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Integral ideals and maximal ideals in BL-algebras

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Abstract. In this paper, we introduce the concept of integral ideals in BL-algebras. With respect to concept, we give some related results. In particular, we prove that an ideal is integral ideal if and only if is Boolean and (prime)maximal ideal. Also, we prove that a BL-algebra is an integral BL-algebra if and only if trivial ideal $\{0\}$ is an integral ideal. Moreover, we study relation between integral ideals and obstinate filters in BL-algebras by using the set of complement elements. Also, we describe relationship between maximal ideals in BL-algebras and locally finite MV-algebras.

Key words and phrases. BL-algebra, integral ideal, Boolean ideal, maximal ideal.

1. Introduction

BL-algebras are the algebraic structure for Hájek basic logic [6] in order to investigate many valued logic by algebraic means. His motivations for introducing BL-algebras were of two kinds. The first one was providing an algebraic counterpart of a propositional logic, called Basic Logic, which embodies a fragment common to some of the most important many-valued logics, namely Lukasiewicz Logic, Gödel Logic and Product Logic. This Basic Logic (BL for short) is proposed as "the most general"many-valued logic with truth values in [0,1] and BL-algebras are the corresponding Lindenbaum-Tarski algebras. The second one was to provide an algebraic mean for the study of continuous t-norms (or triangular norms) on [0, 1]. In 1958, Chang [2] introduced the concept of an MV-algebra which is one of the most classes of BL-algebras. Turunen [9] introduced the notion of an implicative filter and a Boolean filter in BL-algebras. Boolean filters are an important class of filters, because the quotient BL-algebra induced by these filters are Boolean algebras. The notion of (fuzzy) ideal has been introduced in many algebraic structures such as lattices, rings, MV-algebras. Ideal theory is very effective tool for studying various algebraic and logical systems. In the theory of MV-algebras, as various algebraic structures, the notion of ideal is at the center, while in BL-algebras, the focus has been on deductive systems also filters. The study of BL-algebras has experienced a tremendous growth over resent years and the main focus has been on filters. In the meantime, several authors have claimed in recent works that the notion of ideals is missing in BL-algebras. In 2013, Lele [7], introduced the notions of (Boolean, prime) ideals and analyzed the relationship between ideals and filters by using the set of complement elements.

Now, in this paper, we introduce the concept of integral ideals in BL-algebras. In particular, we prove that an ideal is integral ideal if and only if is Boolean and (prime) maximal ideal. Also, we prove that a BL-algebra is an integral BL-algebra if

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and only if trivial ideal $\{0\}$ is an integral ideal. Moreover, we study relation between integral ideals and obstinate filters in BL-algebras by using the set of complement elements. Finally, we describe relationship between maximal ideals in BL-algebras and locally finite MV-algebras and we prove an ideal M of BL-algebra L is a maximal ideal if and only if for all $x \notin M$, there exists $n \in N$, $(x^-)^n \in M$, where N is natural numbers if and only if quotient BL-algebra $\frac{L}{M}$ is a locally finite MV-algebra.

2. Preliminaries

In this section, we give some fundamental definitions and results. For more details, refer to the references.

Definition 2.1. [6] A BL-algebra is an algebra $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ of type $(2,2,2,2,0,0)$ such that
$(BL1)$ $(L, \vee, \wedge, 0, 1)$ is a bounded lattice,
$(BL2)$ $(L, \odot, 1)$ is a commutative monoid,
$(BL3)$ $z \leq x \rightarrow y$ if and only if $x \odot z \leq y$, for all $x, y, z \in L$,
$(BL4)$ $x \wedge y = x \odot (x \rightarrow y)$,
$(BL5)$ $(x \rightarrow y) \vee (y \rightarrow x) = 1$.
We denote $x^n = \overbrace{x \odot ... \odot x}^{n-times}$, if $n > 0$ and $x^0 = 1$. Also, we denote $\overbrace{(x \rightarrow (...(x \rightarrow (x \rightarrow}^{n-times}$ $y)))...)$ by $x^n \rightarrow y$, for all $x, y \in L$.

A BL-algebra L is called a Gödel algebra, if $x^2 = x \odot x = x$, for all $x \in L$ and BL-algebra L is called an MV-algebra, if $(x^-)^- = x$, for all $x \in L$, where $x^- = x \rightarrow 0$. A BL-algebra L is called a Boolean algebra, if $x \vee x^- = 1$, for all $x \in L$. Moreover, BL-algebra L is called an integral BL-algebra, if $x \odot y = 0$, then $x = 0$ or $y = 0$, for all $x, y \in L$. Also, if L is an integral BL-algebra, then $x \rightarrow 0 = 0$ or $x \rightarrow 0 = 1$, for all $x \in L$.

Proposition 2.2. ([3],[4]) In any BL-algebra the following hold:
$(BL6)$ $x \leq y$ if and only if $x \rightarrow y = 1$,
$(BL7)$ $y \leq x \rightarrow y$, and $x \odot y \leq x, y$
$(BL8)$ $x \leq y$ implies $y \rightarrow z \leq x \rightarrow z$ and $z \rightarrow x \leq z \rightarrow y$,
$(BL9)$ $x^- = 1$ if and only if $x = 0$,
$(BL10)$ $1 \rightarrow x = x$, $x \rightarrow x = 1$ and $x \rightarrow 1 = 1$,
$(BL11)$ $x^{---} = x^-$, $x \leq x^{--}$ and $x \odot x^- = 0$,
$(BL12)$ $(x \wedge y) \leq x, y$,
$(BL13)$ $y \rightarrow x \leq (z \rightarrow y) \rightarrow (z \rightarrow x)$,
$(BL14)$ $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$,
for all $x, y, z \in L$.

The following theorems and definitions are from ([1], [3], [5], [6], [7], [8]) and we refer the reader to them, for more details.

Definition 2.3. Let L be a BL-algebra and F be a nonempty subset of L. Then
(i) F is called a *filter* of L, if $x \odot y \in F$, for any $x, y \in F$ and if $x \in F$ and $x \leq y$ then $y \in F$, for all $x, y \in L$. Proper filter F is called a *maximal filter* of L, if it is not properly contained in any other proper filter of L. Moreover, proper filter F is a

maximal filter if and only if for all $x \notin F$, there exists $n \in N$, $(x^n)^- \in F$.
(ii) F is called an *implicative filter* of L, if $1 \in F$ and for all $x, y, z \in L$,

$$x \to (y \to z) \in F \text{ and } x \to y \in F \text{ imply } x \to z \in F.$$

(iii) F is called a *fantastic filter*, if $1 \in F$ and for all $x, y, z \in L$,

$$z \to (y \to x) \in F \text{ and } z \in F \text{ imply } (((x \to y) \to y) \to x) \in F.$$

(v) F is called an *positive implicative filter* of L, if $1 \in F$ and for all $x, y, z \in L$,

$$x \to ((y \to z) \to y) \in F \text{ and } x \in F \text{ imply } y \in F.$$

(vi) Proper filter F is called a *Boolean filter*, if for all $x \in L$, $x \vee x^- \in F$.
(vii) Proper filter F is called an *obstinate filter*, if for all $x, y \in L$,

$$x, y \notin F \text{ imply } x \to y \in F \text{ and } y \to x \in F.$$

$(viii)$ Proper filter F is called an *integral filter*, if for all $x, y \in L$,

$$(x \odot y)^- \in F \ \text{ implies } (x)^- \in F \ \text{ or } \ (y)^- \in F.$$

Definition 2.4. Let L be a BL-algebra and I be a nonempty subset of L. Then
(i) I is called an *ideal* of L, if $x \oslash y = x^- \to y \in I$, for any $x, y \in I$ and if $y \in I$ and $x \leq y$ then $x \in I$, for all $x, y \in L$. Moreover, a set I containing 0 of L is an ideal if and only if for all $x, y \in L$, $x^- \odot y \in I$ and $x \in I$ imply $y \in I$.
(ii) A proper ideal I of L is called *prime ideal* of L, if $x \wedge y \in I$ implies $x \in I$ or $y \in I$, for all $x, y \in L$. Moreover, I is a prime ideal, if it satisfies for all $x, y \in L$, $(x \to y)^- \in I$ or $(y \to x)^- \in I$.
(iii) Proper ideal I is called a *maximal ideal* of L, if it is not properly contained in any other proper ideal of L.
(iv) An ideal I of L is called a *Boolean ideal*, if $x \wedge x^- \in I$, for all $x \in L$.

Definition 2.5. Let L be a BL-algebra and X any subset of L. Then the set of complement elements (with respect to X) is denoted by $N(X)$ and is defined by

$$N(X) = \{x \in L \mid x^- \in X\}$$

Theorem 2.6. Let $F \subseteq G$, where F and G are filters of L and F be an integral filter. Then G is an integral filter of L.

Theorem 2.7. Let F be a filter of BL-algebra L. Then the binary relation $\equiv_F$ on L which is defined by

$$x \equiv_F y \ \text{ if and only if } x \to y \in F \ \text{ and } y \to x \in F$$

is a congruence relation on L. Define $\cdot$, $\rightharpoonup$, $\sqcup$, $\sqcap$ on $\dfrac{L}{F}$, the set of all congruence classes of L, as follows:

$$[x] \cdot [y] = [x \odot y],\ [x] \rightharpoonup [y] = [x \to y],\ [x] \sqcup [y] = [x \vee y],\ [x] \sqcap [y] = [x \wedge y].$$

Then $(\dfrac{L}{F}, \cdot, \rightharpoonup, \sqcup, \sqcap, [0], [1])$ is a BL-algebra which is called quotient BL-algebra with respect to F.

Theorem 2.8. Let F be a proper filter of L. Then F is an integral filter if and only if $\dfrac{L}{F}$ is an integral BL-algebra.

Theorem 2.9. Let I be an ideal of BL-algebra L. Then the binary relation $\equiv_I$ on L which is defined by

$$x \equiv_I y \text{ if and only if } x^- \odot y \in I \text{ and } y^- \odot x \in I$$

is a congruence relation on L. Define $\cdot$, $\rightharpoonup$, $\sqcup$, $\sqcap$ on $\dfrac{L}{I}$, the set of all congruence classes of L, as follows:

$$[x] \cdot [y] = [x \odot y],\ [x] \rightharpoonup [y] = [x \to y],\ [x] \sqcup [y] = [x \vee y],\ [x] \sqcap [y] = [x \wedge y].$$

Then $(\frac{L}{I}, \cdot, \rightharpoonup, \sqcup, \sqcap, [0], [1])$ is a BL-algebra which is called quotient BL-algebra with respect to I. In addition, it is clear $[x]^{--} = [x]$, for all $x \in L$. Consequently, the quotient BL-algebra via any ideal is always an MV-algebra.

Theorem 2.10. Let F be a filter of BL-algebra L. Then the following conditions are equivalent:
(i) F is a maximal and (Boolean)positive implicative filter,
(ii) F is a maximal and implicative filter,
(iii) F is an obstinate filter,
(iv) F is a fantastic and integral filter,
(v) $\dfrac{L}{F}$ is isomorphic to the simplest Boolean algebra $\{0,1\}$ ($\dfrac{L}{F} \cong \{0,1\}$).

Theorem 2.11. Let F be a filter and I be an ideal of BL-algebra L. Then
(i) The set of complement elements $N(I)$ is a filter.
(ii) The set of complement elements $N(F)$ is the ideal generated by $\overline{F}$.
(iii) I is a Boolean ideal if and only if $N(I)$ is a Boolean filter.
(iv) $N(I)$ is a fantastic filter of L.

3. Integral ideals in BL-algebras

In this section we introduce a new class of ideals that called integral ideals and we give some related results.

Definition 3.1. An ideal I of L is called an *integral ideal*, if for all $x, y \in L$,

$$x \odot y \in I \text{ implies } x \in I \text{ or } y \in I$$

Example 3.2. [7] Let $L = \{0, a, b, c, d, e, f, 1\}$ be such that $0 < a < b < c < 1$, $0 < d < e < f < 1$, $a < e$ and $b < f$. Define $\odot$ and $\to$ as follows:

Table 1. Product Operation

$\odot$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	a	a	a	0	a	a	a
b	0	a	a	b	0	a	a	b
c	0	a	b	c	0	a	b	c
d	0	0	0	0	d	d	d	d
e	0	a	a	a	d	e	e	e
f	0	a	a	b	d	e	e	f
1	0	a	b	c	d	e	f	1

Table 2. Implication Operation

$\to$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	d	1	1	1	d	1	1	1
b	d	f	1	1	d	f	1	1
c	d	e	f	1	d	e	f	1
d	c	c	c	c	1	1	1	1
e	0	c	c	c	d	1	1	1
f	0	b	c	c	d	f	1	1
1	0	a	b	c	d	e	f	1

Then $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a BL-algebra. Let $I = \{0, d\}$ and $J = \{0, a, b, c\}$. Then I and J are integral ideals of L.

Example 3.3. [1] Let $L = \{0, a, b, c, 1\}$. Define $\wedge$,$\vee$,$\odot$ and $\rightarrow$ on L as follows:

Table 3. Meet

$\wedge$	0	c	a	b	1
0	0	0	0	0	0
c	0	c	c	c	c
a	0	c	a	c	a
b	0	c	c	b	b
1	0	c	a	b	1

Table 4. Join

$\vee$	0	c	a	b	1
0	0	c	a	b	1
c	c	c	a	b	1
a	a	a	a	1	1
b	b	b	1	b	1
1	1	1	1	1	1

Table 5. Product

$\odot$	0	c	a	b	1
0	0	0	0	0	0
c	0	c	c	c	c
a	0	c	a	c	a
b	0	c	c	b	b
1	0	c	a	b	1

Table 6. Implication

$\rightarrow$	0	c	a	b	1
0	1	1	1	1	1
c	0	1	1	1	1
a	0	b	1	b	1
b	0	a	a	1	1
1	0	c	a	b	1

Then $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ is a BL-algebra. Let $I = \{0\}$. Then I is an integral ideal of L.

Lemma 3.4. Let I be an ideal of L and $\alpha \in I$. Then $\alpha^- \in N(I)$.

Proof. Let I be an ideal of L and $\alpha \in I$. By $(BL11)$, $\alpha^- \odot \alpha^{--} = 0$, since I is an ideal of L, then $\alpha^{--} \in I$. Therefore, $\alpha^- \in N(I)$. □

Theorem 3.5. Let I be an ideal of L. Then I is an integral ideal if and only if $N(I)$ is an integral filter of L.

Proof. Let I be an integral ideal of L and $(x \odot y)^- \in N(I)$, for $x, y \in L$. Then by Theorem 2.11(i) , $N(I)$ is a filter and $(x \odot y)^{--} \in I$. Since $(x \odot y) \leq (x \odot y)^{--}$ by $(BL11)$, then $(x \odot y) \in I$. Now, since I is an integral ideal, then $x \in I$ or $y \in I$ and so $x^- \in N(I)$ or $y^- \in N(I)$. Therefore, $N(I)$ is an integral filter of L. Conversely, let $N(I)$ be an integral filter of L and $(x \odot y) \in I$, for $x, y \in L$. Then by Lemma 3.4, $(x \odot y)^- \in N(I)$. Now, since $N(I)$ is an integral filter of L, then $x^- \in N(I)$ or $y^- \in N(I)$ and so $x^{--} \in I$ or $y^{--} \in I$. By $(BL11)$, $x \leq x^{--}$ and $y \leq y^{--}$. Hence, $x \in I$ or $y \in I$ and so I is an integral ideal of L. □

Theorem 3.6. Let I be an ideal of L. Then I is an integral ideal if and only if $\dfrac{L}{I}$ is an integral BL-algebra.

Proof. Let I be an integral ideal of L and $[x] \cdot [y] = [0]$, for $[x], [y] \in \dfrac{L}{I}$. Then $[x \odot y] = [0]$ and so by Theorem 2.9, $(x \odot y)^- \odot 0 \in I$ and $(x \odot y) \odot 0^- \in I$. Hence, $x \odot y \in I$ and since I is an integral ideal of L, then $x \in I$ or $y \in I$. Therefore, $[x] = [0]$ or $[y] = [0]$ and so $\dfrac{L}{I}$ is an integral BL-algebra. Conversely, let $\dfrac{L}{I}$ is an integral BL-algebra and $x \odot y \in I$. Then $[x \odot y] = [0]$ and so $[x] = [0]$ or $[y] = [0]$. Therefore, $x \in I$ or $y \in I$ and so I is an integral ideal of L. □

Theorem 3.7. Let I be an integral ideal of L. Then
(i) I is a Boolean ideal of L.
(ii) I is a prime ideal of L.
(iii) $\frac{L}{I} = \{[0], [1]\}$.
(iv) $\frac{L}{N(I)} \cong \{0, 1\}$.

Proof. (i) Let I be an integral ideal of L. Then by Theorem 3.5, $N(I)$ is an integral filter of L. Moreover, by Theorem 2.11(iv), $N(I)$ is a fantastic filter of L and so by Theorem 2.10, $N(I)$ is a maximal and positive implicative filter of L, hence, $N(I)$ is a maximal Boolean filter of L. Therefore, by Theorem 2.11(iii), I is a Boolean ideal of L.
(ii) Let I be an integral ideal and $x \wedge y \in I$, for $x, y \in L$. Then By $(BL4)$ $x \wedge y = x \odot (x \to y) \in I$ and so $x \in I$ or $x \to y \in I$. Since $y \leq x \to y$ and I is an ideal, then $x \in I$ or $y \in I$. Therefore, I is a prime ideal of L.
(iii) Since I is an integral ideal of L, then by Theorem 3.6, $\frac{L}{I}$ is an integral BL-algebra. Now, by Theorem 2.9, $\frac{L}{I}$ is an integral MV-algebra. Hence, for all $[0] \neq [x] \in \frac{L}{I}$, $[x]^- = [0]$ and so $[x] = [x]^{--} = [1]$. Therefore, $\frac{L}{I} = \{[0], [1]\}$.
(iv) Since I is an integral ideal of L, then $N(I)$ is a maximal and positive implicative filter of L and so by Theorem 2.10, $\frac{L}{N(I)} \cong \{0, 1\}$. $\square$

The following example shows that the converse of Theorem 3.7(i) and (ii), is not correct in general.

Example 3.8. [7] Let $L = \{0, a, b, 1\}$, where $0 < a < b < 1$. Let $x \wedge y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$ and operations $\odot$ and $\to$ are defined as the following tables:

Table 7. Product

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

Table 8. Implication

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Then $(L, \vee, \wedge, \odot, \to, 0, 1)$ is a BL-algebra. Now, let $I = \{0\}$. Then I is a prime ideal of L, but it is not an integral ideal. Because, $a \odot a = 0 \in I$ and $a \notin I$. Moreover, since $a \wedge a^- = a \wedge a = a \notin I$, then I is not a Boolean ideal of L.

Theorem 3.9. Let $I \subseteq J$, where I and J be two ideals of L and I be an integral ideal of L. Then J is an integral ideal, too.

Proof. Let $I \subseteq J$, where I and J be two ideals of L . Then $N(I) \subseteq N(J)$. Since if $x \in N(I)$, then $x^- \in I \subseteq J$ and so $x^- \in J$. Hence, $x \in N(J)$ and so $N(I) \subseteq N(J)$. Now, since I is an integral ideal, then by Theorem 3.5, $N(I)$ is an integral filter and since $N(I) \subseteq N(J)$, then by Theorem 2.6, $N(J)$ is an integral filter. Now, by Theorem 3.5, J is an integral ideal of L. $\square$

Proposition 3.10. Let I be a nontrivial ideal and F be a filter of integral BL-algebra L. Then $I = L$ and $N(F) = \{0\}$.

Proof. Let $0 \neq x \in I$, since L is an integral BL-algebra. Then $x^- = 0$ and so for all $z \in L$, $x^- \odot z = 0 \in I$. Since $x \in I$ an I is an ideal, then $z \in I$. Hence, $I = L$. Also, since $x^- = 0 \notin F$, for all $0 \neq x \in L$ and $0^- = 1 \in F$, then $N(F) = \{0\}$. Therefore, by Theorem 2.11(i) and (ii), $N(I)$ is a maximal filter of L and $N(F)$ is minimal ideal of L. □

The following theorem describes the relationship between integral ideals and integral BL-algebras.

Theorem 3.11. In any BL-algebra L, the following conditions are equivalent:
(i) $\{0\}$ is an integral ideal of L,
(ii) $\{1\}$ is an integral filter of L,
(ii) L is an integral BL-algebra.

Proof. (i) $\Rightarrow$ (ii) Let $\{0\}$ be an integral ideal of L and $(x \odot y)^- \in \{1\}$, for $x, y \in L$. Then $(x \odot y)^- = 1$ and so by ($BL9$), $x \odot y = 0$. Hence, $x \in \{0\}$ or $y \in \{0\}$ and so by ($BL9$), $x^- = 1$ or $y^- = 1$. Therefore, $\{1\}$ is an integral filter of L.
(i) $\Rightarrow$ (ii) Let $\{1\}$ be an integral filter of L and $x \odot y \in \{0\}$, for $x, y \in L$. Then $x \odot y = 0$ and so by ($BL9$), $(x \odot y)^- = 1$. Hence, $x^- = 1$ or $y^- = 1$ and so ($BL9$), $x = 0$ or $y = 0$. Hence, $x \in \{0\}$ or $y \in \{0\}$. Therefore, $\{0\}$ is an integral ideal of L.
(i) $\Rightarrow$ (iii) Let $\{0\}$ be an integral ideal of L and $x \odot y = 0$, for $x, y \in L$. Then $x \odot y \in \{0\}$ and so $x \in \{0\}$ or $y \in \{0\}$. Therefore, $x = 0$ or $y = 0$ and so L is an integral BL-algebra.
(iii) $\Rightarrow$ (i) Let L be an integral BL-algebra and $x \odot y \in \{0\}$, for $x, y \in L$. Then $x \odot y = 0$ and so $x = 0$ or $y = 0$. Hence, $x \in \{0\}$ or $y \in \{0\}$. Therefore, $\{0\}$ is an integral ideal of L. □

Definition 3.12. [6] Let L_1 and L_2 be two BL-algebras. Then the map $f : L_1 \to L_2$ is called a BL- homomorphism if and only if it satisfies the following conditions, for every $x, y \in L_1$:
(i) $f(0) = 0$,
(ii) $f(x \odot y) = f(x) \odot f(y)$,
(iii) $f(x \to y) = f(x) \to f(y)$.
Moreover, if J is an ideal of L_2, then $f^{-1}(J)$ is an ideal of L_1.

Proposition 3.13. Let L_1, L_2 be two BL-algebras, $\phi : L_1 \to L_2$ be a BL-homomorphism and J be an integral ideal of L_2. Then $\phi^{-1}(J)$ is an integral ideal of L_1.

Proof. Let $x \odot y \in \phi^{-1}(J)$, for $x, y \in L_1$. Then $\phi(x \odot y) \in J$ and so $\phi(x) \odot \phi(y) \in J$. Since J is an integral ideal of L_2, then $\phi(x) \in J$ or $\phi(y) \in J$ and so $x \in \phi^{-1}(J)$ or $y \in \phi^{-1}(J)$. Therefore, $\phi^{-1}(J)$ is an integral ideal of L_1. □

4. Relation among integral ideals, Boolean ideals, maximal ideals and obstinate filters in BL-algebras

In this section, we study relation between integral ideals, Boolean ideals, maximal ideals and obstinate filters in BL-algebras. Also, we describe relationship between integral ideals and prime ideals in BL-algebras.

Theorem 4.1. Let I be an ideal of L. Then I is an integral ideal if and only if $N(I)$ is an obstinate filter of L.

Proof. Let I be an integral ideal of L. Then by Theorem 3.5, $N(I)$ is an integral filter of L. Moreover, by Theorem 2.11(iv), $N(I)$ is a fantastic filter of L. Hence, by Theorem 2.10(iii), $N(I)$ is an obstinate filter of L. Conversely, let $N(I)$ be an obstinate filter of L. Then by Theorem 2.10(v), $\dfrac{L}{N(I)} \cong \{0,1\}$ and so $\dfrac{L}{N(I)}$ is an integral BL-algebra. Hence, by Theorem 2.8, $N(I)$ is an integral filter and so by Theorem 3.5, I is an integral ideal of L. $\square$

Theorem 4.2. Let F be a proper filter of L. Then F is an integral filter if and only if $N(F)$ is an integral ideal of L.

Proof. Let F be an integral filter of L. Then by Theorem 2.11(ii), $N(F)$ is an ideal of L. Let $x \odot y \in N(F)$, for $x, y \in L$. Then $(x \odot y)^- \in F$ and since F is an integral filter, then $x^- \in F$ or $y^- \in F$ and so $x \in N(F)$ or $y \in N(F)$. Therefore, $N(F)$ is an integral ideal of L. Conversely, let $N(F)$ be an integral ideal of L and $(x \odot y)^- \in F$, for $x, y \in L$. Then $x \odot y \in N(F)$ and so $x \in N(F)$ or $y \in N(F)$. Hence, $x^- \in F$ or $y^- \in F$. Therefore, F is an integral filter of L. $\square$

Corollary 4.3. If F is an obstinate filter of L, then $N(F)$ is an integral ideal of L.

Proof. Let F be an obstinate filter of L. Then by Theorem 2.10, F is an integral filter and so by Theorem 4.2, $N(F)$ is an integral ideal of L. $\square$

Corollary 4.4. If F is a fantastic filter of L, which $N(F)$ is an integral ideal of L, then F is an obstinate filter of L.

Proof. Let F be a fantastic filter of L, which $N(F)$ is an integral ideal of L. Then by Theorem 4.2, F is an integral filter and so by Theorem 2.10, F is an obstinate filter of L. $\square$

Theorem 4.5. Let I be an ideal of L. Then the following conditions are equivalent:
(i) I is an integral ideal of L,
(ii) I is a prime and Boolean ideal of L,
(iii) I is a proper ideal and for all $x \in L$, $x \in I$ or $x^- \in I$.

Proof. (i) $\Rightarrow$ (ii) It follows from Theorem 3.7(i) and (ii).
(ii) $\Rightarrow$ (i) Let I be a prime and Boolean ideal of L and $x \odot y \in I$ for $x, y \in L$. If $x \notin I$ and $y \notin I$, since I is a Boolean ideal, then $x \wedge x^- \in I$ and $y \wedge y^- \in I$ and since I is a prime ideal of L, then $x^- \in I$ and $y^- \in I$. Hence, $x \in N(I)$ and $y \in N(I)$. Now, by Theorem 2.11(i), $N(I)$ is a filter and so $x \odot y \in N(I)$. Therefore, $(x \odot y)^- \in I$ and so $[(x \odot y)^-] = [0]$. Moreover, since $x \odot y \in I$, then $[x \odot y] = [0]$. Now, since $\dfrac{L}{I}$ is an MV-algebra, then $[(x \odot y)^-]^- = [0]^- = [1]$ and so $[x \odot y] = [1]$, which is impossible. Therefore, $x \in I$ or $y \in I$ and so I is an integral ideal of L.
(ii) $\Rightarrow$ (iii) let I be a prime and Boolean ideal of L. Then I is a proper ideal of L and $x \wedge x^- \in I$, for all $x \in L$. Now, since I is prime ideal, then $x \in I$ or $x^- \in I$, for all $x \in L$.
(iii) $\Rightarrow$ (ii) Since by ($BL12$), $x \wedge x^- \leq x$ and $x \wedge x^- \leq x^-$ and I is a proper ideal and for all $x \in L$, $x \in I$ or $x^- \in I$, then $x \wedge x^- \in I$, for all $x \in L$. Therefore, I is

a Boolean ideal of L. Now, if $(x \to y)^- \in I$, for $x, y \in L$, then I is a prime ideal and if $(x \to y)^- \notin I$, then $x \to y \in I$. Now, by $(BL8)$, $x \to 0 \leq x \to y$ and so $x \to 0 \in I$. Since $x \leq y \to x$, by $(BL7)$ and so by $(BL8)$, $(y \to x) \to 0 \leq x \to 0$. Hence, $(y \to x)^- \in I$. Therefore, I is a prime ideal of L. □

Lemma 4.6. Let L be a BL-algebra. Then $x \odot (y \to z) \leq y \to (x \odot z)$, for all $x, y, z \in L$.

Proof. Since for all $x, y, z \in L$, $x \odot z \leq x \odot z$ and by $(BL3)$, $x \leq z \to (x \odot z)$, then by $(BL14)$, $z \to (x \odot z) \leq (y \to z) \to (y \to (x \odot z))$ and so $x \leq (y \to z) \to (y \to (x \odot z))$. Therefore, by $(BL3)$,

$$x \odot (y \to z) \leq y \to (x \odot z).$$

□

Definition 4.7. [4] Let $x \in L$. If there exists a smallest positive integer number n such that $x^n = 0$, then we say the order of x is n and we denote by $ord(x) = n$ and we say is $ord(x) = \infty$, if no such n exist $x^n = 0$. L is called a *locally finite* BL-algebra, if, $ord(x) < \infty$, for all $x \in L \backslash \{1\}$.

Theorem 4.8. Let M be a proper ideal of L. Then the following conditions are equivalent:
(i) M is a maximal ideal of L,
(ii) for all $x \notin M$, there exists $n \in N$, $(x^-)^n \in M$.
(iii) $\dfrac{L}{M}$ is a locally finite MV-algebra.

Proof. $(i) \Rightarrow (ii)$ Let M be a maximal ideal of L and $x \notin M$. Define a subset D of L by

$$D = \{z \in L \mid \text{for some}\, y \in M \text{ and } \, n \in N, (x^-)^n \odot z \leq y\}$$

Since by $(BL11)$, $x^- \odot x = 0 \leq 0$, then $x \in D$ and if $z \in M$, then by $(BL7)$, $x^- \odot z \leq z$ and so $z \in D$. Hence, $M \subset D \subseteq L$. Now, we show that D is an ideal of L. Let $z, w \in L$, $z \leq w$ and $w \in D$. Then there exist $y \in M$ and $n \in N$ such that $(x^-)^n \odot w \leq y$. Since $z \leq w$, then by $(BL7)$, $(x^-)^n \odot z \leq (x^-)^n \odot w$ and so $(x^-)^n \odot z \leq y$. Therefore, $z \in D$. Now, let $z, w \in D$. Then there exist $y_1, y_2 \in M$ and $m, n \in N$ such that $(x^-)^n \odot z \leq y_1$ and $(x^-)^m \odot w \leq y_2$. By $(BL7)$, $(x^-)^{m+n} \odot z \leq y_1$ and $(x^-)^{m+n} \odot w \leq y_2$ and since $y_1, y_2 \in M$ and M is an ideal of L, then $y_1^- \to y_2 \in M$. Now, by Lemma 4.6 and $(BL8)$,

$$(x^-)^{m+n} \odot (z^- \to w) \leq z^- \to (((x^-)^{m+n} \odot w)) \leq z^- \to y_2$$

And so by $(BL8)$,

$$(z^- \to y_2) \to (y_1^- \to y_2) \leq (x^-)^{m+n} \odot (z^- \to w) \to (y_1^- \to y_2)$$

Moreover, by $(BL13)$ and $(BL14)$,

$$z \to y_1 \leq y_1^- \to z^- \leq (z^- \to y_2) \to (y_1^- \to y_2)$$

And since $(x^-)^{m+n} \odot z \leq y_1$, then by $(BL3)$ $(x^-)^{m+n} \leq z \to y_1$ and so

$$\begin{aligned} (x^-)^{m+n} &\leq z \to y_1 \\ &\leq (z^- \to y_2) \to (y_1^- \to y_2) \\ &\leq ((x^-)^{m+n} \odot (z^- \to w)) \to (y_1^- \to y_2) \end{aligned}$$

Hence,
$$(x^-)^{m+n} \leq ((x^-)^{m+n} \odot (z^- \to w)) \to (y_1^- \to y_2)$$
And so by $(BL3)$,
$$(x^-)^{m+n} \odot ((x^-)^{m+n} \odot (z^- \to w)) \leq (y_1^- \to y_2)$$
Therefore,
$$(x^-)^{2(m+n)} \odot (z^- \to w) \leq (y_1^- \to y_2)$$
Since $(y_1^- \to y_2) \in M$ and $2(m+n) \in N$, then $(z^- \to w) \in D$ and so D is an ideal of L. Now, since M is a maximal ideal of L and $M \subset D \subseteq L$, then $D = L$ and so $1 \in D$. Hence, there exist $y \in M$ and $n \in N$ such that $(x^-)^n = (x^-)^n \odot 1 \leq y$ and since $y \in M$, then $(x^-)^n \in M$.

$(ii) \Rightarrow (iii)$ Let $[1] \neq [x] \in \dfrac{L}{M}$. Then $x^- \odot 1 \notin M$, since if $x^- \odot 1 \in M$, then $x \odot 1^- = 0 \in M$ and so $[x] = [1]$. Hence, $x^- \notin M$ and so by (ii), there exist $n \in N$ such that $((x^-)^-)^n \in M$. By $(BL11)$, $x \leq x^{--}$ and so $x^n \leq (x^{--})^n$. Hence, $x^n \in M$ and so $[x]^n = [0]$. Therefore, $\dfrac{L}{M}$ is a locally finite MV-algebra.

$(iii) \Rightarrow (ii)$ Let $\dfrac{L}{M}$ be a locally finite MV-algebra. Then for all $[1] \neq [x] \in \dfrac{L}{M}$, there exist $n \in N$ such that $[x]^n = [0]$ and so $x^n \in M$. Now, let $x \notin M$. Then $[x] \neq [0]$ and so $[x]^- \neq [1]$. Since if $[x]^- = [1]$, then $[x]^{--} = [0]$ and since $\dfrac{L}{M}$ is an MV-algebra, then $[x] = [x]^{--} = [0]$, which is impossible. Hence, $[x]^- \neq [1]$ and so there exist $n \in N$, $([x]^-)^n = [0]$. Hence, $[(x^-)^n] = [0]$ and so $(x^-)^n \in M$.

$(iii) \Rightarrow (i)$ J be an ideal of L and $M \subset J \subseteq L$. Then there exist $x \in J$ which $x \notin M$. Hence, there exist $n \in N$ such that $(x^-)^n \in M \subset J$ and so $(x^-)^n \in J$. Therefore, $[(x^-)^n] = [x^-]^n = [0]$. Moreover, since $x \in J$, then $[x] = [0]$ and so $[x^-] = [x]^- = [1]$. Hence, $[x^-]^n = [1]$ and so $[0] = [1]$. Therefore, $1 \odot 0^- = 1 \in J$ and so $J = L$. Thus, M is a maximal ideal of L. □

Theorem 4.9. Let I be an integral ideal of L. Then I is a maximal and Boolean ideal of L.

Proof. Let I be an integral ideal of L and $x \notin I$. Then by Theorem 4.5, I is a Boolean and $x^- \in I$ and so by Theorem 4.8, I is a maximal ideal of L. □

Proposition 4.10. [7] Let I be an ideal of L with $x \in L$, but $x \notin I$. Then there exists a prime ideal P of L containing I with $x \notin P$.

Theorem 4.11. Let M be a maximal ideal of L. Then M is a prime ideal of L.

Proof. Let M be a maximal ideal of L. Since $1 \notin M$, then by Proposition 4.10, there exists a prime ideal P of L containing M with $1 \notin P$. Hence, $M \subseteq P \subset L$ and since M is a maximal ideal of L, then $M = P$. Therefore, M is a prime ideal of L. □

In the following theorem we describe relationship among integral ideals, maximal ideals, prime ideals and Boolean ideals in a BL-algebras.

Theorem 4.12. Let I be an ideal of L. Then the following conditions are equivalent:
(i) I is an integral ideal of L,
(ii) I is a prime and Boolean ideal of L,

(iii) I is a maximal and Boolean ideal of L,
(iv) I is a proper ideal and for all $x \in L$, $x \in I$ or $x^- \in I$.

Proof. It follows from Theorem 4.5, Theorem 4.9 and Theorem 4.11. □

Proposition 4.13. Let F be a maximal filter of L. Then $N(F)$ is a maximal ideal of L.

Proof. Let F be a maximal filter of L. Then by Theorem 2.11(ii), $N(F)$ is an ideal of L. Let $x \notin N(F)$. Then $x^- \notin F$, since if $x^- \in F$, then $x \in N(F)$ which is impossible. Now, since F is a maximal ideal of L and $x^- \notin F$, then there exists $n \in N$, $((x^-)^n)^- \in F$ and so $(x^-)^n \in N(F)$. Therefore, by Theorem 4.8, $N(F)$ is a maximal ideal of L. □

The following example shows that the converse of Proposition 4.13(i), is not correct in general.

Example 4.14. Let L be BL-algebra of Example 3.2 and $F = \{1, c\}$. Then F is a filter and $N(F) = \{0, d\}$. Now, since $N(F)$ is an integral ideal, then by Theorem 4.9, $N(F)$ is a maximal ideal of L. But, since $F \subset G = \{1, a, b, c, e, f\}$ and G is a proper filter of L, then F is not a maximal filter of L. By another way, since $a \notin F$ and for all $n \in N$, $(a^-)^n = d^n = d \notin F$, then by Theorem 4.8, F is not a maximal filter of L. Therefore, the converse of Proposition 4.13(i), is not correct in general.

Theorem 4.15. Let I be an ideal of L. Then I is a maximal ideal of L if and only if $N(I)$ is a maximal filter of L.

Proof. Let I be a maximal ideal of L and $x \notin N(I)$. Then $x^- \notin I$, since if $x^- \in I$, then $x \in N(I)$ which is impossible. Hence, $x^- \notin I$ and so by Theorem 4.8, there exists $n \in N$, $((x^-)^-)^n \in I$. Now, by $(BL11)$, $x \leq x^{--}$ and so $x^n \leq (x^{--})^n$. Hence, $x^n \in I$ and so by Lemma 3.4, $(x^n)^- \in N(I)$. Therefore, $N(I)$ is a maximal filter of L. Conversely, let $N(I)$ be a maximal filter of L and $x \notin I$. Then $x^- \notin N(I)$. Since if $x^- \in N(I)$, then $x^{--} \in I$ and so by $(BL11)$, $x \in I$ which is impossible. Hence, $x^- \notin N(I)$ and so there exists $n \in N$, $((x^-)^n)^- \in N(I)$ and so $((x^-)^n)^{--} \in I$. Therefore , $(BL11)$, $(x^-)^n \in I$ and so by Theorem 4.8, I is a maximal ideal of L. □

5. Conclusion

The results of this paper are be devoted to study a new class of ideals that is called integral ideals. We presented a characterization and several important properties of integral ideals. In particular, we prove that an ideal is integral ideal if and only if is Boolean and (prime)maximal ideal. Also, we proved that a BL-algebra is an integral BL-algebra if and only if trivial ideal $\{0\}$ is an integral ideal. Moreover, we studied relation between integral ideals and obstinate filters in BL-algebras by using the set of complement elements. Also, we described relationship between maximal ideals in BL-algebras and locally finite MV-algebras and we proved an ideal M of BL-algebra L is a maximal ideal if and only if for all $x \notin M$, there exists $n \in N$, $(x^-)^n \in M$ if and only if quotient BL-algebra $\dfrac{L}{M}$ is a locally finite MV-algebra.

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