Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 42(1), 2015, Pages 13–26
ISSN: 1223-6934

Periodic solution for some parabolic degenerate equation with critical growth with respect to the gradient

Fatima Aqel and Nour Eddine Alaa

Abstract. This work is concerned with the existence of periodic solution for a parabolic degenerate equation with critical growth on the gradient and Dirichlet boundary condition. The aim will be achieved by applying some recent results. The first result that we are based on is the existence of solutions for quasilinear elliptic degenerate systems with L^1 data and nonlinearity in the gradient [1] and the second one is the existence of weak periodic solutions of some quasilinear parabolic systems with data measures [2].

2010 Mathematics Subject Classification. 35K59, 35K59, 35K57, 35K55, 34C25, 74G25, 35D30, 46T12.
Key words and phrases. Quasilinear equations, periodic solutions, sub and super solutions.

1. Introduction

The existence of periodic solutions for coupled systems, semilinear and quasilinear parabolic equations under either Dirichlet or Neumann boundary conditions has been investigated by several authors ([2],[3],[4],[5],[6],...) by different methods such as the theory of monotone operators, the Poincaré method, the Leray-Schauder fixed point theory and the method of sub and super-solutions. As we can see, the periodic behavior of solutions of parabolic boundary value problems could be arise from many biological, ecological, chemical engineering and physical systems. The typical problem, which is going to be discussed in the present paper, is to study the existence of a periodic solution for the following model

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + a(t,x)|\nabla u|^2 = f & \text{in } \ Q_T, \\ u = 0 & \text{on } \Sigma_T, \\ u(0) = u(T) & \text{in } \ \Omega, \end{cases} \tag{1}$$

where Ω is an open bounded subset of $\mathbb{R}^N$, $N \geq 1$, with smooth boundary $\partial\Omega$. For given $T > 0$, we set $Q_T = (0,T) \times \Omega$, $\Sigma_T = (0,T) \times \partial\Omega$, $-\Delta$ denotes the Laplacian operator on L^1 with Dirichlet boundary condition and a, $f : (0,T) \times \Omega \to [0, +\infty)$ are two measurable functions, which are periodic in the time t with period $T > 0$.
In recent years, periodic problems for degenerate parabolic equations have been the subject of extensive study; see $[1 - 6]$ and references therein.

In order to describe our results and relate them to others in the early literature, we mention that many researches have studied the existence of periodic solutions for different kinds of equations. One of those studies is the periodic solutions of semilinear parabolic equations where H. Amann used some methods of functional analysis, namely fixed point theorem in Banach spaces to prove the existence and

This paper has been presented at Congrès MOCASIM, Marrakech, 19-22 November 2014.

multiplicity result for periodic solutions of semilinear differential equations of the second order.

Krasnosel'skii [15] in the case of parabolic equations, it turns out that the Poincaré operator is compact in suitable function spaces. Moreover, by involving the strong maximum principle for linear parabolic equations, it can be shown that it is strongly increasing in some closed subspace of $C^{2+\nu}$, $0<\nu<1$.
We are interested particularly in the case where a, f are irregular.

The purpose of the paper is to present a result of existence of at least one weak periodic solution to problem (1) under quite general assumptions on the functions a and f.

In this paper, we have organized our work as follows. In section 2, we introduce the notations that will be used throughout the paper, we also recall some results obtained previously and needed here. In section 3 we present our problem and we exhibit the main result. The proof is given in the section 4, where we begin with an approximating problem, then we give some a priori estimates, and we end by passing to the limit.

2. Necessary conditions for existence

Throughout the paper we shall assume:
$H1)$ f is a positive function such that $f\in L^1(Q_T)$.
$H2)$ $a\in L^1_{loc}(Q_T)$ such that $a\geq 0$.

The object of this paper is to investigate the existence, and almost periodicity of bounded solutions. First of all, we introduce the definition of a weak periodic solution.

Definition 2.1. A function u is said to be a *weak periodic solution* of problem (1) if

$$\begin{cases} u\in C([0,T];L^1(\Omega))\cap L^1(0,T;W_0^{1,1}(\Omega)), \\ a(t,x)|\nabla u|\in L^1_{loc}(Q_T), \\ \dfrac{\partial u}{\partial t}-\Delta u+a(t,x)|\nabla u|^2=f & \text{in } D^{'}(Q_T), \\ u(0)=u(T) & \text{in } L^2(\Omega). \end{cases} \tag{2}$$

Definition 2.2. We call *periodic subsolution* (resp. *supersolution*) of (1) a function u satisfying (2) with "$=$" replaced by "$\leq$" (resp. $\geq$).

3. Main result

In this section we state the main result of this work.
Let's begin by the definition of truncated function $T_k\in C^2$

$$\begin{aligned} T_k(r)=r \quad & \text{if} \quad 0\leq r\leq k, \\ T_k(r)\leq k+1 \quad & \text{if} \quad r\geq k, \\ 0\leq T_k'(r)\leq 1 \quad & \text{if} \quad r\geq 0, \\ T_k'(r)=0 \quad & \text{if} \quad r\geq k+1, \\ 0\leq -T_k''(r)\leq C(k)\,.. & \end{aligned} \tag{3}$$

For example, the function T_k can be defined as

$$\begin{gathered} T_k(r) = r \quad \text{in} \quad [0,k], \\ T_k(r) = \frac{1}{2}(r-k)^4 - (r-k)^3 + r \quad \text{in} \quad [k, k+1], \\ T_k(r) = k + \frac{1}{2} \quad \text{for} \quad r > k+1\,. \end{gathered} \tag{4}$$

Also, we set

$$\tau^{1,2}(Q_T) = \{v : (0,T) \times \Omega \to (0,T) \times \mathbf{R}^N \text{ measurable, such that } T_k(v) \in L^2(0,T; H^1(\Omega)) \text{ for all } k > 0\}.$$

Then, we introduce the function

$$j_k(v) = \int_0^v T_k(s)ds. \tag{5}$$

Now, we can enunciate the main theorem of this section.

Theorem 3.1. *We assume that the hypothesis (H1) and (H2) are satisfied. If it exists a function $\theta \in \tau(Q_T)$ and a function sequence $\theta_n \in L^\infty(Q_T)$ such that*

$$\begin{cases} 0 \leqslant a \leqslant \theta \quad \text{in } Q_T, \\ \theta_n \to \theta \quad \text{almost everywhere in } Q_T, \\ \nabla T_k(\theta_n) \to \nabla T_k(\theta) \quad \text{strongly in } L^2(Q_T), \\ \lim\limits_{k\to\infty} \sup\limits_n \int_\Omega (|\nabla T_k(\theta_n)|^2) = 0, \\ \dfrac{\partial \theta_n}{\partial t} \in L^2(0,T; H^{-1}(\Omega)). \end{cases} \tag{6}$$

Then the problem (1) has a non-negative weak periodic solution.

Remark 3.1. Let $\dfrac{\partial a}{\partial t} - \Delta a = \mu$ denotes the Radon measure. Then, we take $\theta = a$ and θ_n solution of

$$\begin{cases} \dfrac{\partial \theta_n}{\partial t} - \Delta \theta_n = \mu_n \quad \text{in } Q_T, \\ \theta_n = 0 \quad \text{in } \Sigma_T, \\ \theta_n(0,x) = \theta_{0,n}(x) \quad \text{in } \Omega, \end{cases} \tag{7}$$

where $\mu_n \in C_0^\infty(Q_T)$, $\mu_n \to \mu$ in $M_b(Q_T)$ and $\mu_n \le \mu$.

4. Proof of the main result

Now we prove our main existence result which reads as follows.

4.1. The approximate problem. First, we define an approximated equation of (1). For this, we need to truncate the functions a and f by considering the sequence a_n and f_n defined by

$$a_n = \min_{n \geqslant 0}\{a, \theta_n\}, \tag{8}$$

and

$$f_n \in C_0^\infty(Q_T), f_n \to f \quad \textit{in } L^1(Q_T) \quad \textit{and } f_n \leqslant f. \tag{9}$$

Then, we approximate (1) by the following problem

$$\begin{cases} u_n \in C([0,T]; L^\infty(\Omega)) \cap L^\infty(0,T; W_0^{1,\infty}(\Omega)), \\ \dfrac{\partial u_n}{\partial t} - \Delta u_n + a_n(t,x)|\nabla u_n|^2 = f_n \quad \text{in } D'(Q_T), \\ u_n(0) = u_n(T) \quad \textit{in } \Omega. \end{cases} \tag{10}$$

On the other hand, let $a_n \in L^\infty(Q_T)$. Furthermore, 0 is a subsolution of problem (10) and ω_n is a solution of the following linear problem

$$\begin{cases} u_n \in L^\infty(0,T;W_0^{1,\infty}(\Omega)), \\ \dfrac{\partial \omega_n}{\partial t} - \Delta\omega_n = f_n \quad \text{in } D'(Q_T), \\ \omega_n(0) = \omega_n(T) \quad \text{in } \Omega, \end{cases} \tag{11}$$

is a supersolution. So the problem (10) admits a positive solution u_n such that for all $n \geq 0$, $0 \leq u_n \leq \omega_n$ (for more details, see [12] and [**?**]).

4.2. A priori estimates. In order to prove the Theorem 2.1, we propose to pass to the limit in (10). For this, we will need to prove some certain estimates on the nonlinearity and also on the gradient.

Lemma 4.1. *Let $u,\ w \in L^2(0,T;H_0^1(\Omega)) \cap C([0,T];L^2(\Omega)) \cap L^\infty(Q_T)$*

$$\begin{cases} 0 \leq u \leq w \ \text{in } Q_T, \\ (w-u)_t - \Delta(w-u) \geq 0 \ \text{in } D'(Q_T), \\ w_t - \Delta w \geq 0 \ \text{in } D'(Q_T), \\ u(0) = u(T) \ \text{in } L^2(\Omega), \\ w(0) = w(T) \ \text{in } L^2(\Omega). \end{cases}$$

Then

$$\int_{Q_T} |\nabla u|^2 \leq C \int_{Q_T} |\nabla w|^2. \tag{12}$$

Lemma 4.2. *Let $u,\ w \in L^2(0,T;H_0^1(\Omega)) \cap C([0,T];L^2(\Omega)) \cap L^\infty(Q_T)$ such that $0 \leq w_n - u_n \leq w - u$ in Q_T and $u(0) = u(T)$ in $L^2(\Omega)$, $w(0) = w(T)$ in $L^2(\Omega)$*

$$\begin{aligned} &w_n \to w \ \text{strongly in } L^2(0,T;H_0^1(\Omega)), \\ &u_n \to u \ \text{weakly in } L^2(0,T;H_0^1(\Omega)), \end{aligned} \tag{13}$$

and we have

$$\begin{cases} 0 \leq w_n - u_n \leq w - u \ \text{in } Q_T, \\ (w_n - u_n)_t - \Delta(w_n - u_n) = \rho_n \ \text{in } D'(Q_T), \\ u_n(0) = u_n(T) \ \text{in } L^2(\Omega), \\ w_n(0) = w_n(T) \ \text{in } L^2(\Omega), \\ \rho_n \in L^1(Q_T),\ \rho_n \geq 0 \ \text{and } ||\rho_n||_{L^1(Q_T)} \geq C, \end{cases}$$

where C is a positive constant independent of n. Then

$$u_n \to u \ \text{strongly in } L^2(0,T;H_0^1(\Omega)). \tag{14}$$

Proof of Lemma 4.1. First, we define the Lebesgue-Steklov average u^h of the solution. For $h > 0$, we have

$$u^h(t,x) = \frac{1}{h}\int_t^{t+h} u(s,x)ds$$

this function is well-defined in the space $H_0^1(\Omega)$. We have

$$\begin{aligned}\int_0^{T-h}\int_\Omega |(w^h-u^h)|^2 dxdt &= \int_0^{T-h}\int_\Omega -(w^h-u^h)\Delta(w^h-u^h)dxdt\\ &= \int_0^{T-h}\int_\Omega (w^h-u^h)[\partial_t(w^h-u^h)-\Delta(w^h-u^h)dxdt]\\ &\quad -\int_0^{T-h}\int_\Omega (w^h-u^h)\partial_t(w^h-u^h)\\ &= \int_0^{T-h}\int_\Omega (w^h-u^h)[\partial_t(w^h-u^h)-\Delta(w^h-u^h)dxdt]+\varepsilon_1(h),\end{aligned}$$

where

$$\begin{aligned}\varepsilon_1(h) &= -\int_0^{T-h}\int_\Omega (w^h-u^h)\partial_t(w^h-u^h) = -\int_0^{T-h}\int_\Omega \partial_t(w^h-u^h)^2 dxdt\\ &= \int_\Omega -(w^h-u^h)^2(T-h)+(w^h-u^h)^2(0)dxdt.\end{aligned}$$

Then by passing to the limit, we get

$$\lim_{h\to 0}\varepsilon_1(h)=\lim_{h\to 0}\int_\Omega -(w^h-u^h)^2(T-h)+(w^h-u^h)^2(0)dxdt=0,$$

consequently

$$\begin{aligned}\int_0^{T-h}\int_\Omega |\nabla(w^h-u^h)|^2dxdt &\le \int_0^{T-h}\int_\Omega w^h[\partial_t(w^h-u^h)-\Delta(w^h-u^h)dxdt]+\varepsilon_1(h)\\ &\le -\int_0^{T-h}\int_\Omega (w^h-u^h)\partial_t w^h+\int_\Omega [(w^h-u^h)w^h]_0^{T-h}\\ &\quad +2\int_0^{T-h}\int_\Omega \nabla w^h\nabla(w^h-u^h)dxdt+\int_0^{T-h}\int_\Omega (w^h-u^h)\Delta w^h dxdt+\varepsilon_1(h),\end{aligned}$$

hence

$$\begin{aligned}\int_0^{T-h}\int_\Omega |\nabla(w^h-u^h)|^2dxdt &\le -\int_0^{T-h}\int_\Omega (w^h-u^h)[\partial_t w^h-\Delta w^h]\\ &\quad +2\int_0^{T-h}\int_\Omega \nabla w^h\nabla(w^h-u^h)dxdt+\varepsilon_1(h)+\varepsilon_2(h).\end{aligned}$$

Since $\partial_t w^h-\Delta w^h\ge 0$, we get

$$\int_0^{T-h}\int_\Omega |\nabla(w^h-u^h)|^2dxdt\le 2\int_0^{T-h}\int_\Omega \nabla w^h\nabla(w^h-u^h)dxdt+\varepsilon_1(h)+\varepsilon_2(h),$$

where

$$\varepsilon_2(h)=\int_\Omega [(w^h-u^h)w^h]_0^{T-h},$$

which implies that

$$\begin{aligned}\lim_{h\to 0}\varepsilon_2(h) &= \lim_{h\to 0}\int_\Omega [(w^h-u^h)w^h]_0^{T-h}dx\\ &= \lim_{h\to 0}\int_\Omega [(w^h-u^h)(T-h)w^h(T-h)-(w^h-u^h)(0)w^h(0)]dx\\ &= 0.\end{aligned}$$

It follows that

$$\int_0^{T-h}\int_\Omega |\nabla(w^h-u^h)|^2dxdt \le 2\int_0^{T-h}\int_\Omega |\nabla w^h|^2+\frac{1}{2}\int_0^{T-h}\int_\Omega |\nabla(w^h-u^h)|^2 + \varepsilon_1(h)+\varepsilon_2(h),$$

which implies that

$$\int_0^{T-h}\int_\Omega |\nabla(w^h-u^h)|^2dxdt \le 4\int_0^{T-h}\int_\Omega |\nabla w^h|^2+2(\varepsilon_1(h)+\varepsilon_2(h)).$$

Finally, we deduce

$$\int_{Q_T}|\nabla u|^2dxdt \le 12\int_{Q_T}|\nabla w|^2.$$

□

Proof of Lemma 4.2. To prove that u_n converges strongly in $L^2(0,T;H_0^1(\Omega)$, it suffice that

$$\lim_{n\to\infty}\int_{Q_T}|\nabla u_n|^2dxdt \le \int_{Q_T}|\nabla u|^2, \tag{15}$$

For $h>0$, let uh,w^h and u_n^h, w_n^h the sequences given by $0\le w_n^h-u_n^h\le w^h-u^h$. First, we have

$$\begin{aligned}\lim_{n\to\infty}\int_{Q_T}|\nabla(w_n-u_n)|^2dxdt &= \lim_{h\to0}\lim_{n\to\infty}\int_{Q_{T-h}}|\nabla(w_n^h-u_n^h)|^2dxdt\\ &= \lim_{h\to0}\lim_{n\to\infty}\int_{Q_{T-h}}(w_n^h-u_n^h)[\frac{\partial(w_n^h-u_n^h)}{\partial t}-\Delta(w_n^h-u_n^h)dxdt]\\ &\le \lim_{h\to0}\lim_{n\to\infty}\int_{Q_{T-h}}(w^h-u^h)[\frac{\partial(w_n^h-u_n^h)}{\partial t}-\Delta(w_n^h-u_n^h)dxdt]\\ &\le \lim_{h\to0}\lim_{n\to\infty}\int_{Q_{T-h}}[(w^h-u^h)\frac{\partial(w_n^h-u_n^h)}{\partial t}+\nabla(w^h-u^h)\nabla(w_n^h-u_n^h)]dxdt.\end{aligned}$$

By using the weak convergence of u_n in $L^2(0,T;H_0^1(\Omega))$ to u which implies that $u_n^h\rightharpoonup u^h$ weakly in $L^2(0,T;H_0^1(\Omega))$ and the strong convergence of w_n in $L^2(0,T;H_0^1(\Omega))$ which give us that $w_n^h\to w^h$ strongly in $L^2(0,T;H_0^1(\Omega))$. Then we obtain

$$\begin{aligned}\lim_{n\to\infty}\int_{Q_T}|\nabla(w_n-u_n)|^2dxdt &\le \lim_{h\to0}\int_{Q_{T-h}}[(w^h-u^h)\frac{\partial(w^h-u^h)}{\partial t}+|\nabla(w^h-u^h)|^2]dxdt\\ &\le \lim_{h\to0}(\int_\Omega[(w^h-u^h)^2]_0^{T-h}dx+\int_{Q_{T-h}}|\nabla(w^h-u^h)|^2dxdt)\\ &\le \int_{Q_T}|\nabla(w-u)|^2dxdt.\end{aligned}$$

Finally we obtain the desired result (14). □

Now, we need to prove that $a_n(t,x)|\nabla u_n|^2$ is bounded in $L^1(Q_T)$.

Lemma 4.3. *Let (u_n) and (a_n) be sequences previously defined. Then we have*
i) $\int_{Q_T}a_n(t,x)|\nabla u_n|^2\le C_1$.

Proof. *i*) By integrating the equation satisfied by u_n over Q_T and multiplying by a test function φ which is equal to $\varphi = 1$, we obtain

$$\int_{Q_T} \frac{\partial u_n}{\partial t} - \Delta u_n + a_n(t,x)|\nabla u_n|^2 = \int_{Q_T} f_n.$$

As u_n is periodic, $u_n = 0$ on Σ_T, and by using (10), we have

$$\int_{Q_T} a_n(t,x)|\nabla u_n|^2 \leq \int_{Q_T} f,$$

hence, we obtain the desired result. □

Lemma 4.4. *Let u_n be a solution of* (10)*, then*

$$\lim_{h\to 0}\lim_{k\to+\infty}\sup_n\left(\frac{1}{k}\int_{Q_{T-h}} |\nabla T_k(u_n^h)|^2\right) = 0.$$

Proof. By using the Lebesgue-Steklov average and multiplying (1) by the truncated function $T_k(u_n^h)$, then integrating on Q_T, we get for every $0 < M < k$,

$$\int_{Q_{T-h}} \frac{\partial j_k(u_n^h)}{\partial t} + \int_{Q_{T-h}} |\nabla T_k(u_n^h)|^2 + \int_{Q_{T-h}} F_n^h T_k(u_n^h) = \int_{Q_{T-h}} f_n^h T_k(u_n^h),$$

where $F_n^h = \frac{1}{h}\int_t^{t+h} a(s,x)|\nabla u|^2 ds \geq 0$
which yields to

$$\int_{Q_{T-h}} \frac{\partial j_k(u_n^h)}{\partial t} + \int_{Q_{T-h}} |\nabla T_k(u_n^h)|^2 \leq \int_{Q_{T-h}} f^h T_k(u_n^h).$$

By using the assumptions on $T_k(u_n^h)$ and $j_k(u_n^h)$, we obtain

$$\int_{Q_{T-h}} |\nabla T_k(u_n^h)|^2 \leq \int_{Q_{T-h}\cap[u_n^h\leq M]} f^h T_k(u_n^h) + k\int_{Q_{T-h}\cap[u_n^h>M]} f^h T_k(u_n^h) + \varepsilon(n,h).$$

where

$$\lim_{h\to 0}\lim_{n\to+\infty}\varepsilon(n,h) = \lim_{h\to 0}\lim_{n\to+\infty}\int_{Q_{T-h}} \frac{\partial j_k(u_n^h)}{\partial t} = 0,$$

therefore

$$\int_{Q_{T-h}} |\nabla T_k(u_n^h)|^2 \leq M\int_{Q_{T-h}} f^h + k\int_{Q_{T-h}} f^h \chi_{[u_n^h>M]} + \varepsilon(n,h).$$

On the other hand, we have

$$|[u_n^h > M]| = \int_{[u_n^h>M]} dxdt \leq \frac{1}{M}||u_n^h||_{L^1(Q_T)} \leq \frac{C}{M}.$$

which yields

$$\lim_{M\to+\infty}(\sup_n |[u_n^h > M]|) = 0.$$

On the other hand, since $f^h \in L^1(Q_T)$, we have for each $\epsilon > 0$ there exists δ such that, for all $E \subset Q_T$

$$|E| < \delta,\ \int_E f^h \leq \epsilon,$$

by taking into account the previous result, we obtain that for each $\epsilon > 0$, then there exists M_ϵ such that, for all $M \geq M_\epsilon$

$$\sup_n\left(\int_{Q_{T-h}} f^h \chi_{[u_n^h>M]}\right) \leq \frac{\epsilon}{2},$$

we set $M = M_\epsilon$ and for $k \to +\infty$, we get

$$\lim_{k\to+\infty}\lim_{h\to 0}\sup_n\left(\frac{1}{k}\int_{Q_{T-h}}|\nabla T_k(u_n^h)|^2\right) = 0.$$

□

4.3. Convergence. In this paragraph, we need to show the existence of u which is a limit of u_n and also a solution of

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + a(t,x)|\nabla u|^2 = f \text{ in } D'(Q_T), \\ u = 0 \text{ in } \Sigma_T, \\ u(0) = u(T) \text{ in } \Omega. \end{cases} \tag{16}$$

Let Ω be an open bounded subset of $\mathbb{R}^N$ with smooth boundary and u solution of (10). Then for every $T > 0$, the application

$$(u_n(0), g_n) \in L^1(\Omega)\times L^1(Q_T) \to u_n \in L^1(Q_T),$$

is compact, where $g_n = f_n - a_n(t,x)|\nabla u_n|^2$.
Moreover, it's continuous from $L^1(\Omega)\times L^1(Q_T)$ into $C([0,T];L^1(\Omega))$. For more details, we refer you to see this paper (Baras & Pierre)[12].
Then we have

$$\begin{aligned} &u_n \to u \quad \text{in } L^1(Q_T), \\ &\nabla u_n \to \nabla u \quad \text{in } \ L^1(Q_T), \end{aligned} \tag{17}$$

ii) From Lemma 4.1 and 4.2, we have

$$\begin{aligned} &T_k(u_n) \text{ converges to } T_k(u) \text{ weakly in } L^2(0,T;H_0^1(\Omega)), \\ &T_k(u_n) \text{ converges to } T_k(u) \text{ strongly in } L^2(0,T;H_0^1(\Omega)). \end{aligned}$$

Since $T_k(u_n)$ is bounded in $L^\infty(Q_T)$, we conclude the existence of $T_k(u) \in L^\infty(Q_T)$ such that

$$T_k(u_n) \text{ converges to } T_k(u) \text{ weak- * in } L^\infty(Q_T).$$

By Lemma 4.3, we know that $a_n(t,x)|\nabla u_n|^2$ is uniformly bounded in $L^1(Q_T)$ and nonnegative. Moreover

$$a_n(t,x)|\nabla u_n|^2 \to a(t,x)|\nabla u|^2 \ \text{ a.e in } Q_T.$$

Then there exists λ non negative measure, see Schwartz [14], such that

$$\lim_{n\to+\infty}\frac{\partial u_n}{\partial t} - \Delta u_n + a_n(t,x)|\nabla u_n|^2 = \frac{\partial u}{\partial t} - \Delta u + a(t,x)|\nabla u|^2 + \lambda \ \text{ in } \ D'(Q_T). \tag{18}$$

However

$$\lim_{n\to+\infty}\frac{\partial u_n}{\partial t} - \Delta u_n + a_n(t,x)|\nabla u_n|^2 = f \ \text{ in } \ L^1(Q_T), \tag{19}$$

consequently

$$\frac{\partial u}{\partial t} - \Delta u + a(t,x)|\nabla u|^2 + \lambda = f, \tag{20}$$

where $\lambda \geq 0$, which yields to the following inequality

$$\frac{\partial u}{\partial t} - \Delta u + a(t,x)|\nabla u|^2 \leq f.$$

but, it stills to prove that we can also establish an inequality in the opposite sense, which means that

$$\frac{\partial u}{\partial t} - \Delta u + a(t,x)|\nabla u|^2 \geq f, \tag{21}$$

for this, we will introduce the function $H \in C^1(\mathbf{R})$, such that

$$0 \leq H(s) \leq 1$$
$$H(s) = \begin{cases} 0 \text{ if } & |s| \geq 1, \\ 1 \text{ if } & |s| \leq \frac{1}{2}. \end{cases}$$

Next, we define the test function Φ as follows

$$\Phi = \psi exp(-\theta_n u_n^h) H(\frac{\theta_n}{k}) H(\frac{u_n^h}{k}),$$

where H is the function defined previously and $\psi \leq 0$, $\psi \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q_T)$ and $\partial_t \psi \in L^2(0,T;H^{-1}(\Omega)) + L^1(Q_T)$.
Now, we multiply our equation by Φ and we integrate over $(0, T-h) \times \Omega$, we obtain

$$\int_{Q_{T-h}} f_n \Phi = \int_{Q_{T-h}} [\frac{\partial u_n}{\partial t} - \Delta u_n + a_n(t,x)|\nabla u_n|^2]\Phi.$$

Then, we obtain

$$\begin{aligned}
\int_{Q_{T-h}} f_n^h \Phi = & \int_{Q_{T-h}} [-u_n \frac{\partial \psi}{\partial t} + \nabla u_n \nabla \psi] exp(-\theta_n u_n^h) H(\frac{\theta_n}{k}) H(\frac{u_n^h}{k}) \\
& + \int_{Q_{T-h}} (a_n(t,x) - \theta_n)|\nabla u_n|^2 \psi \exp(-\theta_n u_n^h) H(\frac{\theta_n}{k}) H(\frac{u_n^h}{k}) \\
& + \int_{Q_{T-h}} \theta_n (\nabla u_n - \nabla u_n^h) \nabla u_n \psi \exp(-\theta_n u_n^h) H(\frac{\theta_n}{k}) H(\frac{u_n^h}{k}) \\
& + \int_{Q_{T-h}} u_n [\theta_n \frac{\partial u_n^h}{\partial t} + u_n^h \frac{\partial \theta_n}{\partial t}] \psi \exp(-\theta_n u_n^h) H(\frac{\theta_n}{k}) H(\frac{u_n^h}{k}) \\
& - \frac{1}{k} \int_{Q_{T-h}} u_n \frac{\partial \theta_n}{\partial t} \psi \exp(-\theta_n u_n^h) H'(\frac{\theta_n}{k}) H(\frac{u_n^h}{k}) - \\
& - \frac{1}{k} \int_{Q_{T-h}} u_n \frac{\partial u_n^h}{\partial t} \psi \exp(-\theta_n u_n^h) H(\frac{\theta_n}{k}) H'(\frac{u_n^h}{k}) \\
& - \int_{Q_{T-h}} \psi u_n^h \nabla u_n \nabla \theta_n \exp(-\theta_n u_n^h) H(\frac{\theta_n}{k}) H(\frac{u_n^h}{k}) + \\
& + \frac{1}{k} \int_{Q_{T-h}} \psi \nabla u_n \nabla \theta_n \exp(-\theta_n u_n^h) H'(\frac{\theta_n}{k}) H(\frac{u_n^h}{k}) \\
& + \frac{1}{k} \int_{Q_{T-h}} \psi \nabla u_n^h \nabla u_n \exp(-\theta_n u_n^h) H(\frac{\theta_n}{k}) H'(\frac{u_n^h}{k}) + \int_\Omega [u_n \Phi]_0^{T-h} \\
& = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8 + I_9 + I_{10}.
\end{aligned}$$

Now, we study each term. For this we fix k and h and we pass to the limit as $n \to +\infty$, we obtain

$$\begin{aligned}
\lim_{n\to+\infty} I_1 = & \lim_{n\to+\infty} \int_{Q_{T-h}} [-T_k(u_n) \frac{\partial \psi}{\partial t} + \nabla T_k(u_n) \nabla \psi] exp(-\theta_n u_n^h) H(\frac{\theta_n}{k}) H(\frac{u_n^h}{k}) \\
= & \int_{Q_{T-h}} [-u \frac{\partial \psi}{\partial t} + \nabla u \nabla \psi] exp(-\theta u^h) H(\frac{\theta}{k}) H(\frac{u^h}{k}).
\end{aligned}$$

Since

$$\frac{\partial\psi}{\partial t}exp(-\theta_n u_n^h)H(\frac{\theta_n}{k})H(\frac{u_n^h}{k}) \to \frac{\partial\psi}{\partial t}exp(-\theta u^h)H(\frac{\theta}{k})H(\frac{u^h}{k}) \text{ strongly in } L^2(Q_T),$$

and we have

$$T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } L^2(0,T;H_0^1(\Omega)),$$

$$\nabla\psi\exp(-\theta_n u_n^h)H(\frac{\theta_n}{k})H(\frac{u_n^h}{k}) \to \nabla\psi\exp(-\theta u^h)H(\frac{\theta}{k})H(\frac{u^h}{k})$$

converges strongly in $L^2(Q_T)$.

For the second term I_2, we have, since $0 \le a_n \le \theta_n$ and $\psi \le 0$, we get

$$(a_n(t,x)-\theta_n)|\nabla u_n|^2\psi\exp(-\theta_n u_n^h)H(\frac{\theta_n}{k})H(\frac{u_n^h}{k}) \ge 0.$$

Therefore by Fatou's lemma, we obtain

$$\lim_{n\to+\infty} I_2 \ge \int_{Q_{T-h}} (a(t,x)-\theta)|\nabla u|^2\psi\exp(-\theta u^h)H(\frac{\theta}{k})H(\frac{u^h}{k}).$$

Let us pass to the third term I_4, we have

$$\lim_{n\to+\infty} I_3 = \lim_{n\to+\infty}\int_{Q_{T-h}} \theta_n(\nabla u_n - \nabla u_n^h)\nabla T_k(u_n)\psi\exp(-\theta_n u_n^h)H(\frac{\theta_n}{k})H(\frac{u_n^h}{k})$$

$$= \int_{Q_{T-h}} \theta(\nabla u - \nabla u^h)\nabla T_k(u)\psi\exp(-\theta u^h)H(\frac{\theta}{k})H(\frac{u^h}{k}),$$

where

$$\nabla T_k(u_n) \rightharpoonup \nabla T_k(u) \text{ in } L^2(Q_T).$$

and

$$\nabla(u_n - u_n^h) \rightharpoonup \nabla(u_n - u_n^h) \text{ in } L^2(0,T;H_0^1(\Omega))).$$

Concerning the term I_4, we have

$$\lim_{n\to+\infty} I_4 = \lim_{n\to+\infty}\int_{Q_{T-h}} u_n[\theta_n\frac{\partial u_n^h}{\partial t} + u_n^h\frac{\partial\theta_n}{\partial t}]\psi\exp(-\theta_n u_n^h)H(\frac{\theta_n}{k})H(\frac{u_n^h}{k})$$

$$= \int_{Q_{T-h}} u[\theta\frac{\partial u^h}{\partial t} + u^h\frac{\partial\theta}{\partial t}]\psi\exp(-\theta u^h)H(\frac{\theta}{k})H(\frac{u^h}{k}).$$

Where u_n^h is differentiable in time for all $h > 0$, and its derivative equal to $\frac{u_n(t+h)-u_n(t)}{h}$. Then we have

$$\psi\exp(-\theta_n u_n^h)H(\frac{\theta_n}{k})H(\frac{u_n^h}{k}) \to \psi\exp(-\theta u^h)H(\frac{\theta}{k})H(\frac{u^h}{k}) \text{ strongly in } L^2(Q_T),$$

and

$$\frac{\partial u_n^h}{\partial t} \to \frac{\partial u^h}{\partial t} \text{ strongly in } L^2(0,T;H_0^1(\Omega)),$$

$$\frac{\partial\theta_n}{\partial t} \to \frac{\partial\theta}{\partial t} \text{ strongly in } L^2(Q_T).$$

Also, we have

$$\psi\exp(-\theta_n^h u_n^h)H(\frac{\theta_n^h}{k})H(\frac{u_n^h}{k}) \to \psi\exp(-\theta^h u^h)H(\frac{\theta^h}{k})H(\frac{u^h}{k})$$

converges weak * in $L^\infty(Q_T)$.

For I_5 we have

$$\lim_{n\to+\infty} I_5 = \lim_{n\to+\infty} -\frac{1}{k}\int_{Q_{T-h}} u_n \frac{\partial\theta_n}{\partial t}\psi \exp(-\theta_n u_n^h) H^{'}(\frac{\theta_n}{k}) H(\frac{u_n^h}{k})$$

$$= -\frac{1}{k}\int_{Q_{T-h}} u\frac{\partial\theta}{\partial t}\psi \exp(-\theta u^h) H^{'}(\frac{\theta}{k}) H(\frac{u^h}{k}).$$

The same thing for I_6, we obtain that

$$\lim_{n\to+\infty} I_6 = \lim_{n\to+\infty} -\frac{1}{k}\int_{Q_{T-h}} u_n \frac{\partial u_n^h}{\partial t}\psi \exp(-\theta_n u_n^h) H(\frac{\theta_n}{k}) H^{'}(\frac{u_n^h}{k})$$

$$= -\frac{1}{k}\int_{Q_{T-h}} u\frac{\partial u^h}{\partial t}\psi \exp(-\theta u^h) H(\frac{\theta}{k}) H^{'}(\frac{u^h}{k}).$$

For I_7, we get

$$\lim_{n\to+\infty} I_7 = \lim_{n\to+\infty} -\int_{Q_{T-h}} \psi u_n^h \nabla T_k(u_n)\nabla T_k(\theta_n) \exp(-\theta_n u_n^h) H(\frac{\theta_n}{k}) H(\frac{u_n^h}{k})$$

$$= -\int_{Q_{T-h}} \psi u^h \nabla u\nabla\theta \exp(-\theta u^h) H(\frac{\theta}{k}) H(\frac{u^h}{k}),$$

since

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in } L^2(0,T;H_0^1(\Omega)),$$
$$T_k(\theta_n) \to T_k(\theta) \quad \text{strongly in } L^2(0,T;H_0^1(\Omega)).$$

For investigating the terms I_8 and I_9, we will need to use Lemma 4.4 and the fourth hypothesis of the main theorem. Also we need $0 \le H(s) \le 1$ and $\exp(-\theta_n u_n) \le 1$. We obtain

$$I_8 \le \left[\frac{1}{k}\int_{Q_{T-h}} \psi|\nabla T_k(u_n)|^2 \exp(-\theta_n u_n^h) H^{'}(\frac{\theta_n}{k}) H(\frac{u_n^h}{k})\right]^{\frac{1}{2}}$$

$$\times \left[\frac{1}{k}\int_{Q_{T-h}} \psi|\nabla T_k(\theta_n)|^2 \exp(-\theta_n u_n^h) H^{'}(\frac{\theta_n}{k}) H(\frac{u_n^h}{k})\right]^{\frac{1}{2}}$$

$$\le \left[||\psi||_{L^\infty(Q_T)}\frac{1}{k}\int_{Q_T} |\nabla T_k(u_n)|^2\right]^{\frac{1}{2}} \times \left[||\psi||_{L^\infty(Q_T)}\frac{1}{k}\int_{Q_T} |\nabla T_k(\theta_n)|^2\right]^{\frac{1}{2}}$$

$$\le \left[||\psi||_{L^\infty(Q_T)}\rho_k\right]^{\frac{1}{2}} \times \left[||\psi||_{L^\infty(Q_T)}\beta_k\right]^{\frac{1}{2}},$$

where

$$\rho_k = \sup_n(\frac{1}{k}\int_{Q_T} |\nabla T_k(u_n)|^2), \qquad \beta_k = \sup_n(\frac{1}{k}\int_{Q_T} |\nabla T_k(\theta_n)|^2).$$

Similarly to the proof of Lemma 4.4, we get

$$\lim_{k\to+\infty} \rho_k = 0, \qquad \lim_{k\to+\infty} \beta_k = 0.$$

Finally, we conclude

$$\lim_{k\to+\infty} \sup_n (I_8) = 0.$$

Similarly for I_9, we obtain

$$
\begin{aligned}
I_9 &\le \left[\frac{1}{k}\int_{Q_{T-h}} \psi|\nabla T_k(u_n^h)|^2 \exp(-\theta_n u_n^h) H(\frac{\theta_n}{k}) H^{'}(\frac{u_n^h}{k})\right]^{\frac{1}{2}} \\
&\times \left[\frac{1}{k}\int_{Q_{T-h}} \psi|\nabla T_k(u_n)|^2 \exp(-\theta_n u_n^h) H(\frac{\theta_n}{k}) H^{'}(\frac{u_n^h}{k})\right]^{\frac{1}{2}} \\
&\le \left[||\psi||_{L^\infty(Q_T)}\frac{1}{k}\int_{Q_{T-h}} |\nabla T_k(u_n^h)|^2\right]^{\frac{1}{2}} \times \left[||\psi||_{L^\infty(Q_T)}\frac{1}{k}\int_{Q_T} |\nabla T_k(u_n)|^2\right]^{\frac{1}{2}} \\
&\le \left[||\psi||_{L^\infty(Q_T)}\rho_{1,k}\right]^{\frac{1}{2}} \times \left[||\psi||_{L^\infty(Q_T)}\beta_{1,k}\right]^{\frac{1}{2}},
\end{aligned}
$$

where $\rho_{1,k} = \sup_n(\frac{1}{k}\int_{Q_{T-h}} |\nabla T_k(u_n^h)|^2)$ and $\beta_{1,k} = \sup_n(\frac{1}{k}\int_{Q_T} |\nabla T_k(u_n)|^2)$. From Lemma 4.4 we have $\lim_{h\to 0}\lim_{k\to+\infty}\rho_{1,k} = 0$ and $\lim_{k\to+\infty}\beta_{1,k} = 0$. Thus

$$
\lim_{h\to 0}\lim_{k\to+\infty}\sup_n(I_9) = 0.
$$

Now we pass to I_{10}, we have $u_n \to u$ in $C([0,T];L^2(\Omega))$. Since Φ is bounded in $L^\infty(Q_T)$, we conclude the existence of $\psi\exp(-\theta u^h)H(\frac{\theta}{k})H(\frac{u^h}{k}) \in L^\infty(Q_T)$ such that $(\psi\exp(-\theta_n u_n^h)H(\frac{\theta_n}{k})H(\frac{u_n^h}{k}))(0)$ and $(\psi\exp(-\theta u^h)H(\frac{\theta}{k})H(\frac{u^h}{k}))(T-h)$ are bounded in $L^\infty(\Omega)$ which yields to

$$
\begin{aligned}
\lim_{n\to+\infty} I_{10} &= \lim_{n\to+\infty}\int_\Omega [T_k(u_n)\psi\exp(-\theta_n u_n^h)H(\frac{\theta_n}{k})H(\frac{u_n^h}{k})]_0^{T-h} \\
&= \int_\Omega [u(T-h)(\psi\exp(-\theta u^h)H(\frac{\theta}{k})H(\frac{u^h}{k}))(T-h) \\
&\qquad - u(0)(\psi\exp(-\theta u^h)H(\frac{\theta}{k})H(\frac{u^h}{k}))(0)]
\end{aligned}
$$

Then, we obtain

$$
\begin{aligned}
&\int_{Q_{T-h}} [-u\frac{\partial\psi}{\partial t} + \nabla u\nabla\psi] exp(-\theta u^h)H(\frac{\theta}{k})H(\frac{u^h}{k}) \\
&\quad + \int_{Q_{T-h}} (a(t,x)-\theta)|\nabla u^h|^2\psi\exp(-\theta^h u^h)H(\frac{\theta^h}{k})H(\frac{u^h}{k}) \\
&\quad + \int_{Q_{T-h}} u[\theta\frac{\partial u^h}{\partial t} + u^h\frac{\partial\theta}{\partial t}]\psi\exp(-\theta u^h)H(\frac{\theta}{k})H(\frac{u^h}{k}) \\
&\quad - \frac{1}{k}\int_{Q_{T-h}} u\frac{\partial\theta}{\partial t}\psi\exp(-\theta u^h)H^{'}(\frac{\theta}{k})H(\frac{u^h}{k}) \\
&\quad + \int_{Q_{T-h}} \psi\theta(\nabla u - \nabla u^h)\nabla u\exp(-\theta u^h)H(\frac{\theta}{k})H(\frac{u^h}{k}) \\
&\quad - \frac{1}{k}\int_{Q_{T-h}} u\frac{\partial u^h}{\partial t}\psi\exp(-\theta u^h)H(\frac{\theta}{k})H^{'}(\frac{u^h}{k}) \\
&\quad + \int_\Omega [u(T-h)(\psi exp(-\theta u^h)H(\frac{\theta}{k})H(\frac{u^h}{k}))(T-h)
\end{aligned}
$$

$$
\begin{aligned}
&- u(0)(\psi exp(-\theta u^h)H(\frac{\theta}{k})H(\frac{u^h}{k}))(0)]\\
&- \int_{Q_{T-h}} u^h \nabla u \nabla \theta \psi \exp(-\theta u^h)H(\frac{\theta}{k})H(\frac{u^h}{k}) + \omega^h(\frac{1}{k})\\
&\leq \int_{Q_{T-h}} f^h \psi exp(-\theta u^h)H(\frac{\theta}{k})H(\frac{u^h}{k}),
\end{aligned}
$$

where $\omega^h(\frac{1}{k})$ is a quantity that tends to 0 as $k \to +\infty$ and $h \to 0$.
Next, we may define the following equation

$$
\psi = -\phi \exp(\theta u^h)H(\frac{\theta}{k})H(\frac{u^h}{k}),
$$

where $\phi \geq 0$ and belongs to $D(Q_T)$. Let us replace ψ by its value in the previous inequality, we obtain

$$
\begin{aligned}
&\int_{Q_{T-h}} u\frac{\partial \phi}{\partial t}H^2(\frac{\theta}{k})H^2(\frac{u^h}{k}) + \int_{Q_{T-h}} u\phi u^h\frac{\partial \theta}{\partial t}H^2(\frac{\theta}{k})H^2(\frac{u^h}{k})\\
&+ \int_{Q_{T-h}} u\theta\frac{\partial u^h}{\partial t}\phi H^2(\frac{\theta}{k})H^2(\frac{u^h}{k}) + \frac{2}{k}\int_{Q_{T-h}} u\frac{\partial \theta}{\partial t}\phi H^{'}(\frac{\theta}{k})H^2(\frac{u^h}{k})H(\frac{\theta}{k})\\
&+ \frac{2}{k}\int_{Q_{T-h}} u\frac{\partial u^h}{\partial t}\phi H^{'}(\frac{u^h}{k})H(\frac{u^h}{k})H^2(\frac{\theta}{k}) - \int_{Q_{T-h}} \nabla u \nabla \phi H^2(\frac{\theta}{k})H^2(\frac{u^h}{k})\\
&- \int_{Q_{T-h}} \phi u^h \nabla u \nabla \theta H^2(\frac{\theta}{k})H^2(\frac{u^h}{k}) - \int_{Q_{T-h}} (a-\theta)|\nabla u|^2 \phi H^2(\frac{\theta}{k})H^2(\frac{u^h}{k})\\
&- \frac{1}{k}\int_{Q_{T-h}} \phi \nabla u \nabla \theta H^{'}(\frac{\theta}{k})H^2(\frac{u^h}{k})H(\frac{\theta}{k}) - \frac{1}{k}\int_{Q_{T-h}} \phi \nabla u^h \nabla u H^2(\frac{\theta^h}{k})H(\frac{u^h}{k})H^{'}(\frac{u^h}{k})\\
&- \int_{Q_{T-h}} \phi\theta(\nabla u - \nabla u^h)\nabla u H^2(\frac{\theta^h}{k})H^2(\frac{u^h}{k}) - \int_{Q_{T-h}} \theta \nabla u^h \nabla u \phi H^2(\frac{\theta^h}{k})H^2(\frac{u^h}{k})\\
&- \int_{Q_{T-h}} u\theta\frac{\partial u^h}{\partial t}\phi H^2(\frac{\theta}{k})H^2(\frac{u^h}{k}) - \int_{Q_{T-h}} u^h\frac{\partial \theta}{\partial t}u\phi H^2(\frac{\theta}{k})H^2(\frac{u^h}{k})\\
&- \int_{\Omega}[u\phi H^2(\frac{\theta}{k})H^2(\frac{u^h}{k})]_0^{T-h} + \int_{Q_{T-h}} \phi u^h \nabla u \nabla \theta H^2(\frac{\theta}{k})H^2(\frac{u^h}{k}) + \omega^h(\frac{1}{k})\\
&\leq \int_{Q_{T-h}} -f\phi H^2(\frac{\theta}{k})H^2(\frac{u^h}{k}).
\end{aligned}
$$

By simplifying some terms and developing some calculations, we get

$$
\begin{aligned}
&\int_{Q_{T-h}} u\frac{\partial \phi}{\partial t}H^2(\frac{\theta}{k})H^2(\frac{u^h}{k}) + \frac{2}{k}\int_0^{T-h} < \frac{\partial \theta}{\partial t}, u\phi H^{'}(\frac{\theta}{k})H^2(\frac{u^h}{k})H(\frac{\theta}{k}) >\\
&+ \frac{2}{k}\int_{Q_{T-h}} u\frac{\partial u^h}{\partial t}\phi H^{'}(\frac{u^h}{k})H(\frac{u^h}{k})H^2(\frac{\theta}{k}) - \int_{Q_{T-h}} \nabla u \nabla \phi H^2(\frac{\theta}{k})H^2(\frac{u^h}{k})\\
&- \int_{Q_{T-h}} a|\nabla u|^2 \phi H^2(\frac{\theta}{k})H^2(\frac{u^h}{k}) - \int_{\Omega}[u\phi H^2(\frac{\theta}{k})H^2(\frac{u^h}{k})]_0^{T-h} + \omega^h(\frac{1}{k})\\
&\leq \int_{Q_{T-h}} -f\phi H^2(\frac{\theta}{k})H^2(\frac{u^h}{k}),
\end{aligned}
$$

and by tending $k \to +\infty$ on the second and the third terms, we find

$$\lim_{k\to+\infty}\frac{2}{k}\int_0^{T-h} < \frac{\partial\theta}{\partial t}, u\phi H^{'}(\frac{\theta}{k})H^2(\frac{u^h}{k})H(\frac{\theta}{k}) >= 0 \text{ and}$$

$$\lim_{k\to+\infty}\frac{2}{k}\int_{Q_{T-h}} u\frac{\partial u^h}{\partial t}\phi H^{'}(\frac{u^h}{k})H(\frac{u^h}{k})H^2(\frac{\theta}{k}) = 0,$$

where $\dfrac{\partial\theta}{\partial t} \in L^2(0,T;H^{-1}(\Omega))$ and $\dfrac{\partial u^h}{\partial t} = \dfrac{u(t+h)-u(t)}{h} \in L^2(0,T;H_0^1(\Omega))$.
Finally, by using the fact that

$$\lim_{k\to\infty} H(\frac{\theta}{k}) = 1, \quad , \lim_{k\to\infty} H(\frac{u^h}{k}) = 1,$$

and then by letting $h \to 0$, we conclude that for every $\phi \geq 0$ and belongs to $D(Q_T)$

$$\int_{Q_T}[-\Delta u + a(t,x)|\nabla u|^2]\phi - \int_{Q_T} u\frac{\partial\phi}{\partial t} + \int_\Omega [u\phi]_0^T \geq \int_{Q_T} f\phi.$$

References

[1] N. Alaa, A. Mouida, S. Mesbahi, and W. Bouarifi, Existence of solutions for quasilinear elliptic degenerate systems with L^1 Data and nonlinearity in the gradient, *Electronic Journal of Differential Equations* **2013** (2013), 1–13.

[2] N. Alaa and M.Iguernane, Weak periodic solutions of some quasilinear parabolic equations with data measure, *J. of Inequalities in Pure and Applied Mathematics* **3** (2002), no. 3, Article 46.

[3] N. Alaa, N. Idrissi Fatmi, and M. Pierre, Quasilinear elliptic degenered equations with nonlinearity in the gradient and L^1 data, *IJMS* (2011) **8**, no. S11, 62–69.

[4] H. Amann, Periodic solutions of semilinear parabolic equations, In: L. Cesari, R. Kannan, R. Weinberger (Eds.), *Nonlinear Analysis: A Collection of Papers in Honor of Erich H. Rothe*, Academic Press, New York (1978), 1–29.

[5] D.W. Bange, Periodic solution of a quasilinear parabolic differential equation, *J. Differential Equation* **17** (1975), 61–72.

[6] M. Badii, Periodic solutions for a nonlinear parabolic equation with nonlinear boundary conditions, *Rend. Sem. Mat. Univ. Pol. Torino* **67** (2009), no. 3, 341–349.

[7] P. Hess, *Periodic-parabolic Boundary Value Problem and Positivity*, Pitman Res. Notes Math Ser. **247**, Longman Scientific and Technical, New York, 1991.

[8] F.E. Browder, Periodic solutions of nonlinear equations of evolution in infinite dimensional spaces, In: A.K. Aziz (ed.), *Lectures in Differential equations* **1**, Van Nostrand-Reinhold, Princeton, New Jersey, 1969.

[9] J.S. Kolesov, A test for the existence of periodic solutions to parabolic equations, *Soviet Math. Dokl* **7** (1966), 1318–1320.

[10] H. Amann and M.G. Grandall, On some existence theorems for semi linear equation equations, *Indiana Univ.Math.J.* **27** (1978), 779–790.

[11] J. Deuel and P. Hess, Nonlinear parabolic boundary value problems with upper and lower solutions, *Israel Journal of Mathematics* **29** (1978), no. 1, 92–104.

[12] P. Baras and M. Pierre, Problèmes paraboliques semi-linéaires avec données mesures, *Applicable Analysis* **18** (1984), 11–149.

[13] P.L. Lions, Résolution de problèmes elliptiques quasilinéaires, *Arch. Rational Mech. Anl.* **74** (1980), no. 4, 335–353.

[14] J.T. Schwartz, *Nonlinear Functional Analysis*, Gordon and Breach Science Publishers, New York-London, Paris, 1969.

[15] M.A. Krasnosel'skii, *Translation along Trajectories of Differential Equations*, Amer. Math. Soc. Translations of Math. Monographs **19**, Amer. Math. Soc, Providence, Rhode Island, 1968.

(Fatima Aqel, Nour Eddine Alaa) Laboratory LaMSAFA, Faculty of Science and Technology, University Cadi Ayyad, B.P. 549, Av. Abdelkarim Elkhattabi, Marrakech - 40000, Morocco

E-mail address: aqel.fatima@gmail.com, n.alaa@uca.ma