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Global existence of weak solutions for parabolic triangular reaction diffusion systems applied to a climate model

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Abstract. Over the years, reaction-diffusion systems have attracted the attention of a great number of investigators and were successfully developed on the theoretical backgrounds. Not only it has been studied in biological and chemical fields, some investigations range as far as economics, semiconductor physics, and star formation. Recently particular interests have been on the impact of environmental changes, such as climate. This work is devoted to the existence of weak solutions for $m \times m$ reaction-diffusion systems arises from an energy balance climate model. We consider a time evolution model for the climate obtained via energy balance. This type of climate model, independently introduced in 1987 by V. Jentsch [29], has a spatial global nature and involves a relatively long-time scale. Our study concerns the global existence of periodic solutions of the nonlinear parabolic problem. The originality of this study persists in the fact that the non-linearities of our system have critical growth with respect to the gradient of solutions. For this reason new techniques will used to show the global existence. This is our main goal in this article.

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1. Introduction

Energy balance models are at the bottom end of the hierarchy of climate models. Since the climate system is probably the most complex system physical and mathematical theories are tried on, this hierarchy consists of a big number of models of highly different complexity. General circulation models at the top end are based on most of our knowledge about physical and chemical processes in the atmosphere, the oceans, and their interface we can describe in mathematical equations. Of course, using the increasing power of modern computational facilities, there is no way to do more realistic simulations and climate predictions than by using these models. The best models show how a process works and then predict what may follow. That's what we try to describe in this article. Various methods have been proposed for the study of the existence and qualitative property of solutions. Most of the work in the earlier literature is devoted to quasi-linear elliptic systems under either Dirichlet or Newmann boundary conditions [7], [8], [16], [30], and [25], all these works examine the classical solutions. In recent years attention has been given to weak solutions of elliptic systems under linear boundary conditions, and different methods for the existence problem have been used [1], [2], [3], [5], [6], [10], [11], [34], [14], [21], [24], [22], [23], etc.

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In this paper, we prove the existence of solutions for the reaction-diffusion systems of the form

$$\left\{\begin{array}{lll} \frac{\partial u_i}{\partial t} - d_i \Delta u_i = f_i\left(t, x, u, \nabla u\right) & \text{in } Q_T & \\ u_i\left(0, x\right) = u_{i,0} & \text{in } \Omega & , \quad \text{for } 1 \leq i \leq m \\ u_i = 0 & \text{on } \Sigma_T & \end{array}\right. \tag{1}$$

where $u = (u_1, \ldots, u_m)$, $\nabla u = (\nabla u_1, \ldots, \nabla u_m)$, $f = (f_1, \ldots, f_m)$, $m \geq 2$ and Ω is an open bounded subset of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $Q_T =]0, T[\times \Omega$, $\Sigma_T =]0, T[\times \partial\Omega$, $T > 0$, $-\Delta$ denotes the Laplacian operator on $L^1(\Omega)$ with Dirichlet boundary conditions, d_i, $1 \leq i \leq m$, are positive constants, and the non-linearities f_i, $1 \leq i \leq m$, have critical growth with respect to $|\nabla u|$. Moreover, these following main properties hold:

• The positivity of the solution is preserved with time, which is ensured by

$$\left\{\begin{array}{l} f_i\left(\hat{u}_i\right) \geq 0, \text{ where} \\ \hat{u}_i = (t, x, u_1, \ldots, u_{i-1}, 0, u_{i+1}, \ldots, u_m, p_1, \ldots, p_{i-1}, 0, p_{i+1}, \ldots, p_m), \\ \text{for all } 1 \leq i \leq m, \ (u, p) \in \left(\mathbb{R}^+\right)^m \times \mathbb{R}^{Nm} \text{ and for a.e. } (t, x) \in Q_T \\ u_{i,0} \geq 0, \text{ for all } 1 \leq i \leq m \end{array}\right. \tag{2}$$

• The total mass of the components $u_1, \ldots, u_m$ is controlled with time, which is ensured by

$$\left\{\begin{array}{l} \sum\limits_{1 \leq i \leq r} f_i\left(t, x, u, p\right) \leq 0, \text{ for all } 1 \leq r \leq m, \\ \text{for all } (u, p) \in \left(\mathbb{R}^+\right)^m \times \mathbb{R}^{Nm} \text{ and a.e. } (t, x) \in Q_T \end{array}\right. \tag{3}$$

Let us know that if the non-linearities f do not dependent on the gradient (system 1) is semi-linear), the existence of global positive solutions have been obtained by Hollis [17], Hollis and Morgan [18] and Martin and Pierre [21]. One can see that in all of these works, the triangular structure, namely hypotheses (3) plays an important role in the study of semi-linear systems. Indeed, if (3) does not hold, Pierre and Schmitt [35] proved blow up in finite time of the solutions to some semi-linear reaction-diffusion systems.

Where $f = (f_1, f_2)$ depends on the gradient, Alaa and Mounir [4] solved the problem where the triangular structure is satisfied and the growth of f_1 and f_2 with respect to $|\nabla u_1|$, $|\nabla u_2|$ is sub-quadratic.

$$\left\{\begin{array}{l} \text{there exists } 1 \leq p < 2, \ C : [0, \infty)^2 \to [0, \infty) \text{ non-decreasing such that} \\ |f_1| + |f_2| \leq C\left(|u_1|, |u_2|\right)\left(1 + |\nabla u_1|^p + |\nabla u_2|^p\right) \end{array}\right.$$

About the critical growth with respect to the gradient ($p = 2$), we recall that for the case of a single equation ($d_1 = d_2$ and $f_1 = f_2$), existence results have been proved for the elliptic case in [2] and [10]. The corresponding parabolic equations have also been studied by many authors; see for instance [2], [12], [13], [15], and [20].

This work represents a generalization to the parabolic case study we did in the elliptic case (see [7]) for these systems of arbitrary order. This passage in parabolic case, needs new approaches and also technical difficulties to be overcome. We will explain in detail here.

We found a good idea to present our work as follows: we start initially with an introduction that presents the state of the art of the area studied and some recall the main results obtained previously. This will highlight the contribution of our work and its originality. In the second section we give the definition of the notion of solution used here. We then present the main results of this work. In the last section, we give the proof of global existence of our reaction diffusion system. This is done in three steps: in the first we truncate the system, the latter we give suitable estimates on the approximate solutions and in the last step we show the convergence of the approximating system by using the technics introduced by Boccardo et al. [12] and Dall'Aglio and Orsina [15].

2. Statement of the result

2.1. Assumptions. First, we have to clarify in which sense we want to solved problem (1) .

Definition 2.1. We say that $(u_1, \dots, u_m)$ is a solution of (1) if, for all $1 \le i \le m$

$$\begin{cases} u_i \in C\left([0,T]; L^1(\Omega)\right) \cap L^1\left(0,T; W_0^{1,1}(\Omega)\right) \\ f_i(t,x,u,\nabla u) \in L^1(Q_T) \\ u_i(t) = S_{d_i}(t)\, u_0 + \int_0^t S_{d_i}(t-s)\, f_i(.,s,u(s),\nabla u(s))\, ds, \ \forall t \ge 0 \end{cases} \tag{4}$$

where S_{d_i}, $1 \le i \le m$, denote the semi-groups in $L^1(\Omega)$ generated by $-d_i\Delta$ with Dirichlet boundary conditions.

Let us, now introduce for f the hypotheses, for all $1 \le i \le m$

$$f_i :]0,T[\, \times \Omega \times \mathbb{R}^m \times \mathbb{R}^{mN} \to \mathbb{R} \text{ are measurable} \tag{5}$$

$$f_i : \mathbb{R}^m \times \mathbb{R}^{mN} \to \mathbb{R} \text{ are locally Lipschitz continuous} \tag{6}$$

namely

$$\sum_{1\le i\le m} |f_i(x,t,u,p) - f_i(x,t,\hat{u},\hat{p})| \le K(r) \left(\sum_{1\le i\le m} |u_i - \hat{u}_i| + \sum_{1\le i\le m} \|p_i - \hat{p}_i\| \right)$$

for a.e. (t,x) and for all $0 \le |u_i|, |\hat{u}_i|, \|p_i\|, \|\hat{p}_i\| \le r$.

$$|f_1(t,x,u,\nabla u)| \le C_1(|u_1|) \left(F_1(t,x) + \|\nabla u_1\|^2 + \sum_{2\le j\le m} \|\nabla u_j\|^{\alpha_j} \right) \tag{7}$$

where $C_1 : [0,+\infty) \to [0,+\infty)$ is non-decreasing, $F_1 \in L^1(Q_T)$ and $1 \le \alpha_j < 2$.

$$|f_i(t,x,u,\nabla u)| \le C_i \left(\sum_{j=1}^{i} |u_j| \right) \left(F_i(t,x) + \sum_{1\le j\le m} \|\nabla u_j\|^2 \right), \ 2 \le i \le m \tag{8}$$

where $C_i : [0,+\infty) \to [0,+\infty)$ is non-decreasing, $F_i \in L^1(Q_T)$ for all $2 \le i \le m$.
Example. A typical example where the result of this paper can be applied is

$$\begin{cases} \dfrac{\partial u_i}{\partial t} - d_i \Delta u_i = \sum\limits_{1\le j\le i} a_{ij} \dfrac{u_j}{\sum\limits_{1\le k\le m} u_k} |\nabla u_j|^2 + f_i(t,x) & \text{in } Q_T \\ u_i(0,x) = u_{i,0} & \text{in } \Omega \\ u_i = 0 & \text{on } \Sigma_T \end{cases}, \text{ for } 1 \le i \le m$$

The main result

Theorem 2.1. *Assume that* (2), (3) *and* $(5)-(8)$ *hold. If* $u_{i,0} \in L^2(\Omega)$, *for all* $1 \leq i \leq m$, *then there exists a positive global solution* $u = (u_1, \ldots, u_m)$ *of system* (1). *Moreover,* $u_1, \ldots, u_m \in L^2\left(0, T; H_0^1(\Omega)\right)$.

Before giving the proof of this theorem, let us define the following functions. Given a real positive number k, we set

$$T_k(s) = \max\{-k, \min(k, s)\} \quad \text{and} \quad G_k(s) = s - T_k(s)$$

We remark that for $0 \leq s \leq k$, $T_k(s) = s$ and $T_k(s) = k$ for $s > k$.

Proof of Theorem 2.1

Approximating scheme. For every function h defined from $\mathbb{R}^+ \times \Omega \times \mathbb{R}^m \times \mathbb{R}^{mN}$ into $\mathbb{R}$, we associate $\hat{\varphi} = \hat{\varphi}(t, x, u, p)$ such that

$$\hat{\varphi} = \begin{cases} \varphi(t, x, u_1, \ldots, u_m, p_1, \ldots, p_m) & \text{if } u_i \geq 0,\ 1 \leq i \leq m \\ \varphi(t, x, u_1, \ldots, u_{i-1}, 0, u_{i+1}, \ldots, u_m, p_1, \ldots, p_m) & \text{if } u_i \leq 0 \text{ and } u_j \geq 0,\ j \neq i \\ \varphi(t, x, 0, \ldots, 0, p_1, \ldots, p_m) & \text{if } u_i \leq 0,\ 1 \leq i \leq m \end{cases}$$

and consider the system

$$\begin{cases} \dfrac{\partial u_i}{\partial t} - d_i \Delta u_i = \hat{f}_i(t, x, u, \nabla u) & \text{in }]0, +\infty[\, \times \Omega \\ u_i(0, x) = u_{i,0} & \text{in } \Omega \\ u_i = 0 & \text{on }]0, +\infty[\, \times \partial\Omega \end{cases}, \text{ for } 1 \leq i \leq m. \tag{9}$$

It is obviously seen, by the structure of $\hat{f}_i$, $1 \leq i \leq m$, that systems (1) and (9) are equivalent on the set where $u_i \geq 0$, $1 \leq i \leq m$. Consequently, to prove Theorem 2.1, we have to show that problem (9) has a weak solution which is positive.

To this end, we define ψ_n a truncation function by $\psi_n \in C_c^\infty(\mathbb{R})$, $0 \leq \psi_n \leq 1$, and

$$\psi_n(z) = \begin{cases} 1 & \text{if } |z| \leq n \\ 0 & \text{if } |z| \geq n+1 \end{cases}$$

and the mollification with respect to (t, x) is defined as follows. Let $\rho \in C_c^\infty\left(\mathbb{R} \times \mathbb{R}^N\right)$ such that

$$\text{supp}\rho \subset B(0, 1)\ ,\ \int \rho = 1\ ,\ \rho \geq 0 \text{ on } \mathbb{R} \times \mathbb{R}^N$$

and $\rho_n(y) = n^N \rho(ny)$. One can see that

$$\rho_n \in C_c^\infty\left(\mathbb{R} \times \mathbb{R}^N\right),\ \text{supp}\rho_n \subset B\left(0, \frac{1}{n}\right),\ \int \rho_n = 1,\ \rho_n \geq 0 \text{ on } \mathbb{R} \times \mathbb{R}^N.$$

We also consider non-decreasing sequences $u_{i,0}^n \in C_c^\infty(\Omega)$ such that

$$u_{i,0}^n \to u_{i,0} \text{ in } L^2(\Omega),\ 1 \leq i \leq m$$

and define for all (t, x, u, p) in $\mathbb{R}^+ \times \Omega \times \mathbb{R}^m \times \mathbb{R}^{mN}$ and $1 \leq i \leq m$,

$$f_{i,n}(t, x, u, p) = \left[\psi_n\left(\sum_{1 \leq j \leq m} (|u_j| + \|p_j\|)\right) f_i(t, x, u, p)\right] * \rho_n(t, x).$$

Note that these functions enjoy the same properties as f_i, $1 \leq i \leq m$, moreover they are Hölder continuous with respect to (t, x) and $|f_{i,n}| \leq M_n$, $1 \leq i \leq m$, where

M_n is a constant depending only on n (these estimates can be derived from (4), the properties of the convolution product, and the fact that $\int \rho_n = 1$).

Let us now consider the truncated system

$$\begin{cases} \dfrac{\partial u_{i,n}}{\partial t} - d_i \Delta u_{i,n} = f_{i,n}\left(x,t,u_n,\nabla u_n\right) & \text{in } Q_T \\ u_{i,n}\left(0,x\right) = u_{i,0}^n & \text{in } \Omega \\ u_{i,n} = 0 & \text{on } \Sigma_T \end{cases}, \text{ for } 1 \leq i \leq m. \tag{10}$$

It is well known that problem (10) has a global classical solution (see [19], theorem 7.1, p. 591) for the existence and ([20], Corollary of Theorem 4.9, p. 341) for the regularity of solutions. It remains to show the positivity of the solutions.

Lemma 2.2. *Let $u_n = (u_{1,n},\ldots,u_{m,n})$ be a classical solution of* (10) *and suppose that $u_{1,0}^n,\ldots,u_{m,0}^n \geq 0$. Then $u_{1,n},\ldots,u_{m,n} \geq 0$.*

Proof. See [4], Lemma 1, p 537. □

2.2. A priori estimates. The hypotheses (2) and (3) allowed the following lemma.

Lemma 2.3. *There exists a constant M depending on $\sum\limits_{1\leq j\leq m} \|u_{j,0}\|_{L^1(\Omega)}$ such that*

$$\int_\Omega \left(\sum_{1\leq j\leq m} u_{j,n}\left(t\right) \right) \leq M, \text{ for all } t \in \left[0,T\right]. \tag{11}$$

Proof. We consider the equation satisfied by $\sum\limits_{1\leq j\leq m} u_{j,n}$

$$\frac{\partial}{\partial t}\left(\sum_{1\leq j\leq m} u_{j,n} \right) - \sum_{1\leq j\leq m} d_j \Delta u_{j,n} = \sum_{1\leq j\leq m} f_{j,n}$$

Hypothesis (3) implies

$$\frac{\partial}{\partial t}\left(\sum_{1\leq j\leq m} u_{j,n} \right) \leq \sum_{1\leq j\leq m} d_j \Delta u_{j,n}$$

Since $u_{j,n} \geq 0$ for all $1 \leq j \leq m$ and the operator Δ is dissipative on $L^1\left(\Omega\right)$, then

$$\int_\Omega \Delta u_{j,n} \leq 0 \text{ for all } 1 \leq j \leq m \tag{12}$$

Hence

$$\int_\Omega \frac{\partial}{\partial t}\left(\sum_{1\leq j\leq m} u_{j,n} \right) \leq 0$$

Integrating this inequality on $\left[0,t\right]$, for all $0 < t < T$, yields

$$\int_\Omega \left(\sum_{1\leq j\leq m} u_{j,n}\left(t\right) \right) \leq \int_\Omega \left(\sum_{1\leq j\leq m} u_{j,0} \right) = \sum_{1\leq j\leq m} \|u_{j,0}\|_{L^1(\Omega)}$$

This ends the proof of the lemma. □

Lemma 2.4. *There exists a constant R_1 depending on $\sum_{1\le j\le m} \|u_{j,0}\|_{L^1(\Omega)}$, such that*

$$\sum_{1\le j\le m} \int_\Omega |f_{j,n}(x,t,u_n,\nabla u_n)| \le R_1.$$

Proof. Considering the equations satisfied by $u_{i,n}$, $1\le i\le m$, we can write

$$-f_{i,n} = -\frac{\partial u_{i,n}}{\partial t} + d_i \Delta u_{i,n}$$

Integrating on Q_T and using (12), the positivity of the solutions yield

$$-\int_{Q_T} f_{i,n} \le \int_\Omega u_{i,0}dx \quad \text{for all } 1\le i\le m$$

Hence by hypothesis (3)

$$\int_{Q_T} |f_{1,n}| \le \int_\Omega u_{1,0}dx = \|u_{1,0}\|_{L^1(\Omega)} \tag{13}$$

Similarly, we get by hypothesis (3) for all $2\le j\le m$

$$\int_{Q_T}\left|\sum_{1\le i\le j} f_{i,n}\right| = \int_{Q_T}\left(-\sum_{1\le i\le j} f_{i,n}\right) \le \sum_{1\le i\le j}\int_\Omega u_{i,0}dx = \sum_{1\le i\le j}\|u_{i,0}\|_{L^1(\Omega)}$$

Then

$$\int_{Q_T} |f_{j,n}| \le \sum_{1\le i\le j}(j-i+1)\|u_{i,0}\|_{L^1(\Omega)}.$$

□

Lemma 2.5. (*i*) *There exists a constant R_2 depending on k and $\sum_{1\le i\le m}\|u_{i,0}\|_{L^1(\Omega)}$, such that for all $1\le j\le m$,*

$$\int_{Q_T} |\nabla T_k(u_{j,n})|^2 \le R_2.$$

(*ii*) *There exists a constant R_3 depending on $\sum_{1\le j\le r}\|u_{j,0}\|_{L^2(\Omega)}$ such that for all $2\le r\le m$,*

$$\int_{Q_T}\left|\nabla T_k\left(\sum_{1\le j\le r} u_{j,n}\right)\right|^2 \le R_3$$

Proof. (*i*) We multiply the j^{th} equation in (10) by $T_k(u_{j,n})$ and we integrate on Q_T, we obtain

$$\int_\Omega\int_0^T \frac{\partial}{\partial t}u_{j,n}T_k(u_{j,n}) + d_j\int_{Q_T}|\nabla T_k(u_{j,n})|^2 \le k\int_{Q_T}|f_{j,n}|$$

Then

$$\int_\Omega S_k(u_{j,n}(T)) + d_j\int_{Q_T}|\nabla T_k(u_{j,n})|^2 \le k\int_{Q_T}|f_{j,n}| + \int_\Omega S_k(u_{j,n}(0))$$

where $S_k(r) = \int_0^r T_k(s)\,ds$. Since

$$S_k(u_{j,n}(T)) \ge 0 \text{ and for all } r\ge 0,\ \ |S_k(r)| \le \frac{k^2}{2} + k(r-k)^+$$

by using the result of Lemma 2.4, we have

$$\begin{aligned} d_j \int_{Q_T} \left|\nabla T_k\left(u_{j,n}\right)\right|^2 &\leq k \int_{Q_T} \left|f_{j,n}\right| + \int_\Omega \left(\frac{k^2}{2} + k\left(u_{j,n}(0) - k\right)^+\right) \\ &\leq kR_1 + \frac{k^2}{2}|\Omega| + \int_\Omega \left(u_{j,n}(0) - k\right)^+ \leq C\left(k, \left\|u_{j,n}(0)\right\|\right) \end{aligned}$$

(ii) We consider the equation satisfied by $\sum_{1\leq j\leq r} u_{j,n}$, $2 \leq r \leq m$, and we use hypothesis (3), we get

$$\frac{\partial}{\partial t}\left(\sum_{1\leq j\leq r} u_{j,n}\right) - d_r\Delta\left(\sum_{1\leq j\leq r} u_{j,n}\right) + \left(\sum_{1\leq j\leq r-1} \left(d_r - d_j\right)\Delta u_{j,n}\right) = \sum_{1\leq j\leq r} f_{j,n}$$

We denote $U_{r,n} = \sum_{1\leq j\leq r} u_{j,n}$, we obtain

$$\frac{\partial U_{r,n}}{\partial t} - d_r\Delta U_{r,n} + \sum_{1\leq j\leq r-1} \left(d_r - d_j\right)\Delta u_{j,n} \leq 0$$

Now, we multiply by $T_k\left(U_{r,n}\right)$ and integrate on Q_T,we obtain

$$\int_{Q_T} T_k\left(U_{r,n}\right)\frac{\partial U_{r,n}}{\partial t} + d_r\int_{Q_T}\left|\nabla T_k\left(U_{r,n}\right)\right|^2 + \sum_{1\leq j\leq r-1}\left(d_r - d_j\right)\int_{Q_T}\nabla T_k\left(U_{r,n}\right)\nabla T_k\left(u_{j,n}\right) \leq 0$$

Then

$$d_r\int_{Q_T}\left|\nabla T_k\left(U_{r,n}\right)\right|^2 + \sum_{1\leq j\leq r-1}\left(d_r - d_j\right)\int_{Q_T}\nabla T_k\left(U_{r,n}\right)\nabla T_k\left(u_{j,n}\right) \leq \frac{1}{2}\int_\Omega U_{r,n}^2(0)$$

Which gives us

$$d_r\int_{Q_T}\left|T_k\left(U_{r,n}\right)\right|^2 \leq \frac{1}{2}\int_\Omega U_{r,n}^2(0) + \sum_{1\leq j\leq r-1}\left(\left|d_r - d_j\right|\int_{Q_T}\left|\nabla T_k\left(U_{r,n}\right)\right|.\left|\nabla T_k\left(u_{j,n}\right)\right|\right)$$

Using Young's inequality, we have

$$d_r\int_{Q_T}\left|T_k\left(U_{r,n}\right)\right|^2 \leq \frac{1}{2}\int_\Omega U_{r,n}^2(0) + \sum_{1\leq j\leq r-1}\left(\frac{1}{2}\left|d_r - d_j\right|\int_{Q_T}\left[\left|T_k\left(U_{r,n}\right)\right|^2 + \left|\nabla T_k\left(u_{j,n}\right)\right|^2\right]\right)$$

Then

$$\left(d_r - \frac{1}{2}\sum_{1\leq j\leq r-1}\left|d_r - d_j\right|\right)\int_{Q_T}\left|\nabla T_k\left(U_{r,n}\right)\right|^2 \leq$$

$$\frac{1}{2}\int_\Omega U_{r,n}^2(0) + \sum_{1\leq j\leq r-1}\left(\frac{1}{2}\left|d_r - d_j\right|\int_{Q_T}\left|\nabla T_k\left(u_{j,n}\right)\right|^2\right)$$

By (i), we have

$$\left(d_r - \varepsilon_k\right)\int_{Q_T}\left|\nabla T_r\left(U_{k,n}\right)\right|^2 \leq \frac{1}{2}\int_\Omega U_{r,n}^2(0) + \frac{1}{2}R_2\sum_{2\leq j\leq r-1}\left|d_r - d_j\right|$$

□

Lemma 2.6. *There exists a constant R_4 depending on $\sum_{1\le j\le m} \|u_{j,0}\|_{L^2(\Omega)}$ and $d_1,\ldots,d_m$ such that*

$$\int_{Q_T} |f_{j,n}(x,t,u_n,\nabla u_n)| \left(\sum_{1\le r\le m} (m-r+1)\, u_{k,n} \right) \le R_4, \text{ for all } 1\le j\le m.$$

Proof. Set, for all $2\le r\le m$

$$R_{r,n} = -\sum_{1\le j\le r} f_{j,n}$$

and

$$\theta_n = \sum_{1\le r\le m} (m-r+1)\, u_{r,n} \quad , \quad z_n = \sum_{1\le r\le m} (m-r+1)\, d_r u_{r,n}$$

We have by hypothesis (3)

$$R_{r,n} \ge 0, \text{ for all } 2\le r\le m.$$

Combining the equations of system (10), we have

$$\frac{\partial}{\partial t}\theta_n - \Delta z_n + |f_{1,n}| + \sum_{2\le r\le m} R_{r,n} = 0$$

Multiplying by θ_n and integrating on Q_T yield

$$\frac{1}{2}\int_\Omega \theta_n^2(T) + \int_{Q_T} \nabla z_n \nabla\theta_n + \int_{Q_T} \theta_n |f_{1,n}| + \sum_{2\le r\le m} \int_{Q_T} \theta_n R_{r,n} = \frac{1}{2}\int_\Omega \theta_n^2(0)$$

Then

$$\int_{Q_T} \nabla z_n \nabla\theta_n + \int_{Q_T} \theta_n |f_{1,n}| + \sum_{2\le r\le m} \int_{Q_T} \theta_n R_{r,n} \le \frac{1}{2}\int_\Omega \theta_n^2(0)$$

Hence

$$\int_{Q_T} \theta_n |f_{1,n}| + \sum_{2\le r\le m} \int_{Q_T} \theta_n R_{r,n} \le \frac{1}{2}\int_\Omega \theta_n^2(0) + \int_{Q_T} |\nabla z_n| . |\nabla\theta_n|$$

Using Young's inequality, we conclude that

$$\int_{Q_T} \theta_n |f_{1,n}| + \sum_{2\le r\le m} \int_{Q_T} \theta_n R_{r,n} \le \frac{1}{2}\int_\Omega \theta_n^2(0) + \frac{1}{2}\int_{Q_T} \left[|\nabla z_n|^2 + |\nabla\theta_n|^2 \right] \le C$$

Then

$$\int_{Q_T} \theta_n |f_{1,n}| \le C \text{ and } \sum_{2\le r\le m} \int_{Q_T} \theta_n \left| \sum_{1\le j\le r} f_{j,n} \right| \le C \text{ for all } 2\le r\le m \tag{14}$$

We have by (14)

$$\int_{Q_T} \theta_n |f_{2,n}| \le \int_{Q_T} \theta_n |f_{1,n}+f_{2,n}| + \int_{Q_T} \theta_n |f_{1,n}| \le \hat{C}$$

and for all $2\le k\le m$,we have

$$\int_{Q_T} \theta_n |f_{k,n}| \le \int_{Q_T} \theta_n \left| \sum_{1\le j\le k} f_{j,n} \right| + \int_{Q_T} \theta_n \left| \sum_{1\le j\le k-1} f_{j,n} \right| \le \hat{C}.$$

Which gives us the result. ∎

2.3. Convergence. Our objective is to show that $u_n = (u_{1,n}, \dots, u_{m,n})$ converges to some $u = (u_1, \dots, u_m)$ solution of the problem (4). The sequences $u_{1,0}^n, \dots, u_{m,0}^n$ are uniformly bounded in $L^1(\Omega)$ (since they converge in $L^2(\Omega)$), and by Lemma 2.4, the non-linearities $f_{1,n}, \dots, f_{m,n}$ are uniformly bounded in $L^1(Q_T)$. Then according to a result in [9] the applications

$$\left(u_{i,0}^n, f_{i,n}\right) \to u_{i,n}\ ,\ 1 \le i \le m$$

are compact from $L^1(\Omega) \times L^1(Q_T)$ into $L^1\left(0, T; W_0^{1,1}(\Omega)\right)$.

Therefore, we can extract a subsequence, still denoted by $(u_{1,n}, \dots, u_{m,n})$, such that

$$\begin{array}{ll} (u_{1,n}, \dots, u_{m,n}) \to (u_1, \dots, u_m) & \text{in } L^1\left(0, T; W_0^{1,1}(\Omega)\right) \\ (u_{1,n}, \dots, u_{m,n}) \to (u_1, \dots, u_m) & \text{a.e. in } Q_T \\ (\nabla u_{1,n}, \dots, \nabla u_{m,n}) \to (\nabla u_1, \dots, \nabla u_m) & \text{a.e. in } Q_T \end{array}$$

Since $f_{1,n}, \dots, f_{m,n}$ are continuous, we have

$$f_{i,n}(t, x, u_n, \nabla u_n) \to f_i(t, x, u, \nabla u) \quad \text{a.e. in } Q_T\ ,\ 1 \le i \le m$$

This is not sufficient to ensure that $(u_1, \dots, u_m)$ is a solution of (4). In fact, we have to prove that the previous convergence are in $L^1(Q_T)$. In view of the Vitali theorem, to show that $f_{i,n}(t, x, u_n, \nabla u_n)$, $1 \le i \le m$, converges to $f_i(t, x, u, \nabla u)$ in $L^1(Q_T)$, is equivalent to proving that $f_{i,n}(t, x, u_n, \nabla u_n)$, $1 \le i \le m$ are equi-integrable in $L^1(Q_T)$.

Lemma 2.7. *$f_{i,n}(t, x, u_n, \nabla u_n)$, for all $1 \le i \le m$, are equi-integrable in $L^1(Q_T)$.*

The proof of this lemma requires the following result based on some properties of two time-regularization denoted by u_γ and u_σ $(\gamma,\ \sigma > 0)$ which we define for a function $u \in L^2\left(0, T; H_0^1(\Omega)\right)$ such that $u(0) = u_0 \in L^2(\Omega)$ (for more details see [4]). In the following we will denote by $\omega(\varepsilon)$ a quantity that tends to zero as ε tends to zero, and $\omega^\sigma(\varepsilon)$ a quantity that tends to zero for every fixed σ as ε tends to zero.

Lemma 2.8. *Let (u_n) be a sequence in $L^2\left(0, T; H_0^1(\Omega)\right) \cap C([0, T])$ such that $u_n(0) = u_0^n \in L^2(\Omega)$ and $(u_n)_t = \rho_{1,n} + \rho_{2,n}$ with $\rho_{1,n} \in L^2\left(0, T; H^{-1}(\Omega)\right)$ and $\rho_{2,n} \in L^1(Q_T)$. Moreover assume that u_n converges to u in $L^2(Q_T)$, and u_0^n converges to $u(0)$ in $L^2(\Omega)$.*

Let Ψ be a function in $C^1([0, T])$ such that $\Psi \ge 0$, $\Psi' \le 0$, $\Psi(T) = 0$. Let φ be a Lipschitz increasing function in $C^0(\mathbb{R})$ such that $\varphi(0) = 0$.

Then for all $k,\ \gamma > 0$,

$$\begin{aligned} &\left\langle \rho_{1n}, \Psi\varphi\left(T_k(u_n) - T_k(u_m)_\gamma\right)\right\rangle + \int_{Q_T} \rho_{2n}\Psi\varphi\left(T_k(u_n) - T_k(u_m)_\gamma\right) \\ &\quad \ge \ \omega^{\gamma,n}\left(\frac{1}{m}\right) + \omega^\gamma\left(\frac{1}{n}\right) + \int_\Omega \Psi(0)\,\Phi\left(T_k(u) - T_k(u)_\gamma\right)(0)\,dx \\ &\qquad - \int_\Omega G_k(u)(0)\,\Psi(0)\,\varphi\left(T_k(u) - T_k(u)_\gamma\right)(0)\,dx \end{aligned}$$

where $\Phi(t) = \int_0^t \varphi(s)\,ds$ and $G_k(s) = s - T_k(s)$.

Proof. See [4], Lemma 7, p 544. □

Lemma 2.9. *Suppose that $u_{j,n}$, u_j, $1 \le j \le m$, are as above.*

i) If

$$|f_{1,n}| \leq C_1\left(|u_{1,n}|\right)\left(F_1\left(t,x\right)+\left|\nabla u_{1,n}\right|^2+\sum_{2\leq j\leq m}\left|\nabla u_j\right|^{\alpha_j}\right) \tag{15}$$

where $C_1:[0,+\infty)\to[0,+\infty)$ *is non-decreasing,* $F_1\in L^1\left(Q_T\right)$ *and* $1\leq\alpha_j<2$.

Then for each fixed k

$$\lim_{n\to\infty}\int_{Q_T}\left|\nabla T_k\left(u_{1,n}\right)-\nabla T_k\left(u_1\right)\right|^2\chi_{\left[\sum\limits_{1\leq j\leq m}u_{j,n}\leq k\right]}=0.$$

ii) If

$$\left|f_{i,n}\left(t,x,u,\nabla u\right)\right|\leq C_i\left(\sum_{j=1}^{i}|u_j|\right)\left(F_i\left(t,x\right)+\sum_{1\leq j\leq m}\left|\nabla u_j\right|^2\right),\ 2\leq i\leq m \tag{16}$$

where $C_i:[0,+\infty)\to[0,+\infty)$ *is non-decreasing,* $F_i\in L^1\left(Q_T\right)$ *for all* $2\leq i\leq m$.

Then for each fixed k *and for all* $2\leq i\leq m$

$$\lim_{n\to\infty}\int_{Q_T}\left|\nabla T_k\left(\sum_{1\leq j\leq i}u_{j,n}\right)-\nabla T_k\left(\sum_{1\leq j\leq i}u_j\right)\right|^2\chi_{\left[\sum\limits_{1\leq j\leq m}u_{j,n}\leq k\right]}=0.$$

Proof. (i) This is a direct consequence of the resulting output established in [4] in the case of system 2×2 (see [4], proof of Lemma 6, p 548). The generalization in the case of any system is immediate.

(ii) Let k and γ be positive real numbers, let $\ell\in\mathbb{N}$, and choose Ψ as in previous lemma. Let $\varphi\left(s\right)=s\exp\left(\mu s^2\right)$, with μ to be fixed later. Consider the equation satisfied by $\left(u_{1,n}+u_{2,n}\right)$.

$$\frac{\partial}{\partial t}\left(u_{1,n}+u_{2,n}\right)=d_2\Delta\left(u_{1,n}+u_{2,n}\right)-\left(d_2-d_1\right)\Delta u_{1,n}+f_{1,n}+f_{2,n}$$

and use again $\Psi\varphi\left(T_k\left(u_{1,n}+u_{2,n}\right)-T_k\left(u_{1,n}+u_{2,n}\right)_\gamma\right)$ as a test function, then we will integrate on Q_T. Finally we will use Lemma 2.8 to get the result.

For simplicity, we denote

$$U_{r,n}=\sum_{1\leq j\leq r}u_{j,n}\quad,\quad U_r=\sum_{1\leq j\leq r}u_j\quad\text{for all }2\leq r\leq m$$

Since $\dfrac{\partial}{\partial t}\left(U_{2,n}\right)=\rho_{1,n}^{(2)}+\rho_{2,n}^{(2)}$ where

$$\begin{cases}\rho_{1,n}^{(2)}=d_2\Delta\left(U_{2,n}\right)-\left(d_2-d_1\right)\Delta u_{1,n}\in L^2\left(0,T;H^{-1}\left(\Omega\right)\right)\\ \rho_{2,n}^{(2)}=f_{1,n}+f_{2,n}\in L^1\left(Q_T\right)\end{cases}$$

we have by Lemma 2.8

$$\begin{aligned}&\int_{Q_T}\frac{\partial}{\partial t}\left(U_{2,n}\right)\Psi\varphi\left(T_k\left(U_{2,n}\right)-T_k\left(U_{2,\ell}\right)_\gamma\right)\\&\quad\geq\omega^{\gamma,n}\left(\frac{1}{\ell}\right)+\omega^{\gamma}\left(\frac{1}{n}\right)-\int_\Omega\Psi\left(0\right)\Phi\left(T_k\left(U_2\right)-T_k\left(U_2\right)_\gamma\right)dx\\&\quad\quad-\int_\Omega G_k\left(U_2\right)\left(0\right)\Psi\left(0\right)\varphi\left(T_k\left(U_2\right)-T_k\left(U_2\right)_\gamma\right)\left(0\right)dx\end{aligned}$$

Hence

$$
\begin{aligned}
I+&J+\lambda+\beta \\
&= d_2 \int_{Q_T} \nabla\left(U_{2,n}\right) \Psi \varphi'\left(T_k\left(U_{2,n}\right)-T_k\left(U_{2,\ell}\right)_\gamma\right) \nabla\left(T_k\left(U_{2,n}\right)-T_k\left(U_{2,\ell}\right)_\gamma\right) \\
&\quad-\int_{Q_T}\left(f_{1,n}+f_{2,n}\right) \Psi \varphi\left(T_k\left(U_{2,n}\right)-T_k\left(U_{2,\ell}\right)_\gamma\right) \\
&\quad-\left(d_2-d_1\right) \int_{Q_T}\left(\nabla u_{1,n}-\nabla u_1\right) \Psi \varphi'\left(T_k\left(U_{2,n}\right)-T_k\left(U_{2,\ell}\right)_\gamma\right) \nabla\left(T_k\left(U_{2,n}\right)-T_k\left(U_{2,\ell}\right)_\gamma\right) \\
&\quad-\left(d_2-d_1\right) \int_{Q_T} \nabla u_1 \Psi \varphi'\left(T_k\left(U_{2,n}\right)-T_k\left(U_{2,\ell}\right)_\gamma\right) \nabla\left(T_k\left(U_{2,n}\right)-T_k\left(U_{2,\ell}\right)_\gamma\right) \\
&\leq \omega^{\gamma, n}\left(\frac{1}{\ell}\right)+\omega^\gamma\left(\frac{1}{n}\right)+\int_{\Omega} \Psi(0) \Phi\left(T_k\left(U_2\right)-T_k\left(U_2\right)_\gamma\right) \\
&\quad+\int_{\Omega} G_k\left(U_2\right)(0) \Psi(0) \varphi\left(T_k\left(U_2\right)-T_k\left(U_2\right)_\gamma\right)(0) \\
&\leq \omega^{\gamma, n}\left(\frac{1}{\ell}\right)+\omega^\gamma\left(\frac{1}{n}\right)+\omega\left(\frac{1}{\gamma}\right)
\end{aligned}
$$

since $T_k\left(U_2\right)_\gamma \rightarrow T_k\left(U_2\right)$ in $L^2\left(0, T ; H_0^1(\Omega)\right)$ weakly.

The term I can be written as

$$
\begin{aligned}
I &= d_2 \int_{Q_T} \nabla\left(U_{2,n}\right) \Psi \varphi'\left(T_k\left(U_{2,n}\right)-T_k\left(U_{2,\ell}\right)_\gamma\right) \nabla\left(T_k\left(U_{2,n}\right)-T_k\left(U_{2,\ell}\right)_\gamma\right) \\
&\quad+d_2 \int_{E_n \geq k} \nabla\left(U_{2,n}\right) \Psi \varphi'\left(T_k\left(U_{2,n}\right)-T_k\left(U_{2,\ell}\right)_\gamma\right) \nabla\left(T_k\left(U_{2,n}\right)-T_k\left(U_{2,\ell}\right)_\gamma\right) \\
&= I_1+I_2
\end{aligned}
$$

For I_2, we have

$$
\begin{aligned}
I_2 &= -d_2 \int_{E_n \geq k} \nabla\left(U_{2,n}\right) \Psi \varphi'\left(T_k\left(U_{2,n}\right)-T_k\left(U_{2,\ell}\right)_\gamma\right) \nabla\left(T_k\left(U_{2,\ell}\right)_\gamma\right) \chi_{\left[E_n \geq k\right]} \\
&= \omega^{\gamma, n}\left(\frac{1}{\ell}\right)-d_2 \int_{Q_T} \nabla\left(U_{2,n}\right) \Psi \varphi'\left(T_k\left(U_{2,n}\right)-T_k\left(U_{2,\ell}\right)_\gamma\right) \nabla\left(T_k\left(U_2\right)_\gamma\right) \chi_{\left[E_n \geq k\right]} \\
&= \omega^{\gamma, n}\left(\frac{1}{\ell}\right)-d_2 \int_{Q_T} \nabla\left(U_{2,n}\right) \Psi \varphi'\left(T_k\left(U_{2,n}\right)-T_k\left(U_2\right)_\gamma\right) \nabla\left(T_k\left(U_2\right)_\gamma\right) \chi_{\left[E_n \geq k\right]} \chi_{[E \geq k]} \\
&\quad-d_2 \int_{Q_T} \nabla\left(U_{2,n}\right) \Psi \varphi'\left(T_k\left(U_{2,n}\right)-T_k\left(U_2\right)_\gamma\right) \nabla\left(T_k\left(U_2\right)_\gamma\right) \chi_{\left[E_n \geq k\right]} \chi_{[E<k]} \\
&= \omega^{\gamma, n}\left(\frac{1}{\ell}\right)+I_{2.1}+I_{2.2}
\end{aligned}
$$

For $I_{2.1}$, we have by Hölder's inequality

$$
\left|I_{2.1}\right| \leq d_2\left\|\nabla\left(U_{2,n}\right) \varphi'\left(T_k\left(U_{2,n}\right)-T_k\left(U_2\right)_\gamma\right)\right\|_{L^2\left(Q_T\right)}\left\|\nabla\left(T_k\left(U_2\right)_\gamma\right) \chi_{[E \geq k]}\right\|_{L^2\left(Q_T\right)}
$$

Using the fact that $\varphi'\left(T_k\left(U_{2,n}\right)-T_k\left(U_2\right)_\gamma\right) \leq \varphi'(2k)$ and Lemma 2.5, we obtain

$$
\left|I_{2.1}\right| \leq d_2 C\left\|\nabla\left(T_k\left(U_2\right)_\gamma\right) \chi_{[E \geq k]}\right\|_{L^2\left(Q_T\right)}=\omega\left(\frac{1}{\gamma}\right)
$$

since $T_k(U_2)_\gamma \to T_k(U_2)$ in $L^2\left(0,T;H_0^1(\Omega)\right)$ and $\nabla T_k(U_2)\chi_{[E\geq k]} = 0$ a.e. in Q_T. Now we study the term $I_{2.2}$

$$\begin{aligned} I_{2.2} &= -d_2\int_{Q_T} \nabla(U_{2,n})\,\Psi\varphi'\left(T_k(U_{2,n}) - T_k(U_2)_\gamma\right)\nabla\left(T_k(U_2)_\gamma\right)\chi_{[E_n\geq k]}\chi_{[E<k]} \\ &= \omega^\gamma\left(\frac{1}{n}\right) \end{aligned}$$

since $\chi_{[E_n\geq k]}\chi_{[E<k]} \to 0$ a.e. in Q_T. Thus

$$I_2 \geq \omega^{\gamma,n}\left(\frac{1}{\ell}\right) + \omega^\gamma\left(\frac{1}{n}\right) + \omega\left(\frac{1}{\gamma}\right)$$

We investigate I_1

$$\begin{aligned} I_1 &= \omega^{\gamma,n}\left(\frac{1}{\ell}\right) + d_2\int_{Q_T}\nabla T_k(U_{2,n})\,\Psi\varphi'\left(T_k(U_{2,n}) - T_k(U_2)_\gamma\right)\nabla\left(T_k(U_{2,n}) - T_k(U_2)_\gamma\right) \\ &= \omega^{\gamma,n}\left(\frac{1}{\ell}\right) + d_2\int_{Q_T}\nabla T_k(U_2)\,\Psi\varphi'\left(T_k(U_{2,n}) - T_k(U_2)_\gamma\right)\nabla\left(T_k(U_{2,n}) - T_k(U_2)_\gamma\right) \\ &\quad + d_2\int_{Q_T}\nabla\left(T_k(U_{2,n}) - T_k(U_2)\right)\Psi\varphi'\left(T_k(U_{2,n}) - T_k(U_2)_\gamma\right)\nabla\left(T_k(U_{2,n}) - T_k(U_2)_\gamma\right) \\ &= \omega^{\gamma,n}\left(\frac{1}{\ell}\right) + \omega^\gamma\left(\frac{1}{n}\right) \\ &\quad + d_2\int_{Q_T}\nabla T_k(U_2)\,\Psi\varphi'\left(T_k(U_2) - T_k(U_2)_\gamma\right)\nabla\left(T_k(U_2) - T_k(U_2)_\gamma\right) \\ &\quad + d_2\int_{Q_T}\nabla\left(T_k(U_{2,n}) - T_k(U_2)\right)\Psi\varphi'\left(T_k(U_{2,n}) - T_k(U_2)_\gamma\right)\nabla\left(T_k(U_{2,n}) - T_k(U_2)_\gamma\right) \\ &= \omega^{\gamma,n}\left(\frac{1}{\ell}\right) + \omega^\gamma\left(\frac{1}{n}\right) + \omega\left(\frac{1}{\gamma}\right) \\ &\quad + d_2\int_{Q_T}\left|\nabla\left(T_k(U_{2,n}) - T_k(U_2)\right)\right|^2\Psi\varphi'\left(T_k(U_{2,n}) - T_k(U_2)_\gamma\right) \\ &\quad + d_2\int_{Q_T}\nabla\left(T_k(U_{2,n}) - T_k(U_2)\right)\Psi\varphi'\left(T_k(U_{2,n}) - T_k(U_2)_\gamma\right)\nabla\left(T_k(U_2) - T_k(U_2)_\gamma\right) \\ &= \omega^{\gamma,n}\left(\frac{1}{\ell}\right) + \omega^\gamma\left(\frac{1}{n}\right) + \omega\left(\frac{1}{\gamma}\right) \\ &\quad + d_2\int_{Q_T}\left|\nabla\left(T_k(U_{2,n}) - T_k(U_2)\right)\right|^2\Psi\varphi'\left(T_k(U_{2,n}) - T_k(U_2)_\gamma\right) \end{aligned}$$

Hence

$$\begin{aligned} I &\geq \omega^{\gamma,n}\left(\frac{1}{\ell}\right) + \omega^\gamma\left(\frac{1}{n}\right) + \omega\left(\frac{1}{\gamma}\right) \\ &\quad + d_2\int_{Q_T}\left|\nabla\left(T_k(U_{2,n}) - T_k(U_2)\right)\right|^2\Psi\varphi'\left(T_k(U_{2,n}) - T_k(U_2)_\gamma\right) \end{aligned}$$

For J, we have

$$\begin{aligned} J &= \omega^{\gamma,n}\left(\frac{1}{\ell}\right) - \int_{Q_T} (f_{1,n}+f_{2,n})\,\Psi\varphi\left(T_k(U_{2,n}) - T_k(U_2)_\gamma\right) \\ &= \omega^{\gamma,n}\left(\frac{1}{\ell}\right) - \int_{E_n>k} (f_{1,n}+f_{2,n})\,\Psi\varphi\left(T_k(U_{2,n}) - T_k(U_2)_\gamma\right) \\ &\quad - \int_{E_n\le k} (f_{1,n}+f_{2,n})\,\Psi\varphi\left(T_k(U_{2,n}) - T_k(U_2)_\gamma\right) \end{aligned}$$

Then

$$J \ge \omega^{\gamma,n}\left(\frac{1}{\ell}\right) - \int_{E_n\le k} (f_{1,n}+f_{2,n})\,\Psi\varphi\left(T_k(U_{2,n}) - T_k(U_2)_\gamma\right)$$

since $\varphi\left(T_k(U_{2,n}) - T_k(U_2)_\gamma\right) \ge 0$ on $[E_n>k]$, $\Psi \ge 0$ and $-(f_{1,n}+f_{2,n}) \ge 0$ by hypothesis (3). On the other hand

$$\begin{aligned} &\left|\int_{\{E_n\le k\}} (f_{1,n}+f_{2,n})\,\Psi\varphi\left(T_k(U_{2,n}) - T_k(U_2)_\gamma\right)\right| \\ &\le C_1(k)\int_{\{E_n\le k\}} F_1(t,x)\,\Psi\left|\varphi\left(T_k(U_{2,n}) - T_k(U_2)_\gamma\right)\right| \\ &\quad + C_1(k)\sum_{2\le j\le m}\int_{\{E_n\le k\}} |\nabla u_{j,n}|^{\alpha_j}\,\Psi\left|\varphi\left(T_k(U_{2,n}) - T_k(U_2)_\gamma\right)\right| \\ &\quad + C_1(k)\int_{\{E_n\le k\}} |\nabla T_k(u_{1,n})|^2\,\Psi\left|\varphi\left(T_k(U_{2,n}) - T_k(U_2)_\gamma\right)\right| \\ &\quad + C_2(k)\int_{\{E_n\le k\}} F_2(t,x)\,\Psi\left|\varphi\left(T_k(U_{2,n}) - T_k(U_2)_\gamma\right)\right| \\ &\quad + C_2(k)\sum_{1\le j\le m}\int_{\{E_n\le k\}} |\nabla T_k(u_{j,n})|^2\,\Psi\left|\varphi\left(T_k(U_{2,n}) - T_k(U_2)_\gamma\right)\right| \\ &= J_1 + J_2 + J_3 + J_4 + J_5 \end{aligned}$$

We set

$$\begin{aligned} J_1 &= C_1(k)\int_{\{E_n\le k\}} F_1(t,x)\,\Psi\left|\varphi\left(T_k(U_{2,n}) - T_k(U_2)_\gamma\right)\right| \\ &= \omega^{\gamma}\left(\frac{1}{n}\right) + \omega\left(\frac{1}{\gamma}\right) \end{aligned}$$

Similarly for J_4

$$\begin{aligned} J_4 &= C_2(k)\int_{\{E_n\le k\}} F_2(t,x)\,\Psi\left|\varphi\left(T_k(U_{2,n}) - T_k(U_2)_\gamma\right)\right| \\ &= \omega^{\gamma}\left(\frac{1}{n}\right) + \omega\left(\frac{1}{\gamma}\right) \end{aligned}$$

Since $1 \le \alpha_j < 2$, for all $2 \le j \le m$, we have

$$\begin{aligned} J_2 &= C_1(k)\sum_{2\le j\le m}\int_{\{E_n\le k\}} |\nabla u_j|^{\alpha_j}\,\Psi\left|\varphi\left(T_k(U_{2,n}) - T_k(U_2)_\gamma\right)\right| \\ &= \omega^{\gamma}\left(\frac{1}{n}\right) + \omega\left(\frac{1}{\gamma}\right) \end{aligned}$$

and

$$\begin{aligned}
J_3 &= C_1(k)\int_{\{E_n\le k\}} \left|\nabla T_k(u_{1,n})\right|^2 \Psi \left|\varphi\left(T_k(U_{2,n}) - T_k(U_2)_\gamma\right)\right| \\
&= C_1(k)\int_{\{E_n\le k\}} \left|\nabla\left(T_k(u_{1,n}) - T_k(u_1)\right)\right|^2 \Psi \left|\varphi\left(T_k(U_{2,n}) - T_k(U_2)_\gamma\right)\right| \\
&\quad +2C_1(k)\int_{\{E_n\le k\}} \nabla T_k(u_{1,n})\,\nabla T_k(u_1)\,\Psi \left|\varphi\left(T_k(U_{2,n}) - T_k(U_2)_\gamma\right)\right| \\
&\quad -C_1(k)\int_{\{E_n\le k\}} \left|\nabla T_k(u_1)\right|^2 \Psi \left|\varphi\left(T_k(U_{2,n}) - T_k(U_2)_\gamma\right)\right| \\
&= \omega^\gamma\left(\frac{1}{n}\right) + \omega\left(\frac{1}{\gamma}\right) \\
&\quad +C_1(k)\int_{\{E_n\le k\}} \left|\nabla\left(T_k(u_{1,n}) - T_k(u_1)\right)\right|^2 \Psi \left|\varphi\left(T_k(U_{2,n}) - T_k(U_2)_\gamma\right)\right|
\end{aligned}$$

and

$$\begin{aligned}
J_5 &= C_2(k)\sum_{1\le j\le m}\int_{\{E_n\le k\}} \left|\nabla T_k(u_{j,n})\right|^2 \Psi \left|\varphi\left(T_k(U_{2,n}) - T_k(U_2)_\gamma\right)\right| \\
&= \omega^\gamma\left(\frac{1}{n}\right) + \omega\left(\frac{1}{\gamma}\right)
\end{aligned}$$

since

$$\int_{\{E_n\le k\}} \left|\nabla T_k(u_{j,n})\right|^2 \le \liminf_{n\to\infty}\left(\int_{\{E_n\le k\}} \left|\nabla T_k(u_{j,n})\right|^2\right) \le R_2$$

Thus

$$\begin{aligned}
&-\int_{\{E_n\le k\}} (f_{1,n} + f_{2,n})\,\Psi\varphi\left(T_k(U_{2,n}) - T_k(U_2)_\gamma\right) \\
&\qquad \ge \omega^\gamma\left(\frac{1}{n}\right) + \omega\left(\frac{1}{\gamma}\right) \\
&\qquad\quad -C_1(k)\int_{\{u_{1,n}\le k\}} \left|\nabla\left(T_k(u_{1,n}) - T_k(u_1)\right)\right|^2 \Psi \left|\varphi\left(T_k(u_{1,n}) - T_k(u_1)_\gamma\right)\right|
\end{aligned}$$

hence

$$\begin{aligned}
J &\ge \omega^\gamma\left(\frac{1}{n}\right) + \omega\left(\frac{1}{\gamma}\right) \\
&\quad -C_1(k)\int_{\{E_n\le k\}} \left|\nabla\left(T_k(u_{1,n}) - T_k(u_1)\right)\right|^2 \Psi \left|\varphi\left(T_k(U_{2,n}) - T_k(U_2)_\gamma\right)\right|
\end{aligned}$$

For λ, we have

$$\begin{aligned}
\lambda &= -(d_2 - d_1)\int_{Q_T} (\nabla u_{1,n} - \nabla u_1)\,\Psi\varphi'\left(T_k(U_{2,n}) - T_k(U_{2,\ell})_\gamma\right)\nabla\left(T_k(U_{2,n}) - T_k(U_{2,\ell})_\gamma\right) \\
&= \omega^{\gamma,n}\left(\frac{1}{\ell}\right) + \omega^\gamma\left(\frac{1}{n}\right) + \omega\left(\frac{1}{\gamma}\right)
\end{aligned}$$

since $\nabla T_k(u_{1,n}) \to \nabla T_k(u_1)$ strongly in $L^2\left(0,T;H_0^1(\Omega)\right)$. We have

$$\begin{aligned}\beta &= -(d_2-d_1)\int_{Q_T}\nabla u_1\Psi\varphi'\left(T_k(U_{2,n})-T_k(U_{2,\ell})_\gamma\right)\nabla\left(T_k(U_{2,n})-T_k(U_{2,\ell})_\gamma\right)\\ &= \omega^{\gamma,n}\left(\frac{1}{\ell}\right)+\omega^{\gamma}\left(\frac{1}{n}\right)+\omega\left(\frac{1}{\gamma}\right)\end{aligned}$$

Then

$$\begin{aligned}I+J+\lambda+\beta &\geq \omega^{\gamma,n}\left(\frac{1}{\ell}\right)+\omega^{\gamma}\left(\frac{1}{n}\right)+\omega\left(\frac{1}{\gamma}\right)\\ &+d_2\int_{Q_T}\left|\nabla\left(T_k(U_{2,n})-T_k(U_2)\right)\right|^2\Psi\varphi'\left(T_k(U_{2,n})-T_k(U_2)_\gamma\right)\\ &-C_1(k)\int_{\{E_n\leq k\}}\left|\nabla\left(T_k(u_{1,n})-T_k(u_1)\right)\right|^2\Psi\left|\varphi\left(T_k(U_{2,n})-T_k(U_2)_\gamma\right)\right|\end{aligned}$$

We choose

$$\mu \geq \left(\frac{C_1(k)}{2d_2}\right)^2$$

we have

$$d_2\varphi'(s)-C_1(k)\left|\varphi(s)\right| > \frac{d_2}{2}$$

We conclude that

$$\begin{aligned}&\int_{Q_T}\left|\nabla\left(T_k(U_{2,n})-T_k(U_2)\right)\right|^2\Psi d_2\varphi'\left(T_k(U_{2,n})-T_k(U_2)_\gamma\right)-\\ &\quad-\int_{Q_T}\left|\nabla\left(T_k(U_{2,n})-T_k(U_2)\right)\right|^2\Psi C_1(k)\left|\varphi\left(T_k(U_{2,n})-T_k(U_2)_\gamma\right)\right|\\ &\leq\omega^{\gamma,n}\left(\frac{1}{\ell}\right)+\omega^{\gamma}\left(\frac{1}{n}\right)+\omega\left(\frac{1}{\gamma}\right)\end{aligned}$$

Then we have

$$\lim_{n\to\infty}\int_{Q_T}\left|\nabla\left(T_k(U_{2,n})-T_k(U_2)\right)\right|^2\chi_{[U_{2,n}\leq k]}=0.$$

We get step by step by considering the equation satisfied by $U_{r,n}=\sum_{1\leq j\leq r}u_{j,n}$. Arguing in the same way as before, choosing

$$\mu \geq \max\left\{\left(\frac{C_1(k)}{2d_j}\right)^2,\ 1\leq j\leq r\right\}$$

We obtain

$$\begin{aligned}&\int_{Q_T}\left|\nabla\left(T_k(U_{r,n})-T_k(U_r)\right)\right|^2\Psi d_r\varphi'\left(T_k(U_{r,n})-T_k(U_r)_\gamma\right)-\\ &\quad-\int_{Q_T}\left|\nabla\left(T_k(U_{r,n})-T_k(U_r)\right)\right|^2\Psi C_1(k)\left|\varphi\left(T_k(U_{r,n})-T_k(U_r)_\gamma\right)\right|\\ &\leq\omega^{\gamma,n}\left(\frac{1}{\ell}\right)+\omega^{\gamma}\left(\frac{1}{n}\right)+\omega\left(\frac{1}{\gamma}\right)\end{aligned}$$

which shows the desired result. □

Proof of Lemma (2.7). Let A be a measurable subset of Ω, we have

$$\begin{aligned}\int_A |f_{1,n}(t,x,u_n,\nabla u_n)| &= \int_{A\cap[E_n>k]} |f_{1,n}| + \int_{A\cap[E_n\le k]} |f_{1,n}| \\ &\le \int_{A\cap[\theta_n>k]} |f_{1,n}| + \int_{A\cap[E_n\le k]} |f_{1,n}|\end{aligned}$$

with $E_n = \sum_{1\le j\le m} u_{j,n}$ and $\theta_n = \sum_{1\le k\le m} (m-k+1)\, u_{k,n}$

Thanks to Lemma 2.5, we obtain $\forall\varepsilon>0,\ \exists k_0$ such that if $k \ge k_0$ then for all n

$$\int_{A\cap[E_n>k]} |f_{1,n}(t,x,u_n,\nabla u_n)| \le \frac{1}{k}\int_{[E_n>k]} k\,|f_{1,n}| \le \frac{1}{k}\int_{Q_T} E_n\,|f_{1,n}| \le \frac{1}{k}\int_{Q_T} \theta_n\,|f_{1,n}| \le \frac{\varepsilon}{m+2}$$

Hypothesis (7) implies that for all $k > k_0$

$$\begin{aligned}\int_A |f_{1,n}(t,x,u_n,\nabla u_n)| &\le \frac{\varepsilon}{m+2} + C_1(k)\left(\int_A F_1(x,t) + \int_{A\cap[E_n\le k]} |\nabla u_{1,n}|^2\right) \\ &\quad + C_1(k)\sum_{2\le j\le m}\left(\int_{A\cap[E_n\le k]} |\nabla u_{j,n}|^{\alpha_j}\right) \\ &\le \frac{\varepsilon}{m+2} + C_1(k)\left(\int_A F_1(x,t) + \int_{A\cap[E_n\le k]} |\nabla T_k(u_{1,n})|^2\right) \\ &\quad + C_1(k)\sum_{2\le j\le m}\left(\int_{A\cap[E_n\le k]} |\nabla T_k(u_{j,n})|^{\alpha_j}\right)\end{aligned}$$

Using Hölder's inequality for $1 \le \alpha_j < 2$ and Lemma 2.5, we obtain

$$\begin{aligned}C_1(k)\int_{A\cap[E_n\le k]} |\nabla T_k(u_{j,n})|^{\alpha_j} &\le C_1(k)\left(\int_A |\nabla T_k(u_{j,n})|^2\right)^{\frac{\alpha_j}{2}} |A|^{\frac{2-\alpha_j}{2}} \\ &\le C_1(k)\, R_2^{\frac{\alpha_j}{2}}\, |A|^{\frac{2-\alpha_j}{2}} \le \frac{\varepsilon}{m+2}\end{aligned}$$

Whenever $|A| \le \varrho_j$, with $\varrho_j = \left(\frac{\varepsilon}{m+2} C_1^{-1}(k)\, R_2^{-\frac{\alpha_j}{2}}\right)^{\frac{2}{2-\alpha_j}}$, $2 \le j \le m$.

To deal with the second integral we write

$$\int_{A\cap[E_n\le k]} |\nabla T_k(u_{1,n})|^2 \le 2\int_{A\cap[E_n\le k]} |\nabla T_k(u_{1,n}) - \nabla T_k(u_1)|^2 + 2\int_A |\nabla T_k(u_1)|^2$$

According to Lemma 2.5, $|\nabla T_k(u_{1,n}) - \nabla T_k(u_1)|^2\, \chi_{[E_n\le k]}$ is equi-integrable in $L^1(\Omega)$ since it converges strongly to 0 in $L^1(\Omega)$. So, there exists ϱ_{m+1} such that if $|A| \le \varrho_{m+1}$, then

$$2C_1(k)\int_{A\cap[E_n\le k]} |\nabla T_k(u_{1,n}) - \nabla T_k(u_1)|^2 \le \frac{\varepsilon}{m+2}$$

On the other hand $F_1,\ |\nabla T_k(u_1)|^2 \in L^1(\Omega)$, therefore there exists ϱ_{m+2} such that

$$C_1(k)\left(2\int_A |\nabla T_k(u_1)|^2 + \int_A F_1(t,x)\right) \le \frac{\varepsilon}{m+2}$$

whenever $|A| \le \varrho_{m+2}$. Choose $\varrho_0 = \inf\{\varrho_j,\ 2 \le j \le m+2\}$, If $|A| \le \varrho_0$ we obtain

$$\int_A |f_{1,n}(x,u_n,\nabla u_n)| \le \varepsilon$$

Similarly, we get for all $2 \le i \le m$

$$\int_A |f_{i,n}| \le \frac{\varepsilon}{m+2} + C_i(k)\left(\int_A F_i(x,t) + \int_{A\cap[E_n\le k]} \left(6|\nabla u_1|^2 + 6|\nabla T_k(u_{1,n}) - \nabla T_k(u_1)|^2\right)\right)$$
$$+ 8C_i(k)\sum_{2\le r\le m}\left(\int_{A\cap[E_n\le k]} \left|\nabla T_k\left(\sum_{1\le j\le r} u_j\right)\right|^2\right)$$
$$+ 8C_i(k)\sum_{2\le r\le m}\left(\int_{A\cap[E_n\le k]} \left|\nabla T_k\left(\sum_{1\le j\le r} u_{j,n}\right) - \nabla T_k\left(\sum_{1\le j\le r} u_j\right)\right|^2\right)$$

Arguing in the same way as before, we obtain the required result. □

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