. We apply Proposition 4.7 and Proposition 4.8. By Proposition 4.7 there is a stratified graph $\mathcal{G}$ of alphabet L_0 such that *label*$(\mathcal{G}) = \overline{L_0}$. By Proposition 4.8 we deduce that $\epsilon_\sigma(\overline{L_0}) = L_0^+$ therefore L_0^+ is a stratified language of alphabet L_0. The proposition is proved. □

The next propositions shows that the concept of alphabet for stratified languages is the minimal alphabet of a formal language.

Proposition 4.10. *If X is a stratified language of first type of alphabet L_0 then $L_0 \subseteq X \subseteq L_0^+$.*

Proof. There is a stratified graph $\mathcal{G} = (G, L, T, u, f)$ over $G = (S, L_0, T_0, f_0)$ such that $X = \epsilon_\sigma(L)$. But $L \subseteq \overline{L_0}$, where $\overline{L_0}$ is the Peano algebra generated by L_0. It follows that $\epsilon_\sigma(L) \subseteq \epsilon_\sigma(\overline{L_0})$. But $X = \epsilon_\sigma(L)$ and by Proposition 4.7 we have $epsilon_\sigma(\overline{L_0}) = L_0^+$. By Proposition 4.1 we have $L_0 \subseteq X$ and thus $L_0 \subseteq X \subseteq L_0^+$. □

5. Stratified languages of second type

We consider a path $d = ([x_1, ..., x_{n+1}],\ [a_1, ..., a_n])$ in a labeled graph $G = (S, L_0, T_0, f_0)$. Consider the least set $STR(d)$ satisfying the following conditions:

- $([x_i, x_{i+1}], a_i) \in STR(d)$, $i \in \{1, ..., n\}$;

- if $([x_i,...,x_k],b_1) \in STR(d)$ and $([x_k,...,x_r],b_2) \in STR(d)$, where $1 \le i < k < r \le n+1$, then $([x_i,...,x_r],[b_1,b_2]) \in STR(d)$.

The elements of $STR(d)$ are called **structured paths** over d. We denote

$$STR_2(d) = \{\alpha \mid \exists([x_1,\ldots],\alpha) \in STR(d)\}.$$

Let d be a path. We define the mapping $h_d : STR_2(d) \longrightarrow B$, where B is defined in (1), as follows:

- $h_d(x) = x$ for $x \in L_0$;
- $h_d([u,v]) = \sigma(h_d(u), h_d(v))$.

The structured path $d_s \in STR(d)$ is named an **accepted structured path** over $\mathcal{G}$ if $d_s = ([x_1,...,x_{n+1}],c)$ and $h_d(c) \in L$. We denote by $ASP(\mathcal{G})$ the set of all accepted structured paths over $\mathcal{G}$. We denote by R a set of conditions imposed on the accepted structured path. An element of $ASP(\mathcal{G})$ which satisfies R is named **R-accepted structured path**. We denote by $ASP_R(\mathcal{G})$ the set of all R-accepted structured paths.

For every accepted structured path $d = ([x_1,\ldots,x_{n+1}],\sigma(v_1,v_2)) \in ASP(\mathcal{G})$, where $n \ge 2$, there is one and only one $i \in \{2,\ldots,n\}$ such that $([x_1,\ldots,x_i],v_1) \in ASP(\mathcal{G})$ and $([x_i,\ldots,x_{n+1}],v_2) \in ASP(\mathcal{G})$ ([13]). In other words, this property states that every accepted structured path over $\mathcal{G}$ can be broken into two accepted structured paths over $\mathcal{G}$. The number i stated in this property is named the *break index* for the path d and is denoted by $ind(d)$.

An **interpretation** for $\mathcal{G}$ is a tuple

$$\Sigma = (Ob, i, D, \mathcal{P}),$$

where:

- Ob is a finite set of objects such that $Card(Ob) = Card(S)$;
- $i : S \longrightarrow Ob$ is a bijective mapping;
- $D = (Y, *)$ is a partial algebra, Y is called the domain of Σ and $*$ is a partial binary operation on Y;
- $\mathcal{P} = \{Alg_a\}_{a\in L_0}$, where $Alg_a : Ob \times Ob \longrightarrow Y$.

The **valuation mapping** generated by Σ is the mapping $val_\Sigma : ASP_R(\mathcal{G}) \longrightarrow Y$ defined inductively as follows.

$$\begin{cases} val_\Sigma([x,y],a) = Alg_a(i(x),i(y)), \\ \\ val_\Sigma(x(1;n+1),\sigma(v_1,v_2)) = val_\Sigma(x(1;i),v_1) * val_\Sigma(x(i;n+1),v_2), \end{cases}$$

where $i = ind([x_1,...,x_{n+1}],\ \sigma(v_1,v_2))$ and $x(i;j) = [x_i,...,x_j]$.

Consider a stratified graph $\mathcal{G} = (G_0, L, T, u, f)$ over $G_0 = (S, L_0, T_0, f_0)$ and $\Sigma = (Ob, i, D, \mathcal{P})$ an interpretation for $\mathcal{G}$. A pair $(x,y) \in S \times S$ is called **interrogation**. For a given interrogation (x,y) we designate by $ASP_R(x,y)$ the set of all R-accepted structured paths from x to y in $\mathcal{G}$. The **answer mapping** is the mapping

$$Ans : S \times S \longrightarrow Y \cup \{no\},$$

defined as follows: $\begin{cases} Ans(x,y) = no \quad \text{if} \quad ASP_R(x,y) = \emptyset, \\ Ans(x,y) = \{val_\Sigma(d) \mid d \in ASP_R(x,y)\} \quad \text{if} \quad ASP_R(x,y) \neq \emptyset. \end{cases}$

In what follows we consider:

- An alphabet V;

- A stratified graph $\mathcal{G} = (G, L, T, u, f)$, where $G = (S, L_0, T_0, f_0)$ is a labeled graph;
- An interpretation $\Sigma = (V^*, i, D, \mathcal{P})$, where
  - ◦ V^* is the free monoid generated by V;
  - ◦ $i : S \longrightarrow V^*$;
  - ◦ $D = (V^*, *)$ is a partial algebra, where $* : V^* \times V^* \longrightarrow V^*$ is a partial binary operation;
  - ◦ For each $a \in L_0$ we have $Alg_a : V^* \times V^* \longrightarrow V^*$ and $\mathcal{P} = \{Alg_a\}_{a \in L_0}$;
- A subset $M \subseteq S \times S$;
- The set R of restrictions to build the R-accepted structured paths.

Definition 5.1. By definition, the language defined by the sets M and R is the following collection

$$\mathcal{L}(M, R) = \bigcup_{(x,y) \in M} Ans(x, y).$$

The language $\mathcal{L}(M, R)$ is a **stratified language of second type**.

Remark 5.1. We observe that for each $(x, y) \in M$ the set $Ans(x, y) \subseteq V^*$ is a formal language over V, therefore $\mathcal{L}(M, R) \subseteq V^*$ is a formal language over V.

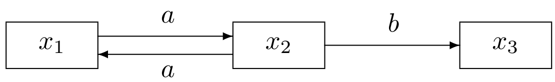

Figure 4. Labeled graph with an infinite loop.

Using the labeled graph from Figure 4 and the stratified graph generated by $u = prod_S$ we consider the following interpretation $\Sigma = (V^*, i, D, \mathcal{P})$, where

- ◦ V^* is the free monoid generated by $V = \{a_1, a_2, a_3\}$;
- ◦ $i : S \longrightarrow V^*$, $i(x_1) = a_1$, $i(x_2) = a_2$, $i(x_3) = a_3$;
- ◦ $D = (V^*, *)$ is a partial algebra, where $* : V^* \times V^* \longrightarrow V^*$ is defined as follows: for every natural number k we take:

$$(a_1^k a_2^k) * a_1 = a_1^{k+1} a_2^k, \tag{21}$$

$$(a_1^{k+1} a_2^k) * a_2 = a_1^{k+1} a_2^{k+1}, \tag{22}$$

$$(a_1^{k+1} a_2^k) * a_2 a_3 = a_1^{k+1} a_2^{k+1} a_3^{k+1}. \tag{23}$$

  We observe that $*$ is in this case a partial operation;
- ◦ The algorithms $Alg_a : V^* \times V^* \longrightarrow V^*$ and $Alg_b : V^* \times V^* \longrightarrow V^*$ are defined as follows:

  $Alg_a(x, y) = x$

  $Alg_b(x, y) = xy$

  and take $\mathcal{P} = \{Alg_a, Alg_b\}$.

We choose the set $M = \{(x_1, x_3)\}$. We have an infinity set of accepted structured paths from x_1 to x_3.

Using the set R of restrictions we impose the restriction to use only the accepted structured paths of the following form:

$$([x_1, x_2, x_1, x_2, \ldots, x_1, x_2, x_3], \sigma(\omega_k, b)),$$

where the pair (x_1, x_2) appears k times and $\omega_k = \sigma(\sigma(\ldots\sigma(a,a),a)\ldots,a)$ contains $2(k-1)$ letters σ and $2k-1$ letters a.

So we obtain: $ASP_R(\mathcal{G}) = ASP_R(x_1, x_3) = \bigcup_{k\geq 1}\{([x_1, x_2, \ldots, x_1, x_2, x_3], \sigma(\omega_k, b))\}$.

It is not difficult to observe that the language generated by the sets M and R is

$$\mathcal{L}(M, R) = \{a_1^k a_2^k a_3^k\}_{k\geq 1}.$$

Remark 5.2. The language $\{a_1^k a_2^k a_3^k\}_{k\geq 1}$ is a context-sensitive language, but is not a context free language in Chomsky hierarchy.

6. Conclusions and open problems

In this paper we consider two families of formal languages generated by stratified graphs. We present several properties of these families. We consider the following open problems:

(1) Study the families of the stratified languages in comparison with the Chomsky classification of the formal languages or in comparison with the Lindenmayer languages.
(2) Study the generative power of the stratified languages.
(3) Compare the generative power of the stratified language of first type with the generative power of the stratified languages of second type.
(4) Study the use of stratified languages as models for natural languages.

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