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Mathematical analysis of a reaction diffusion model for image restoration

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Abstract. In this paper, we propose a modification and a generalization of the theory developed by P. Perona and J. Malik for edge detection and image restoration, from the case of a single equation to the case of nonlinear reaction-diffusion system. We are interested in the existence of weak solutions for this system for which two main properties hold: the positivity of the solutions and the total mass of the components are preserved with time.

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1. Introduction

Image processing is a discipline of computer science and applied mathematics that deals with digital images and their transformations in order to improve their quality or to extract information. There are a large number of applications of image processing in diverse spectrum of activities. One of the most active topics in this field has been restoration of images. A number of different techniques have been proposed for digital image restoration, utilizing a number of different models and assumptions. The restoration of degraded images is an important problem because it allow to recovery lost information from the observed degraded image data.

One of the first attempts to derive a model that incorporates local information from an image within a PDE framework was conducted by Perona and Malik [21], [22]. They proposed a nonlinear diffusion model in order to avoid the blurring of edges and other localization problems presented by linear diffusion models. The model can be written as

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}\left(g\left(|\nabla u|^2\right)\nabla u\right) = 0 \\ u(0,x) = u_0 \\ \dfrac{\partial u}{\partial v} = 0 \end{cases}$$

In this model the diffusivity has to be such that $\lim\limits_{s\to+\infty} g(s) \to 0$ and $\lim\limits_{s\to 0} g(s) \to 1$.

Notwithstanding the practical success of the Perona-Malik model, it presents some serious theoretical problems: (i) None of the classical well-posedness frameworks is applicable to the Perona-Malik model, (Weickert and Schnörr [25]; Nitzberg and Shiota [19]); (ii) Uniqueness and stability with respect to the initial image should not be expected, i.e. solvability is a difficult problem, in general (Kichenassamy [16]; Höllig and Nohel [14]; Höllig [15]; Perona et al. [20]; Catté et al. [11]); (iii) The regularizing effect of the discretization plays too much of an important role in the

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solution (Fröhlich and Weickert [13]; Benhamouda [9]). The latter is perhaps the key element in the success or failure of the model. Most practical applications work very well provided that the numerical schemes stabilize the process through some implicit regularization.

This observation motivated much research towards the introduction of the regularization directly into the PDE to avoid the dependence on the numerical schemes (Catté et al. [11]; Nitzberg and Shiota [19]). A variety of spatial, spatio-temporal, and temporal regularization procedures have been proposed over the years (Barenblatt et al. [8]; Catté et al. [11]; Weickert [24]; Weickert [26]; Whitaker and Pizer [27]; Li and Chen [17]). The one that has attracted much attention is the mathematically sound formulation due to Catté et al. [11]. They proposed replacing the diffusivity $g\left(|\nabla u|^2\right)$ of the Perona-Malik model by a slight variation $g\left(|\nabla u_\sigma|^2\right)$ with $u_\sigma = G_\sigma * u$, where G_σ is a smooth kernel (Gaussian of variance σ^2). Their proposed model is therefore

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}\left(g\left(|\nabla u_\sigma|^2\right)\nabla u\right) = 0 \\ u(0,x) = u_0 \\ \dfrac{\partial u}{\partial \upsilon} = 0 \end{cases}$$

We should note that this spatial regularization model belongs to a class of well-posed problems (existence and uniqueness were proven in Catté et al. [11], and that its successful implementation is contingent on the choosing of an appropriate value for the additional regularization parameter σ. Whitaker and Pizer [27] and Li and Chen [17] suggested making the parameter σ time-dependent, and Benhamouda [9] performed a systematic study of the influence of σ for the one-dimensional case.

Another interesting variation to the Perona-Malik model is the one proposed by Alvarez et al. [5], [6]. They studied a class of nonlinear parabolic differential equations of the form

$$\begin{cases} \dfrac{\partial u}{\partial t} - g\left(|G * \nabla u|\right)|\nabla u| \operatorname{div}\left(\dfrac{\nabla u}{|\nabla u|}\right) = 0 \\ u(0,x) = u_0 \\ \dfrac{\partial u}{\partial \upsilon} = 0 \end{cases}$$

The term $g\left(|G * \nabla u|\right)$ is used for edge enhancement and it controls the speed of the diffusion.

In 2006, the study of Morfu [19] was focused on the contrast enhancement and noise filtering, he considers the Fisher equation which generally allows simulating the transport mechanisms in living cells, but also enhances the contrast and segmenting images. The model proposed by Morfu is

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}\left(g\left(|\nabla u|\right)\nabla u\right) = f(u) \\ u(0,x) = u_0 \\ \dfrac{\partial u}{\partial \upsilon} = 0 \end{cases}$$

where u_0 is the original image to be processed and $f(s) = s(s-a)(1-s)$ with $0 < a < 1$. The Major defects of this model are: (i) Sensitivity to noise; if we increase slightly the noise, the method gives unsatisfactory results. (ii) No results of existence and consistency.

In 2014, the aim of study of Alaa et al. [1] is to modify the model of Morfu [19] by applying a Gaussian filter on the gradient of the noisy image during the calculation of the coefficient of anisotropic diffusion. The proposed model is as follows

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}\left(g\left(\left|\nabla u_\sigma\right|\right)\nabla u\right) = f\left(t,x,u\right) \\ u\left(0,x\right) = u_0 \\ \dfrac{\partial u}{\partial \upsilon} = 0 \end{cases}$$

where $\Omega =]0,1[\times]0,1[$ denotes picture domain with boundary $\partial\Omega$, with Neumann boundary conditions.

In this paper, we assume the system of the form

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}\left(g\left(\left|\nabla G_\sigma * u\right|\right)\nabla u\right) = A\left(t,x,u,v\right) & \text{in } Q_T \\ \dfrac{\partial v}{\partial t} - \operatorname{div}\left(h\left(\left|\nabla G_\sigma * v\right|\right)\nabla v\right) = B\left(t,x,u,v\right) & \text{in } Q_T \\ u\left(0,x\right) = u_0\ ,\ v\left(0,x\right) = v_0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \upsilon} = \dfrac{\partial v}{\partial \upsilon} = 0 & \text{on } \Sigma_T \end{cases} \tag{1.1}$$

Here $Q_T =]0,T[\times\Omega$, $\Sigma_T =]0,T[\times\partial\Omega$, $T > 0$, $\Omega =]0,1[\times]0,1[$ denotes picture domaine with boundary $\partial\Omega$, with Neumann boundary conditions, $u = u\left(t,x\right)$, $v = v\left(t,x\right)$, υ is an outward Normal to domain Ω. The diffusivities $g,\ h : \mathbb{R}^+ \to \mathbb{R}^+$ are smooth decreasing functions with $g\left(0\right) = h\left(0\right) = 1$, $\lim\limits_{t\to+\infty} g\left(t\right) = \lim\limits_{t\to+\infty} h\left(t\right) = 0$. Let $\sigma > 0$, we suppose that G_σ is a Gaussian filter

$$G_\sigma\left(x\right) = \frac{1}{\sqrt{2\pi\sigma}}\exp\left(-\frac{\left|x\right|^2}{4\sigma}\right);\ x \in \mathbb{R}^2 \quad , \quad \left|\nabla G_\sigma * \omega\right| = \left[\sum_{i=1}^{2}\left(\frac{\partial G_\sigma}{\partial x_i} * \tilde{\omega}\right)^2\right]^{\frac{1}{2}}$$

where $\tilde{\omega}$ is a linear continuous extension of ω to $\mathbb{R}^2$.

The nonlinearity $A,\ B$ are regular functions whose nonlinear structure is such that two main properties occur:

- The nonnegativity of the solution (u,v) of (1.1) is preserved with time, which is ensured by

$$A\left(t,x,0,v\right) \geq 0,\ B\left(t,x,u.0\right) \geq 0 \ \text{ for all } u,v \geq 0 \text{ and for a.e. } (t,x) \in Q_T \tag{1.2}$$

- The total mass of the components $u,\ v$ is controlled with time, which is ensured by the structure condition

$$A\left(t,x,u,v\right) \leq 0,\ A\left(t,x,u,v\right) + B\left(t,x,u,v\right) \leq 0 \text{ for all } u,v \geq 0 \text{ and for a.e. } (t,x) \in Q_T \tag{1.3}$$

This work represents a generalization of the theory developed by P. Perona and J. Malik for edge detection and image restoration, from the case of a single equation to the case of nonlinear parabolic reaction-diffusion system. This passage needs new approaches and also technical difficulties to be overcome. We will explain in detail here. We found a good idea to present our work as follows:

We began with this introduction where we describe briefly the nonlinear diffusion model proposed by Catté et al. [11] applied in image processing for restoration and which serves as background for our proposed model generalization, and some reminders of the main results obtained previously. This will highlight the contribution of our work and originality. In the second section we give the definition of the notion of solution used here. We then present the main results of this work. In the last section, we give the proof of global existence of our system, this is done in three steps: in the first we truncate the system, the latter we give suitable estimates on the approximate solutions and in the last step we show the convergence of the approximating system by using the technics introduced by Boccardo [10] and Dall'Aglio and Orsina [12], we can also see Alaa et al. [2], [3], [4], [7].

2. Statement of the result

2.1. Assumptions. Let us now introduce for A and B the hypotheses:

$$A,\ B : Q \times [0,+\infty)^2 \to \mathbb{R} \text{ are measurable} \tag{2.1}$$

$$A,\ B : Q \times [0,+\infty)^2 \to \mathbb{R} \text{ are locally Lipschitz continuous} \tag{2.2}$$

Moreover, we assume

$$|A(t,x,u,v)| + |B(t,x,u,v)| \leq \xi(|u| + |v|) \tag{2.3}$$

where $\xi : [0,+\infty) \to [0,+\infty)$ is non-decreasing, $\xi \in L^1([0,+\infty))$, and

$$A(t,x,0,0),\ B(t,x,0,0) \in L^1(Q_T) \tag{2.4}$$

First, we have to clarify in which sense we want to solve the problem (1.1):

Definition 2.1. We say that (u,v) is a *weak solution* of (1.1) if

$$\begin{cases} u, v \in L^\infty\left(0,T; L^2(\Omega)\right) \cap L^2\left(0,T; H^1(\Omega)\right) \\ A,\ B \in L^1(Q_T) \\ \dfrac{\partial u}{\partial t} - \operatorname{div}\left(g\left(|\nabla G_\sigma * u|\right)\nabla u\right) = A(t,x,u,v) \quad \text{in } D'(Q_T) \\ \dfrac{\partial v}{\partial t} - \operatorname{div}\left(h\left(|\nabla G_\sigma * v|\right)\nabla v\right) = B(t,x,u,v) \quad \text{in } D'(Q_T) \end{cases} \tag{2.5}$$

2.2. The main result. Our main result in this paper is the following existence theorem:

Theorem 2.1. *Assume that* (1.2), (1.3) *and* (2.1) – (2.4) *hold, and* $u_0,\ v_0 \in L^2(\Omega)$ *such as* $u_0,\ v_0 \geq 0$. *Then for all fixed* $T > 0$ *and* $\sigma > 0$, *there exists a weak positive solution* (u,v) *of the system* (1.1). *Moreover,* $u,\ v \in L^\infty\left(0,T; L^2(\Omega)\right) \cap L^2\left(0,T; H^1(\Omega)\right)$.

3. Proof of Theorem 2.2

3.1. Positivity of the solutions.

Lemma 3.1. *Let* (u,v) *be a solution of* (1.1), *then* $u,\ v \geq 0$.

Proof. Consider the function

$$\operatorname{sign}^-(r) = \begin{cases} -1 & \text{if } r < 0 \\ 0 & \text{if } r \geq 0 \end{cases}$$

We build a sequence of convex functions $j_\varepsilon(r)$ such as $j'_\varepsilon(r)$ is bounded and for all $r \in \mathbb{R}$, $j'_\varepsilon(r) \to \operatorname{sign}^-(r)$ when $\varepsilon \to 0$.

Let (u,v) be a solution of (1.1), we multiply both sides of the first equation by $j'_\varepsilon(u)$ and by integrating on $Q_\tau =]0,\tau[\times\Omega$ for $\tau \in [0,T[$, we obtain

$$\int_{Q_\tau} j'_\varepsilon(u)\frac{\partial u}{\partial t}\,dx\,dt + \int_{Q_\tau} g\left(|\nabla G_\sigma * u|\right)\nabla u.\nabla j'_\varepsilon(u)\,dx\,dt = \int_{Q_\tau} A(t,x,u,v) j'_\varepsilon(u)\,dx\,dt$$

We remark that $g\left(|\nabla G_\sigma * u|\right),\ h\left(|\nabla G_\sigma * v|\right) \in L^\infty(0,T;\mathcal{C}^\infty(\Omega))$ because $u,\ v \in L^\infty(0,T;L^2(\Omega))$ and $g,\ h,\ G_\sigma$ are C^∞, and we can show the existence of a C_0 depends only on $G_\sigma, \|u_0\|_{L^2(\Omega)},\ \|v_0\|_{L^2(\Omega)}$ such as

$$\|\nabla G_\sigma * u\|_{L^\infty(Q_T)} + \|\nabla G_\sigma * v\|_{L^\infty(Q_T)} \leq C_0$$

Moreover, as g, h are decreasing, then there exist constants $a = a\left(\sigma, \|u_0\|_{L^2(\Omega)}\right) > 0,\ b = b\left(\sigma, \|v_0\|_{L^2(\Omega)}\right) > 0$, such as:

$$g\left(|\nabla G_\sigma * u|\right) \geq a \quad \text{and} \quad h\left(|\nabla G_\sigma * v|\right) \geq b, \quad \text{a.e. in } (t,x) \in Q_T \tag{3.1}$$

Consequently,

$$\int_\Omega [j_\varepsilon(u)(t) - j_\varepsilon(u)(0)]dx + a\int_{Q_\tau} |\nabla u|^2 j''_\varepsilon(u)\,dt\,dx \leq \int_{Q_\tau} A(t,x,u,v) j'_\varepsilon(u)\,dx\,dt$$

Since $\int_\Omega j_\varepsilon(u)(0)dx = 0$ and $\int_{Q_t} |\nabla u|^2 j''_\varepsilon(u)\,dx\,ds \geq 0$ then we have

$$\int_\Omega j_\varepsilon(u)(t)dx \leq \int_{[u<0]} A(t,x,u,v)j'_\varepsilon(u)\,dx\,dt + \int_{[u\geq 0]} A(t,x,u,v)j'_\varepsilon(u)\,dx\,dt$$

On the set where $u \geq 0$ we have $j'_\varepsilon(u) = 0$ and $\int_{[u\geq 0]} A(t,x,u,v)j'_\varepsilon(u)\,dx\,dt = 0$; therefore

$$\int_\Omega j_\varepsilon(u)(t)dx \leq \int_{[u<0]} A(t,x,u,v)j'_\varepsilon(u)\,dx\,dt$$

When $\varepsilon \to 0$, we obtain

$$\int_\Omega (u)^-(t)dx \leq -\int_{[u\leq 0]} A(t,x,u,v)j'_\varepsilon(u)\,dx\,dt$$

Using (1.3) and the fact that $(u)^-(t) \geq 0$, we obtain $(u)^-(t) = 0$ on Ω; therefore $u \geq 0$ in Q_T.

Similarly, we multiply both sides of the second equation by $j'_\varepsilon(v)$ and by integrating on $Q_\tau =]0,\tau[\times\Omega$ for $\tau \in [0,T[$, and using (1.3), we get $v \geq 0$. See Alaa et al. [1]. □

3.2. An existence result when the nonlinearities $A,\ B$ are bounded.

Lemma 3.2. *Assume that* (1.2), (1.3) *and* (2.1) − (2.4)*, and there exists* $M \geq 0$ *such as for almost* $(t,x) \in Q_T$ *and all* $r,\ s \in \mathbb{R}$

$$|A(t,x,u,v)| + |B(t,x,u,v)| \leq M$$

then for all $u_0,\ v_0 \in L^2(\Omega)$*, the problem* (1.1) *admits a weak solution* (u,v)*. Moreover there exists a constant* C *depending on* $a,\ b,\ M,\ T,\ \|u_0\|_{L^2(\Omega)}$ *and* $\|v_0\|_{L^2(\Omega)}$*, such that*

$$\begin{aligned} \sup_{0<t<T} \|u(t)\|_{L^2(\Omega)} + \|u\|_{L^2(0,T;H^1(\Omega))} &\leq C, \\ \sup_{0<t<T} \|v(t)\|_{L^2(\Omega)} + \|v\|_{L^2(0,T;H^1(\Omega))} &\leq C. \end{aligned}$$

Proof. See Alaa et al. [1], Amann [7], Catté et al. [11] and Zhang [28]. □

3.3. Approximating scheme. We define ψ_n a truncation function by $\psi_n \in C_c^\infty(\mathbb{R})$, $0 \le \psi_n \le 1$, and

$$\psi_n(z) = \begin{cases} 1 & \text{if } |z| \le n \\ 0 & \text{if } |z| \ge n+1 \end{cases}$$

and we truncate the nonlinearities $A,\ B$ by ψ_n, for all (t,x,u,v) in $\mathbb{R}^+ \times \Omega \times \mathbb{R}^2$,

$$\begin{aligned} A_n(t,x,u,v) &= \psi_n(|u|+|v|)\,.A(t,x,u,v) \\ B_n(t,x,u,v) &= \psi_n(|u|+|v|)\,.B(t,x,u,v) \end{aligned}$$

Note that these functions $A_n,\ B_n$ enjoy the same properties as $A,\ B$, moreover they are Hölder continuous with respect to (t,x) and $|A_n| + |B_n| \le M_n$, where M_n is a constant depending only on n, and for almost $(t,x) \in Q_T$, for all $r,\ s \in \mathbb{R}$:

$$A_n(t,x,r,s) \to A(t,x,r,s) \quad \text{and} \quad B_n(t,x,r,s) \to B(t,x,r,s)$$

Let us now consider the truncated system

$$\begin{cases} \dfrac{\partial u_n}{\partial t} - \operatorname{div}\left(g\left(|\nabla G_\sigma * u_n|\right)\nabla u_n\right) = A_n(t,x,u_n,v_n) & \text{in } Q_T \\ \dfrac{\partial v_n}{\partial t} - \operatorname{div}\left(h\left(|\nabla G_\sigma * v_n|\right)\nabla v_n\right) = B_n(t,x,u_n,v_n) & \text{in } Q_T \\ u_n(0,x) = u_0^n\ ,\ v_n(0,x) = v_0^n & \text{in } \Omega \\ \dfrac{\partial u_n}{\partial \upsilon} = \dfrac{\partial v_n}{\partial \upsilon} = 0 & \text{on } \Sigma_T \end{cases} \tag{3.2}$$

Remark 3.1. Since $u_0,\ v_0 \in L^2(\Omega)$ and $|A_n| + |B_n| \le M_n$, the previous lemma 3.2 gives us that the problem (3.2) has a weak solution.

3.4. A priori estimates.

Lemma 3.3. *There exists a constant R_1 depending on $\|u_0\|_{L^1(\Omega)}$, $\|v_0\|_{L^1(\Omega)}$ such that*

$$\int_{Q_T} \left(|A_n(t,x,u_n,v_n)| + |B_n(t,x,u_n,v_n)|\right) \le R_1.$$

Proof. Considering the equations satisfied by u_n and v_n, the positivity of the solutions and the hypothesis (1.3) give us

$$\int_{Q_T} |A_n| \le \int_\Omega u_0\,dx = \|u_0\|_{L^1(\Omega)} \quad \text{and} \quad \int_{Q_T} |B_n| \le 2\|u_0\|_{L^1(\Omega)} + \|v_0\|_{L^1(\Omega)}$$

□

Lemma 3.4. *There exists a constant R_2 depending on $a,\ b$, $\|u_0\|_{L^2(\Omega)}$ and $\|v_0\|_{L^2(\Omega)}$, such that*

$$\int_{Q_T} |\nabla u_n|^2 + \int_{Q_T} |\nabla v_n|^2 \le R_2.$$

Proof. We multiply the first equation in the truncated problem (3.2) by u_n and we integrate on Q_T. We obtain

$$\frac{1}{2}\int_\Omega u_n^2(T)\,dx - \frac{1}{2}\int_\Omega u_n^2(0)\,dx + \int_{Q_T} g\,|\nabla G_\sigma * u_n|\,.\,|\nabla u_n|^2 = \int_{Q_T} A_n.u_n$$

Since, by (3.1), and by hypothesis (1.3) we have

$$\int_{Q_T} |\nabla u_n|^2 \le \frac{1}{2a}\int_\Omega u_n^2(0)\,dx$$

Multiplying by $(u_n + v_n)$ the equation satisfied by $u_n + v_n$, and integrating on Q_T yield

$$\frac{1}{2}\int_\Omega (u_n+v_n)^2(T) - \frac{1}{2}\int_\Omega (u_n+v_n)^2(0) + \int_{Q_T} h(|\nabla G_\sigma * v_n|).|\nabla(u_n+v_n)|^2 +$$

$$+\int_{Q_T} (g(|\nabla G_\sigma * u_n|) - h(|\nabla G_\sigma * v_n|)).\nabla u_n.\nabla(u_n+v_n) = \int_\Omega (A_n+B_n)(u_n+v_n)$$

By (3.1) We have

$$\begin{aligned} b\int_{Q_T} |\nabla(u_n+v_n)|^2 \quad &\leq \quad \frac{1}{2}\int_\Omega (u_0+v_0)^2 + \\ &\quad + \int_{Q_T} (g(|\nabla G_\sigma * u_n|) - h(|\nabla G_\sigma * v_n|)).\nabla u_n.\nabla(u_n+v_n) \end{aligned}$$

Using Young's inequality, we have for all $\varepsilon > 0$, such as $b - \frac{\varepsilon}{2} > 0$

$$\left(b-\frac{\varepsilon}{2}\right)\int_{Q_T} |\nabla(u_n+v_n)|^2 \leq \frac{1}{2}\int_\Omega (u_0+v_0)^2 + \frac{1}{4\varepsilon a}\int_\Omega u_n^2(0)\,dx$$

Then

$$\left(b-\frac{\varepsilon}{2}\right)\int_{Q_T} |\nabla v_n|^2 \leq \frac{1}{2}\int_\Omega (u_0+v_0)^2 + \frac{1}{4\varepsilon a}\int_\Omega u_n^2(0)\,dx + \left|b-\frac{\varepsilon}{2}\right|\int_{Q_T} |\nabla u_n|.|\nabla v_n|$$

Now, using Young's inequality another time, we end the proof of lemma. □

Lemma 3.5. *There exists a constant R_3 depending on $\|u_0\|_{L^2(\Omega)}$, $\|v_0\|_{L^2(\Omega)}$, R_2 such that*

$$\int_{Q_T} (2u_n+v_n)(|A_n(t,x,u_n,v_n)| + |B_n(t,x,u_n,v_n)|) \leq R_3.$$

Proof. Combining the equations of system (3.2), we have

$$\frac{\partial}{\partial t}(2u_n+v_n) - 2\,\mathrm{div}(g(|\nabla G_\sigma * u_n|)\nabla u_n) - \mathrm{div}(h(|\nabla G_\sigma * v_n|)\nabla v_n) = 2A_n + B_n$$

Multiplying by $(2u_n + v_n)$ and integrating on Q_T yield

$$\begin{aligned} \frac{1}{2}\int_\Omega (2u_n+v_n)^2(T) + \ & 2\int_{Q_T} g(|\nabla G_\sigma * u_n|).\nabla u_n.\nabla(2u_n+v_n) + \\ & + \int_{Q_T} h(|\nabla G_\sigma * v_n|).\nabla v_n.\nabla(2u_n+v_n) \\ & = \frac{1}{2}\int_\Omega (2u_n+v_n)^2(0) + \int_{Q_T} (2A_n+B_n)(2u_n+v_n) \end{aligned}$$

Then

$$\begin{aligned} & 2\int_{Q_T} g(|\nabla G_\sigma * u_n|).\nabla u_n.\nabla v_n + 2\int_{Q_T} h(|\nabla G_\sigma * v_n|).\nabla u_n.\nabla v_n \\ & \leq \quad \frac{1}{2}\int_\Omega (2u_0+v_0)^2 + \int_{Q_T} (2A_n+B_n)(2u_n+v_n) \end{aligned}$$

Using Young's inequality and (1.3) we conclude that

$$\int_{Q_T} |2A_n+B_n|.(2u_n+v_n) \leq \frac{1}{2}\int_\Omega (2u_0+v_0)^2 + 2\left(|\nabla u_n|^2 + |\nabla v_n|^2\right)$$

By Lemma 3.4 and (1.3), we obtain

$$\int_{Q_T} [(-A_n) + (-A_n - B_n)] \, . \, (2u_n + v_n) \leq \frac{1}{2} \int_{\Omega} (2u_0 + v_0)^2 + 4R_2$$

We note that $|B_n| \leq |A_n| + |A_n + B_n|$, which gives us the result. □

Lemma 3.6. *(u_n, v_n) is relatively compact in $L^2(Q_T) \times L^2(Q_T)$.*

Proof. Since $\dfrac{\partial u_n}{\partial t} = \operatorname{div}\left(g\left(|\nabla G_\sigma * u_n|\right) \nabla u_n\right) + A_n\left(t, x, u_n, v_n\right)$ and $\dfrac{\partial v_n}{\partial t} = \operatorname{div}\left(h\left(|\nabla G_\sigma * v_n|\right) \nabla v_n\right) + B_n\left(t, x, u_n, v_n\right)$ are bounded in $L^1\left(0, T; (H^1(\Omega))'\right)$. Since $u_n,\ v_n$ are also bounded in $L^2(0,T;H^1(\Omega))$ and that the injection of $H^1(\Omega) \times H^1(\Omega)$ in $L^2(\Omega) \times L^2(\Omega)$ is compact, it follows that (u_n, v_n) is relatively compact in $L^2(Q_T) \times L^2(Q_T)$, see Simon [23]. □

3.5. Convergence. Our objective is to show that (u_n, v_n) converges to some (u, v) solution of the problem (1.1). According to Lemma 3.6, the sequence (u_n, v_n) is relatively compact in $L^2(Q_T) \times L^2(Q_T)$, so we can extract a subsequence still denoted (u_n, v_n) such that

- $u_n \to u,\ v_n \to v$ strongly in $L^2(Q_T)$ and a.e. in Q_T,
- $\nabla G_\sigma * u_n \to \nabla G_\sigma * u,\ \nabla G_\sigma * v_n \to \nabla G_\sigma * v$ strongly in $L^2(Q_T)$ and a.e. in Q_T,
- $g(|\nabla G_\sigma * u_n|) \to g(|\nabla G_\sigma * u|),\ h(|\nabla G_\sigma * v_n|) \to h(|\nabla G_\sigma * v|)$ strongly in $L^2(Q_T)$,
- $A_n(t, x, u_n, v_n) \to A(t, x, u_n, v_n),\ B_n(t, x, u_n, v_n) \to B(t, x, u_n, v_n)$ for a.e. in Q_T.

This is not sufficient to ensure that (u, v) is a weak solution of (1.1). In fact, we have to prove that the previous convergences are in $L^1(Q_T)$. In view of the Vitali theorem, to show that $A_n(t, x, u_n, v_n) \to A(t, x, u, v)$ and $B_n(t, x, u_n, v_n) \to B(t, x, u, v)$ in $L^1(Q_T)$, is equivalent to proving that $\left(A_n(t, x, u_n, v_n)\right)_n$ and $\left(B_n(t, x, u_n, v_n)\right)_n$ are equi-integrable in $L^1(Q_T)$.

Lemma 3.7. *$\left(A_n(t, x, u_n, v_n)\right)_n$ and $\left(B_n(t, x, u_n, v_n)\right)_n$ are equi-integrable in $L^1(Q_T)$.*

Proof. Let E be a measurable subset of Q_T, we have

$$\begin{aligned}
\int_E |A_n(t, x, u_n, v_n)| &= \int\limits_{E \cap \{u_n > k\}} |A_n| + \int\limits_{E \cap \{u_n \leq k, v_n > k\}} |A_n| + \int\limits_{E \cap \{u_n \leq k, v_n \leq k\}} |A_n| \\
&= I_1 + I_2 + I_3
\end{aligned}$$

Thanks to Lemma 3.5, we obtain $\forall \varepsilon > 0,\ \exists k_0$, such that, if $k \geq k_0$ then for all n

$$I_1 \leq \frac{1}{k} \int\limits_{\{u_n > k\}} k. \, |A_n| \leq \frac{1}{k} \int\limits_{Q_T} (2u_n + v_n) \, . \, |A_n| \leq \frac{1}{k} R_3 \leq \frac{\varepsilon}{3}$$

$$I_2 \leq \frac{1}{k} \int\limits_{E \cap \{u_n \leq k, v_n > k\}} k. \, |A_n| \leq \frac{1}{k} \int\limits_{Q_T} (2u_n + v_n) \, . \, |A_n| \leq \frac{1}{k} R_3 \leq \frac{\varepsilon}{3}$$

Now, using hypothesis (2.3), we write

$$I_3 \leq \int\limits_{E \cap \{u_n \leq k, v_n \leq k\}} \xi\left(|u_n| + |v_n|\right) \leq \xi(k, k) \, . \, |E|$$

Choose $|E| \leq \frac{\varepsilon}{3} \frac{1}{\xi(k,k)}$, we obtain

$$\int_E |A_n(t,x,u_n,v_n)| \leq \varepsilon$$

Similarly, we get

$$\int_E |B_n(t,x,u_n,v_n)| \leq \varepsilon$$

We obtain the required result. □

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