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Homogenization of parabolic nonlinear coupled problem in heat exchange

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Abstract. This work deals with the homogenization of heat transfer nonlinear parabolic problem in a periodic composite medium consisting in two-component (fluid/solid). This problem presents some difficulties due to the presence of a nonlinear Neumann condition modeling a radiative heat transfer on the interface between the two parts of the medium and to the fact that the problem is strongly coupled. In order to justify rigorously the homogenization process, we use two scale convergence. For this, we show first the existence and uniqueness of the homogenization problem by topological degree of Leray-Schauder, Then we establish the two scale convergence, and identify the limit problems.

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1. Introduction

The composite material is a macroscopic combination of two or more distinct materials, having a recognizable interface between them. Composites are used not only for their structural properties, but also for electrical, thermal, tribological, and environmental applications. The resulting composite material has a balance of structural properties that is superior to either constituent material alone. This explain the more and more intense use of this composite material in industrials sectors such as transport, buildings and aeronautics.

One of the important challenges is to have an optimized composite material which achieve a particular balance of properties for a given range of applications. This is directly dependent on the temperature evolution imposed during the injection process. Indeed, the control of the optimality of the obtained piece requires knowledge and control of thermal cycle. One of major difficulties in its modeling is the determination of the effective thermal conductivity.

As the experimental measurement is not feasible in presence of the flow and even if we try to solve numerically the Navier-Stokes equations in all structure and coupled system fluid/solid based on physical parameters in the two phases fluid and solid, these lead to extreme computational difficulties.

A natural way to overcome those difficulties is to replace the composite with a kind of equivalent material model. This procedure is usually called periodic homogenization. In many industrial areas, the multiscale nature of the problem is imposed by the microstructure of the material under consideration. As the numerical simulation of the microstructure in detail still infeasible, an upscaled models, describing on an observation scale much larger than the size of the microstructure, is required.

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The periodic homogenization has proven its efficiency for upscaling rigorously mathematical models of multiscale process [5, 15]. From a mathematical point of view the homogenization theory consists in finding the homogenized characteristics and using them to construct the homogenized model approximating the initial one, and giving global description of the physical process [13].

There are many previous contributions on the homogenization of fluid flow in porous media. The elliptic problem in linear case corresponds to different geometries or scalings has been extensively studied (see for example [9]). The homogenization in nonlinear case of elliptic operator in a perforated media was investigated in [3, 12]. Unlike, the studies of the homogenization of nonlinear parabolic problems are still few in number[10]. In this work, the homogenization of nonlinear parabolic problem in a periodic composite medium is investigated. The main goal here is twofold: we first establish an existence result and then perform rigorously the homogenization process. Although the homogenization process is standard, it has still some difficulties in our situation. In fact, the problem is time depending, strongly coupled (fluid/solid) and with nonlinear Neumann condition.

To circumvent these difficulties, first, for the existence result we use the topological degree of Leray-Schauder, which is more powerful and more general and often easier to use than the classical fixed point theorems [14]. We note that in this case, the compactness of the mapping under consideration need a special attention. Indeed the fact that the system is strongly coupled complicate the task. Then the uniqueness of the fixed point is obtained under some assumptions on the non linear function. The second main result is the upscaling of our problem by periodic homogenization. We note that the choice of the correct scaling of the material parameters with the homogenization parameter is very important, as it is well known that this has a large influence on the limit problems. In particular, different scalings may in general lead to different types of limit problems [16]. Moreover, the obtained convergences in periodic homogenization are of weak type. This implies that they are not compatible with nonlinear terms a priori. Thus, in order to characterize the limit problems, additional considerations are required [6, 7, 8].

The paper is organized as follows. In Section 2 the microscale problem is introduced and the mathematical assumptions are stated. The remaining sections contain the details of the rigorous homogenization procedure. More precisely, in Section 3, we show the existence and uniqueness of the homogenization problem. The two scale convergence and the identification of the limit problems and there existence and the uniqueness are established in Section 4.

2. Problem setting

We are interested in a heat transfer problem in periodic porous media Ω, which is an open bounded set of $\mathbb{R}^2$ with Lipschitz boundary, consisting in two-component composite (solid and fluid see Figure 1). Let $\{\epsilon\}$ be a sequence of positive real numbers that tends to zero.

Note by $Y =]0, l_1[\times]0, l_2[$ the representative cell and by Y_f and Y_s two non empty open subsets of Y such that

$$Y = Y_f \cup \overline{Y_s}.$$

Assume that $\Gamma = \partial Y_s$ Lipschitz continuous and Y_f connected .

We define

$$Y_i^k := k_l + Y_i, \quad k \in \mathbb{Z}^2,$$

and

$$\Gamma_k := k_l + \Gamma,$$

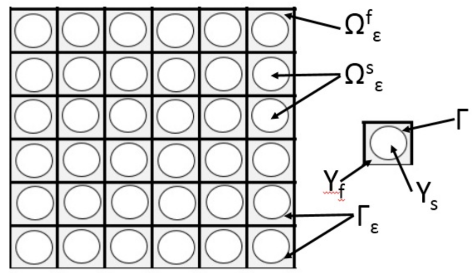

FIGURE 1. Periodic domain and reference cell domain.

where $k_l = (k_1 l_1, k_2 l_2)$ and $i = s, f$. Thanks to this construction we can define the following correspondence between any $x \in \Omega$ and $y \in Y$

$$\forall x \in \Omega, \exists k \in \mathbb{Z}^2 \text{ and } y \in Y \quad \text{such that} \quad x = \epsilon(k_l + y).$$

We define the set Z_ϵ by

$$Z_\epsilon = \left\{ k \in \mathbb{Z}^2 / \epsilon Y_i^k \cap \Omega \neq \emptyset, \, i = s, f \right\}.$$

Assuming that,

$$\partial\Omega \cap (\underset{k \in \mathbb{Z}^2}{\cup} (\epsilon \Gamma_k)) = \emptyset. \tag{1}$$

We can define the two components of Ω and their interface by

$$\Omega_\epsilon^i = \Omega \cap (\underset{k \in Z_\epsilon}{\cup} (\epsilon Y_i^k)), \, i = s, f, \qquad \Gamma_\epsilon = \partial \Omega_\epsilon^s. \tag{2}$$

From (1) it follows that

$$\partial\Omega \cap \Gamma_\epsilon = \emptyset, \tag{3}$$

and from (2) it's clear that

$$\Omega = \Omega_\epsilon^f \cup \Omega_\epsilon^s.$$

Let u_ϵ be a temperature in the domain Ω decomposed as

$$u_\epsilon = \begin{cases} u_\epsilon^f & \text{in} \quad (0,T) \times \Omega_\epsilon^f, \\ u_\epsilon^s & \text{in} \quad (0,T) \times \Omega_\epsilon^s. \end{cases}$$

The u_ϵ is continuous through the interface Γ_ϵ. The radiative transfer between the two parts of the media is modeled by a continuity condition on Γ_ϵ and its expressed as follow

$$u_\epsilon^s = u_\epsilon^f \qquad \text{and} \qquad - K_\epsilon^s \nabla u_\epsilon^s \cdot n_1 = -K_\epsilon^f \nabla u_\epsilon^f \cdot n_2 + \epsilon F(u_\epsilon^f), \; on \; (0,T) \times \Gamma_\epsilon,$$

where n_1 and n_2 are the outward normal vectors on Γ_ϵ, where F is a function expresses radiative exchange transfer on Γ_ϵ.

The aim is to describe the asymptotic behavior, as $\epsilon \to 0$ of the following problem, which models the local evolution of the temperature in the porous medium

$$\begin{cases} l \dfrac{\partial u_\epsilon^f}{\partial t} - \nabla \cdot (K_\epsilon^f \nabla u_\epsilon^f) + V_\epsilon \cdot \nabla u_\epsilon^f = 0, & \text{on} \quad (0,T) \times \Omega_\epsilon^f, \\ \dfrac{\partial u_\epsilon^s}{\partial t} - \nabla \cdot (K_\epsilon^s \nabla u_\epsilon^s) = 0, & \text{on} \quad (0,T) \times \Omega_\epsilon^s, \\ u_\epsilon^s(t,x) = u_\epsilon^f(t,x), & \text{on} \quad (0,T) \times \Gamma_\epsilon, \\ -K_\epsilon^s \nabla u_\epsilon^s . n_1 = -K_\epsilon^f \nabla u_\epsilon^f . n_2 + \epsilon F(u_\epsilon^f), & \text{on} \quad (0,T) \times \Gamma_\epsilon \\ u_\epsilon^f(t,x) = 0, & \text{on} \quad (0,T) \times \partial\Omega, \\ u_\epsilon(0,x) = u_{in}, & \text{on} \quad \Omega. \end{cases} \tag{4}$$

where u_{in} is a given function, and V_ϵ is a given fluid velocity

$$V_\epsilon(x) = V(x, \frac{x}{\epsilon}) \qquad \text{in} \qquad \Omega_\epsilon^f,$$

such that $V(x,y)$ is the solution of the nonlinear Stockes equation in Ω_ϵ^f, which is supposed Y-periodic. We denote by K_ϵ^f the conductivity tensor in the fluid part defined by

$$K_\epsilon^f = K^f(t, x, \frac{x}{\epsilon}), \quad t \in]0, T[\,, \quad x \in \Omega_\epsilon^f,$$

and K_ϵ^s denotes the conductivity tensor in the solid part defined by

$$K_\epsilon^s = K^s(t, x, \frac{x}{\epsilon}), \quad t \in]0, T[\,, \quad x \in \Omega_\epsilon^s,$$

where $K^s(t,x,y),\ K^f(t,x,y)$ are periodic symmetric positive definite tensors defined in the unit cell Y and satisfying

$$\forall v \in \mathbb{R}^2,\ \forall t \in]0, T[\,,\ \forall x \in \Omega,\ \forall y \in Y, \qquad \alpha_1 |v|^2 \leq \sum_{i,j=1}^{2} K^f(t,x,y) v_i v_j,$$

and

$$\forall v \in \mathbb{R}^2,\ \forall t \in]0, T[\,,\ \forall x \in \Omega,\ \forall y \in Y, \qquad \alpha_2 |v|^2 \leq \sum_{i,j=1}^{2} K^s(t,x,y) v_i v_j,$$

for some constants $0 < \alpha_i$ for $i = 1, 2$ and $(\partial_t K_\epsilon^f, \partial_t K_\epsilon^s) \in L^\infty((0,T); L^\infty(\Omega_\epsilon^f)) \times L^\infty((0,T); L^\infty(\Omega_\epsilon^s))$. The function F expresses radiative exchange transfer on Γ_ϵ, which verify the following conditions

(H1) F is continuous Lipschitz, $F(0) = 0$.

(H2) For all t_1, t_2 in $\mathbb{R}$, we have

$$(F(t_1) - F(t_2))(t_1 - t_2) \geq 0.$$

Let us introduce the space

$$W_\epsilon = \{u \in H^1(\Omega_\epsilon^f) / \quad u_{/\partial\Omega} = 0\}$$

equipped with the norm

$$\|u\|_{W_\epsilon} = \|\nabla u\|_{L^2(\Omega_\epsilon^f)}.$$

And $H^1_\sharp(Y)$ is the closure of $C^\infty_\sharp(\mathbb{R}^N)$ for the norm H^1 where

$$C^\infty_\sharp(Y) = \{u \in C^\infty(\mathbb{R}^N) / u \text{ is } Y\text{-periodic}\}.$$

We need also to define the following spaces

$L^p(0,T;W) = \{u : [0,T] \longrightarrow W$ summable, such that $\|u(t)\| \in L^p(0,T)\}$, $\forall\, p,\ 1 \leq p < \infty$, equipped by the norm

$$\|u\|_{L^p(0,T;W)} = (\int_0^T \|u(t)\|_W^p dt)^{\frac{1}{p}},$$

and the space

$$L^\infty(0,T;W) = \inf\left\{C;\ \text{such that } \|u(t)\|_W < C \text{ a.e in } [0,T]\right\}.$$

which is equipped by the norm

$$\|u\|_{L^\infty(0,T;W)} = \sup_{t \in [0,T]} \|u(t)\|_W.$$

Then the variational formulation of the problem (4) is stated:

$$\begin{cases} \text{Find } u_\epsilon = (u_\epsilon^f|_{\Omega_\epsilon^f}, u_\epsilon^s|_{\Omega_\epsilon^s}) \in L^2(0,T;H_0^1(\Omega)) \text{ and } \frac{\partial u_\epsilon}{\partial t} \in L^2(0,T;H^{-1}(\Omega)) \\ \text{such that } \langle \dfrac{\partial u_\epsilon^f}{\partial t}, w_\epsilon^f \rangle_{(W_\epsilon)',W_\epsilon} + \displaystyle\int_{\Omega_\epsilon^f} K_\epsilon^f \nabla u_\epsilon^f \nabla w_\epsilon^f dx + \\ \langle \frac{\partial u_\epsilon^s}{\partial t}, w_\epsilon^s \rangle_{(H^1(\Omega_\epsilon^s))',H^1(\Omega_\epsilon^s)} + \int_{\Omega_\epsilon^s} K_\epsilon^s \nabla u_\epsilon^s \nabla w_\epsilon^s dx + \int_{\Omega_\epsilon^f} V_\epsilon \nabla u_\epsilon^f w_\epsilon^f dx \\ +\epsilon \displaystyle\int_{\Gamma_\epsilon} F(u_\epsilon^f) w_\epsilon^f d\sigma_x = 0, \quad \forall w_\epsilon = (w_\epsilon^f|_{\Omega_\epsilon^f}, w_\epsilon^s|_{\Omega_\epsilon^s}) \in L^2(0,T;H_0^1(\Omega)). \end{cases} \tag{5}$$

In order to obtain the effective model posed in an homogeneous domain with homogenized coefficients we will use the so-called two-scale convergence. For this, we need first to show the existence and uniqueness of the problem (4). The principal difficulties lie in the fact that the model is nonlinear coupled and time dependent. To overcome these difficulties, we use the Leray-Schauder topological degree.

3. Existence and uniqueness of the homogenization problem

In the sequel we will denote by C a non negative generic constant. In order, to show the existence of the problem (5), we use the topological degree of Leray-Schauder. For this, we begin by stated the following Lemma, whose proof is based on the assumption $(H1)-(H2)$ and the fact that $\nabla \cdot V_\epsilon = 0$

Lemma 3.1. *If u_ϵ solution of (5), then it exists a constant $C > 0$, such that*

$$\|u_\epsilon\|_{L^2(0,T;H_0^1(\Omega))} \leq C. \tag{6}$$

Now we are ready to state the existence and uniqueness of the solution of problem (5).

Theorem 3.2. *The problem (5) admits an unique solution in $L^2(0,T;H_0^1(\Omega))$.*

Proof. We define the operator G by

$$\begin{array}{rcl} G \ : L^2(0,T;H_0^1(\Omega)) & \longrightarrow & L^2(0,T;H_0^1(\Omega)) \\ \overline{u}_\epsilon & \longmapsto & u_\epsilon, \end{array}$$

where the u_ϵ is the unique solution of the following obtained thanks to [11]

$$\langle \frac{\partial u_\epsilon^f}{\partial t}, w_\epsilon^f \rangle_{(W_\epsilon)',W_\epsilon} + \int_{\Omega_\epsilon^f} K_\epsilon^f \nabla u_\epsilon^f \nabla w_\epsilon^f dx + \langle \frac{\partial u_\epsilon^f}{\partial t}, w_\epsilon^f \rangle_{(H^1(\Omega_\epsilon^s))',H^1(\Omega_\epsilon^s)} + \int_{\Omega_\epsilon^s} K_\epsilon^s \nabla u_\epsilon^s \nabla w_\epsilon^s dx = -\int_{\Omega_\epsilon^f} V_\epsilon \nabla \overline{u}_\epsilon^f w_\epsilon^f dx - \int_{\Gamma_\epsilon} \epsilon F(\overline{u}_\epsilon^f) w_\epsilon^f d\sigma_x. \tag{7}$$

such that $\frac{\partial u_\epsilon}{\partial t} \in L^2(0,T;H^{-1}(\Omega))$. It is easy then to see that G is well defined.

A fixed point of G is a solution of (5). To prove the existence of a fixed point G, we have to show that G is compact and continuous, and find $R > 0$ such that $\forall \tau\,[0,1]$, there exists no solution of $u - \tau G(u) = 0$ satisfying $\|u\|_{L^2(0,T;H_0^1(\Omega))} = R$.

In order to prove the continuity of the map G, we take the sequence $\overline{u}_{\epsilon,n}$ in $L^2(0,T,H_0^1(\Omega))$, such that $\overline{u}_{\epsilon,n} \underset{n\to\infty}{\longrightarrow} \overline{u}_\epsilon$ in $L^2(0,T,H_0^1(\Omega))$ we have to prove that $G(\overline{u}_{\epsilon,n}) \underset{n\to\infty}{\longrightarrow} G(\overline{u}_\epsilon)$ in $L^2(0,T,H_0^1(\Omega))$.

Let $u_{\epsilon,n}$ (respectively u_ϵ) be the unique solution associated to $\overline{u}_{\epsilon,n}$(respectively $\overline{u}_\epsilon$) for the formulation (7). By subtracting the two weak formulations associated to $u_{\epsilon,n}$ and

u_ϵ, we obtain the following equation

$$\begin{aligned}&\langle\frac{\partial u_\epsilon^f}{\partial t}-\frac{\partial u_{\epsilon,n}^f}{\partial t},w_\epsilon^f\rangle_{(W_\epsilon)',W_\epsilon}+\langle\frac{\partial u_\epsilon^s}{\partial t}-\frac{\partial u_{\epsilon,n}^s}{\partial t},w_\epsilon^s\rangle_{(H^1(\Omega_\epsilon^s))',H^1(\Omega_\epsilon^s)}\\&\quad+\int_{\Omega_\epsilon^f}K_\epsilon^f(\nabla u_\epsilon^f-\nabla u_{\epsilon,n}^f)\nabla w_\epsilon^f dx+\int_{\Omega_\epsilon^s}K_\epsilon^s(\nabla u_\epsilon^s-\nabla u_{\epsilon,n}^f)\nabla w_\epsilon^s dx\\&\quad=\int_{\Omega_\epsilon^f}V_\epsilon\nabla(\overline{u}_{\epsilon,n}^f-\overline{u}_\epsilon^f)w_\epsilon^f dx+\int_{\Gamma_\epsilon}\epsilon(F(\overline{u}_{\epsilon,n}^f)-F(\overline{u}_\epsilon^f))w_\epsilon^f d\sigma_x.\end{aligned}$$

Taking $w_\epsilon^f=u_\epsilon^f-u_{\epsilon,n}^f$, $w_\epsilon^s=u_\epsilon^s-u_{\epsilon,n}^s$ and integrating in t

$$\begin{aligned}&\int_0^T\langle\frac{\partial u_\epsilon^f}{\partial t}-\frac{\partial u_{\epsilon,n}^f}{\partial t},u_\epsilon^f-u_{\epsilon,n}^f\rangle_{(W_\epsilon)',W_\epsilon}dt+\int_0^T\langle\frac{\partial u_\epsilon^s}{\partial t}-\frac{\partial u_{\epsilon,n}^s}{\partial t},u_\epsilon^s-u_{\epsilon,n}^s\rangle_{(H^1(\Omega_\epsilon^s))',H^1(\Omega_\epsilon^s)}dt\\&+\int_0^T\int_{\Omega_\epsilon^f}K_\epsilon^f(\nabla u_\epsilon^f-\nabla u_{\epsilon,n}^f)(\nabla u_\epsilon^f-\nabla u_{\epsilon,n}^f)dxdt+\int_0^T\int_{\Omega_\epsilon^s}K_\epsilon^s(\nabla u_\epsilon^s-\nabla u_{\epsilon,n}^f)(\nabla u_\epsilon^s-\nabla u_{\epsilon,n}^s)dxdt\\&=\int_0^T\int_{\Omega_\epsilon^f}V_\epsilon\nabla(\overline{u}_{\epsilon,n}^f-\overline{u}_\epsilon^f)(u_\epsilon^f-u_{\epsilon,n}^f)dxdt+\int_0^T\int_{\Gamma_\epsilon}\epsilon(F(\overline{u}_{\epsilon,n}^f)-F(\overline{u}_\epsilon^f))(u_\epsilon^f-u_{\epsilon,n}^f)d\sigma_x dt.\end{aligned}$$

Since W_ϵ has continous injection and dense in $L^2(\Omega_\epsilon^f)$ (respectivly $H^1(\Omega_\epsilon^s)$ has continous injection and dense in $L^2(\Omega_\epsilon^s)$), also $L^2(\Omega_\epsilon^f)$ has continous injection in $(W_\epsilon)'$ (respectivly $L^2(\Omega_\epsilon^s)$ has continous injection $(H^1(\Omega_\epsilon^s))'$) and $\frac{\partial u_\epsilon^f}{\partial t}\in L^2(0,T;(W_\epsilon)')$ (respectivly $\frac{\partial u_\epsilon^s}{\partial t}\in L^2(0,T;(H^1(\Omega_\epsilon^s))')$, we have the following results (see for example [11])

$$\frac{1}{2}\|u_\epsilon^f(T)-u_{\epsilon,n}^f(T)\|_{L^2(\Omega_\epsilon^f)}^2=\int_0^T\langle\frac{\partial u_\epsilon^f}{\partial t}-\frac{\partial u_{\epsilon,n}^f}{\partial t},u_\epsilon^f-u_{\epsilon,n}^f\rangle_{(W_\epsilon)',W_\epsilon}dt$$

and

$$\frac{1}{2}\|u_\epsilon^s(T)-u_{\epsilon,n}^s(T)\|_{L^2(\Omega_\epsilon^s)}^2=\int_0^T\langle\frac{\partial u_\epsilon^s}{\partial t}-\frac{\partial u_{\epsilon,n}^s}{\partial t},u_\epsilon^s-u_{\epsilon,n}^s\rangle_{(H^1(\Omega_\epsilon^s))',H^1(\Omega_\epsilon^s)}dt.$$

Then we get

$$\begin{aligned}&\frac{1}{2}\|u_\epsilon^f(T)-u_{\epsilon,n}^f(T)\|_{0,\Omega_\epsilon^f}^2+\frac{1}{2}\|u_\epsilon^s(T)-u_{\epsilon,n}^s(T)\|_{0,\Omega_\epsilon^s}^2\\&\qquad+\alpha_1\|\nabla u_\epsilon^f-\nabla u_{\epsilon,n}^f\|_{L^2(0,T;L^2(\Omega_\epsilon^f))}^2+\alpha_2\|\nabla u_\epsilon^s-\nabla u_{\epsilon,n}^s\|_{L^2(0,T;L^2(\Omega_\epsilon^s))}^2 \qquad (8)\\&\le\int_0^T\int_{\Omega_\epsilon^f}V_\epsilon\nabla(\overline{u}_{\epsilon,n}^f-\overline{u}_\epsilon^f)(u_{\epsilon,n}^f-u_\epsilon^f)dxdt+\epsilon\int_0^T\int_{\Gamma_\epsilon}(F(\overline{u}_{\epsilon,n}^f)-F(\overline{u}_\epsilon^f))(u_\epsilon^f-u_{\epsilon,n}^f)d\sigma_x dt.\end{aligned}$$

We decompose the right term in the above inequality, as following

$$\begin{aligned}I_1&=\int_0^T\int_{\Omega_\epsilon^f}V_\epsilon(\nabla\overline{u}_{\epsilon,n}^f-\nabla\overline{u}_\epsilon^f)(u_\epsilon^f-u_{\epsilon,n}^f)dxdt,\\I_2&=\int_0^T\int_{\Gamma_\epsilon}\int_{\Gamma_\epsilon}\epsilon(F(\overline{u}_{\epsilon,n}^f)-F(\overline{u}_\epsilon^f))(u_\epsilon^f-u_{\epsilon,n}^f)d\sigma_x dt\end{aligned}$$

since we have $V_\epsilon\in H_0^1(\Omega_\epsilon^f)\cap H^2(\Omega_\epsilon^f)$, which means that $V_\epsilon\in L^\infty(\Omega_\epsilon^f)$ and by Hölder inequality, we have

$$|I_1|\le\|V_\epsilon\|_{L^\infty(\Omega_\epsilon^f)}\int_0^T\|u_\epsilon^f(t)-u_{\epsilon,n}^f(t)\|_{L^2(\Omega_\epsilon^f)}\|\nabla\overline{u}_{\epsilon,n}^f(t)-\nabla\overline{u}_\epsilon^f(t)\|_{L^2(\Omega_\epsilon^f)}dt.$$

Using Poincaré inequality and the Hölder inequality, we obtain

$$|I_1|\le C\|u_\epsilon^f-u_{\epsilon,n}^f\|_{L^2(0,T;W_\epsilon)}\|\nabla\overline{u}_{\epsilon,n}^f-\nabla\overline{u}_\epsilon^f\|_{L^2(0,T;L^2(\Omega_\epsilon^f))}.$$

On the other hand, we have

$$|I_2| \leq \int_0^T \int_{\Gamma_\epsilon} \epsilon(F(\overline{u}^f_{\epsilon,n}) - F(\overline{u}^f_\epsilon))(u^f_\epsilon - u^f_{\epsilon,n}) d\sigma_x dt,$$

by using Hölder inequality and the fact that F is Lipschitz, we have

$$|I_2| \leq C\|\overline{u}^f_\epsilon - \overline{u}^f_{\epsilon,n}\|_{L^2(0,T;L^2(\Gamma_\epsilon))}\|u^f_\epsilon - u^f_{\epsilon,n}\|_{L^2(0,T;L^2(\Gamma_\epsilon))}.$$

Then by using the continuity of trace operator, we get

$$|I_2| \leq C\|\overline{u}^f_\epsilon - \overline{u}^f_{\epsilon,n}\|_{L^2(0,T;W_\epsilon)}\|u^f_\epsilon - u^f_{\epsilon,n}\|_{L^2(0,T;W_\epsilon)},$$

then the inequality (8) becomes

$$\begin{aligned}&\frac{1}{2}\|u^f_{\epsilon,n}(T) - u^f_\epsilon(T)\|^2_{0,\Omega^f_\epsilon} + \frac{1}{2}\|u^s_{\epsilon,n}(T) - u^s_\epsilon(T)\|^2_{0,\Omega^s_\epsilon} + \alpha_1\|\nabla u^f_{\epsilon,n} - \nabla u^f_\epsilon\|^2_{L^2(0,T;\Omega^f_\epsilon)}\\ &+ \alpha_2\|\nabla u^s_{\epsilon,n} - \nabla u^s_\epsilon\|^2_{L^2(0,T;\Omega^s_\epsilon)} \leq C\|\overline{u}^f_\epsilon - \overline{u}^f_{\epsilon,n}\|_{L^2(0,T;W_\epsilon)}\|u^f_\epsilon - u^f_{\epsilon,n}\|_{L^2(0,T;W_\epsilon)},\end{aligned}$$

hence

$$\|\nabla u_{\epsilon,n} - \nabla u_\epsilon\|^2_{L^2(0,T;L^2(\Omega))} \leq C\|\overline{u}_\epsilon - \overline{u}_{\epsilon,n}\|_{L^2(0,T;H^1_0(\Omega))}\|u_\epsilon - u_{\epsilon,n}\|_{L^2(0,T;H^1_0(\Omega))}. \quad (9)$$

Since the sequence $\overline{u}_{\epsilon,n}$ converge to $\overline{u}_\epsilon$ in $L^2(0,T,H^1_0(\Omega))$, we conclude that G is continuous.

Now, let us show that G is compact. For this, let $(\overline{u}_{\epsilon,n})_n$ be a bounded sequence in $L^2(0,T;H^1_0(\Omega))$, and let $u_{\epsilon,n} = G(\overline{u}_{\epsilon,n})$ be the unique solution of (7) associated to $\overline{u}_{\epsilon,n}$.

Indeed by taking $w_\epsilon = u_{\epsilon,n}$ and integrating in t, we have

$$\begin{aligned}&\tfrac{1}{2}(\|u^f_{\epsilon,n}(T)\|^2_{L^2(\Omega^f_\epsilon)} + \|u^s_{\epsilon,n}(T)\|^2_{L^2(\Omega^s_\epsilon)}) + \alpha_1\|\nabla u^f_{\epsilon,n}\|^2_{L^2(0,T;L^2(\Omega^f_\epsilon))} + \alpha_2\|\nabla u^s_{\epsilon,n}\|^2_{L^2(0,T;L^2(\Omega^s_\epsilon))}\\ &\leq C\|\nabla\overline{u}^f_{\epsilon,n}\|_{L^2(0,T,L^2(\Omega^f_\epsilon))}\|\nabla u^f_{\epsilon,n}\|_{L^2(0,T,L^2(\Omega^f_\epsilon))} + \tfrac{1}{2}(\|u^f_{in}\|^2_{L^2(\Omega^f_\epsilon)} + \|u^s_{in}\|^2_{L^2(\Omega^s_\epsilon)})\end{aligned} \quad (10)$$

then

$$\|\nabla u_{\epsilon,n}\|^2_{L^2(0,T;L^2(\Omega))} \leq C\|\nabla u_{\epsilon,n}\|_{L^2(0,T;L^2(\Omega))} + C$$

by using the Young inequality, we obtain

$$\|\nabla u_{\epsilon,n}\|^2_{L^2(0,T;L^2(\Omega))} \leq C.$$

Using the fact that $u_{\epsilon,n} \in L^2(0,T;H^1_0(\Omega))$ and $\frac{\partial u_{\epsilon,n}}{\partial t} \in L^2(0,T;H^{-1}(\Omega))$, according to [17], we have

$$\langle \frac{\partial u_{\epsilon,n}}{\partial t}, w_\epsilon\rangle_{(H^{-1}(\Omega))',H^1_0(\Omega)} = \frac{d}{dt}\int_\Omega u_{\epsilon,n} w_\epsilon dx,$$

and from the weak formulation (5), we get

$$\|\frac{\partial u_{\epsilon,n}}{\partial t}\|_{L^2(0,T;H^{-1}(\Omega))} \leq C.$$

Thanks to the compact embedding of $H^1_0(\Omega)$ in $L^2(\Omega)$ and the continous embedding of $L^2(\Omega)$ in $H^{-1}(\Omega)$, we can extract a subsequence denoted again $(u_{\epsilon,n})_n$ which converges in $L^2(0,T;L^2(\Omega))$ (see [11]), and satisfies

$$\|\nabla u^f_{\epsilon,n}\|^2_{L^2(0,T;L^2(\Omega^f_\epsilon))} \leq C.$$

From the first equation of (4), we get

$$\|\frac{\partial u^f_{\epsilon,n}}{\partial t}\|_{L^2(0,T;H^{-1}(\Omega^f_\epsilon))} \leq C.$$

Thanks to the compact embedding of W_ϵ in $H^{1-\delta}(\Omega^f_\epsilon)$ for $0 < \delta < \frac{1}{2}$ and the continuous embedding of $H^{1-\delta}(\Omega^f_\epsilon)$ in $H^{-1}(\Omega^f_\epsilon)$ we can extract a subsequence denoted again $(u_{\epsilon,n})_n$

which converges in $L^2(0,T;H^{1-\delta}(\Omega_\epsilon^f))$ and by using the continuous trace operator from $H^{1-\delta}(\Omega_\epsilon^f)$ in $L^2(\Gamma_\epsilon)$, the subsequence $(u_{\epsilon,n})_n$ converge in $L^2(0,T;L^2(\Gamma_\epsilon))$ (see [11]).

Let $u_{\epsilon,n}$ (respectively $u_{\epsilon,m}$) be the unique solution of the formulation (7) associated to $\overline{u}_{\epsilon,n}$ (respectively $\overline{u}_{\epsilon,m}$). By subtracting the two formulations we obtain the following inequality

$$\begin{aligned}\|\nabla u_{\epsilon,n}-\nabla u_{\epsilon,m}\|^2_{L^2(0,T;L^2(\Omega))} \leq C\|\nabla \overline{u}^f_{\epsilon,n}-\nabla \overline{u}^f_{\epsilon,m}\|_{L^2(0,T;L^2(\Omega_\epsilon^f))}\|u^f_{\epsilon,n}-u^f_{\epsilon,m}\|_{L^2(0,T;L^2(\Omega_\epsilon^f))}\\ +\epsilon\|F(\overline{u}^f_{\epsilon,n})-F(\overline{u}^f_{\epsilon,m})\|_{L^2(0,T;L^2(\Gamma_\epsilon))}\|u^f_{\epsilon,n}-u^f_{\epsilon,m}\|_{L^2(0,T;L^2(\Gamma_\epsilon))}.\end{aligned}$$

Using the fact that F is Lipschitz and the continuity of the trace operator, we get

$$\begin{aligned}\|\nabla u_{\epsilon,n}-\nabla u_{\epsilon,m}\|^2_{L^2(0,T;L^2(\Omega))} \leq C\|\overline{u}^f_{\epsilon,n}-\overline{u}^f_{\epsilon,m}\|_{L^2(0,T;W_\epsilon)}(\|u^f_{\epsilon,n}-u^f_{\epsilon,m}\|_{L^2(0,T;L^2(\Omega_\epsilon^f))}\\ +\|u^f_{\epsilon,n}-u^f_{\epsilon,m}\|_{L^2(0,T;L^2(\Gamma_\epsilon))})\end{aligned}$$

then

$$\begin{aligned}\|\nabla u_{\epsilon,n}-\nabla u_{\epsilon,m}\|^2_{L^2(0,T;L^2(\Omega))} \leq 2C\sup_m(\|\overline{u}^f_{\epsilon,m}\|_{L^2(0,T;W_\epsilon)})(\|u^f_{\epsilon,n}-u^f_{\epsilon,m}\|_{L^2(0,T;L^2(\Omega_\epsilon^f))}\\ +\|u^f_{\epsilon,n}-u^f_{\epsilon,m}\|_{L^2(0,T;L^2(\Gamma_\epsilon))})\end{aligned}$$

the sequence is a Cauchy one in $L^2(0,T;H^1_0(\Omega))$. This end the proof of the compactness of G.

Since G is continuous and compact, to show that G admits a fixed point, we consider the open ball B, defined by:

$$B=\{u_\epsilon\in L^2(0,T;H^1_0(\Omega)),\ \|u_\epsilon\|_{L^2(0,T;H^1_0(\Omega))}<R\}$$

with $R=C+1$. The map G has no fixed point on ∂B. Then $deg[I-G,B,0]$ is defined and independent of τ. By using the theorem of the topological degree of Leray-Schauder, since G_0 corresponding to the trivial problem:

$$\langle\frac{\partial u^f_\epsilon}{\partial t},w^f_\epsilon\rangle_{(W_\epsilon)',W_\epsilon}+\langle\frac{\partial u^s_\epsilon}{\partial t},w^s_\epsilon\rangle_{(H^1(\Omega^s_\epsilon))',H^1(\Omega^s_\epsilon)}+\int_{\Omega^f_\epsilon}K^f_\epsilon\nabla u^f_\epsilon\nabla w^f_\epsilon dx+\int_{\Omega^s_\epsilon}K^s_\epsilon\nabla u^s_\epsilon\nabla w^s_\epsilon dx=0 \quad (11)$$

which has a unique u_ϵ (obtained thanks to [11]), then $deg[I-G_0,B,0]=1$. Therefore $deg[I-G_1,B,0]=1$.

Consequently, there exists $u_\epsilon\in L^2(0,T;H^1_0(\Omega))$, such that $G_1(u_\epsilon)=u_\epsilon$ and moreover $\frac{\partial u_\epsilon}{\partial t}\in L^2(0,T;H^{-1}(\Omega))$. To prove the uniqueness of the solution, we suppose that u^1_ϵ and u^2_ϵ are two solutions of the problem (5), by subtracting the weak formulations associated to the solutions u^1_ϵ and u^2_ϵ and integrating in t, we use the assumption $(H2)$ to find

$$\|u^1_\epsilon-u^2_\epsilon\|_{L^2(0,T;H^1_0(\Omega))}\leq 0.$$

This achieves the proof. □

4. Two scale convergence

To use the two-scale convergence method, we first need to show some a priori estimates on the u_ϵ. Indeed, using the continuity of the trace operator, the assumptions $(H1)-(H2)$, the fact that $\nabla\cdot V_\epsilon=0$ and Young and Gronwall inequalities, we show the following result.

Lemma 4.1. *From the weak formulation* (5), *we have the following estimation:*

$$\|u_\epsilon\|_{L^\infty(0,T;L^2(\Omega))} \leq C \quad (12)$$

$$\|\frac{\partial u_\epsilon}{\partial t}\|_{L^2(0,T;L^2(\Omega))} \leq C \quad (13)$$

$$\|u_\epsilon\|_{L^2(0,T;L^2(\Gamma_\epsilon))} \leq C \quad (14)$$

Now, we recall some classical results on two-scale convergence which can be found in [1, 9]. Then we prove a rigorous homogenization results, using the two-scale convergence method.

Definition 4.1. For all sequence bounded u_ϵ in $L^2(0,T;L^2(\Omega_\epsilon))$ is said converge in two scale sense to the function $u_0(t,x,y)$ in $L^2(0,T;L^2(\Omega,L^2(Y)))$ if there exists subsequence still denoted also u_ϵ, such that, for all Y-periodic function test $\varphi(t,x,y) \in C_0^\infty(0,T;C_0^\infty(\Omega;C_\sharp^\infty(Y)))$, we have :

$$\lim_{\varepsilon\to 0}\int_0^T\int_{\Omega_\epsilon} u_\epsilon(t,x)\varphi(t,x,\frac{x}{\epsilon})dxdt = \frac{1}{\mid Y\mid}\int_0^T\int_\Omega\int_Y u_0(t,x,y)\varphi(t,x,y)dydxdt. \quad (15)$$

In fact, we have the following lemma

Lemma 4.2. *From each bounded sequence u_ϵ the $L^2(0,T;L^2(\Omega_\epsilon))$, we can extract a subsequence and there exists a limit $u_0(t,x,y) \in L^2(0,T;L^2(\Omega,L^2(Y)))$ such that the subsequence two-scale converge to u_0.*

Now, we will show the following convergence result associated to the nonlinear term on the boundary. This result can be used to show the two scale convergence of our problem.

Lemma 4.3. *Let $u_\epsilon \in L^2(0,T,H_0^1(\Omega))$ and $u_0 \in L^2(0,T,L^2(\Omega))$ such that u_ϵ two scale converges to u_0 then*

$$\lim_{\epsilon\to 0}\epsilon\int_0^T\int_{\Gamma_\epsilon} F(u_\epsilon)\phi_\epsilon d\sigma_x dt = \frac{\mid\Gamma\mid}{\mid Y\mid}\int_0^T\int_\Omega F(u_0(t,x))\phi(t,x)dxdt.$$

Proof. Form the Lemma 3.1 and the inequality (13), we have $u_\epsilon \to u_0$ strongly in $L^2((0,T);L^2(\Omega))$. Using the continuity of F we have that $F(u_\epsilon) \to F(u_0)$ strongly in $L^2((0,T);L^2(\Omega))$.

Moreover, since

$$\|\nabla F(u_\epsilon)\|_{L^2((0,T);L^2(\Omega))} = \|\partial_u F(u_\epsilon)\nabla u_\epsilon\|_{L^2((0,T);L^2(\Omega))}$$

is bounded, we deduce the weak convergence of a subsequence $F(u_\epsilon)$ in $L^2(0,T;H^1(\Omega))$ and for any $\phi \in C_0^\infty(0,T;C_0^\infty(\Omega))$ it holds that

$$\phi F(u_\epsilon) \underset{\epsilon\to 0}{\rightharpoonup} \phi F(u_0) \text{ weakly } L^2(0,T;H^1(\Omega)).$$

Put $z_\epsilon = \phi F(u_\epsilon(t))$, according to [7], we have

$$\langle \mu_1^\epsilon, \phi F(u_\epsilon(t))\rangle \to \frac{\mid\Gamma\mid}{\mid Y\mid}\int_\Omega \phi F(u_0(t))dx, \quad (16)$$

for almost every $t \in [0,T]$. Finally, we are in the position to use the Lebesgue's convergence theorem and get

$$\lim_{\epsilon\to 0}\int_0^T\int_{\Gamma_\epsilon} \epsilon F(u_\epsilon)\phi_\epsilon d\sigma_x dt = \frac{\mid\Gamma\mid}{\mid Y\mid}\int_0^T\int_\Omega F(u_0)\phi dxdt.$$

□

Having proved the convergence of the nonlinear term, we are ready to identify the limit problems. This result is stated in the following theorem.

Theorem 4.4. *Let $u_\epsilon \in L^2(0,T;H_0^1(\Omega))$ be the solution of (5). Then, there exist a subsequence denoted again (u_ϵ), $u_0(t,x)$ in $L^2(0,T;H_0^1(\Omega))$ and $u_1(t,x,y)$ in $L^2(0,T;L^2(\Omega;H^1_{\neq}(Y)/\mathbb{R}))$, such that*

- *u_ϵ two scale converges to $u_0(t,x)$ solution of (17).*
- *∇u^s_ϵ two scale converges to $\chi^s(y)(\nabla_x u_0(t,x)+\nabla_y u_1(t,x,y))$*
- *∇u^f_ϵ two scale converges to $\chi^f(y)(\nabla_x u_0(t,x)+\nabla_y u_1(t,x,y))$*

where $\chi^s_\epsilon(x)$ (resp. $\chi^s_\epsilon(x)$) the characteristic function of Ω^s_ϵ (resp. Ω^f_ϵ) and $\chi^s(y)$ (resp. $\chi^f(y)$) that of Y_s (resp. Y_f)

$$\begin{cases} \frac{|Y_f|+|Y_s|}{|Y|}\frac{\partial u_0(t,x)}{\partial t} + \nabla(K^*\nabla u_0(t,x)) + V^*\nabla u_0(t,x) + \\ \qquad\qquad \frac{|\Gamma|}{|Y|}F(u_0(t,x)) = 0, & on\ (0,T)\times\Omega, \\ u_0(t,x) = 0, & in\ (0,T)\times\partial\Omega, \\ u_0(0,x) = u_{in}, & in\ \Omega, \end{cases} \tag{17}$$

and the effective (homogenized) conductivity tensor $K^*(t,x)$ is given by:

$$K^*_{i,j} = \frac{1}{|\,Y\,|}\int_{Y_f} K^f(t,x,y)(e_j+\nabla w^f_j(y))(e_k+\nabla w^f_k(y))dy \tag{18}$$

$$+\frac{1}{|\,Y\,|}\int_{Y_s} K^s(t,x,y)(e_j+\nabla w^s_j(y))(e_k+\nabla w^s_k(y))dy, \quad j,k=1,2,$$

and the homogenized velocity is defined by

$$V^* = \frac{1}{|\,Y\,|}\int_{Y_f} V_0(x,y)dy, \tag{19}$$

where

$$w_j = \begin{cases} w^s_j \ \ in\ \ Y_s, \\ w^f_j \ \ in\ \ Y_f, \end{cases} \tag{20}$$

is solution of the following system:

$$\begin{cases} \nabla(K^s(t,x,y)(e_j+\nabla w^s_j)) = 0, \ in\ Y_s. \\ \nabla(K^f(t,x,y)(e_j+\nabla w^f_j)) = 0, \ in\ Y_f. \\ K^s(t,x,y)(e_j+\nabla w^s_j) = K^f(t,x,y)(e_j+\nabla w^f_j), \ on\ \Gamma \\ w^s_j = w^f_j, \ on\ \Gamma \\ w^f_j \ is\ Y-periodic \\ w^f_j \ is\ Y-periodic. \end{cases} \tag{21}$$

and $u_1(t,x,y)$ denote the first corrector defined by:

$$u_1(t,x,y) = \sum_{j=1}^{2} w_j(y)\frac{\partial u_0(t,x)}{\partial x_j}. \tag{22}$$

Proof. When passing to the limit by the convergence two scale it is necessary to respect the choice of test functions.

Let

$$\phi_\epsilon(t,x) = \phi_0(t,x) + \epsilon\phi_1(t,x,\frac{x}{\epsilon}),$$

where $(\phi_0, \phi_1) \in C_0^\infty(0,T;C_0^\infty(\Omega)) \times C_0^\infty(0,T;C_0^\infty(\Omega, C_\sharp^\infty(Y)))$, and

$$\phi_i = \begin{cases} \phi_i^s \ \text{ in } \ (0,T) \times \Omega_\epsilon^s, \\ \phi_i^f \ \text{ in } \ (0,T) \times \Omega_\epsilon^f, \end{cases}$$

for $i = 0, 1$.

Due to the analogy between u_ϵ^f and u_ϵ^s, the determination of the limit problems will be showed only for u_ϵ^f.

By using a similar technique as in [16], we obtain the following convergence results,

$$\lim_{\epsilon \to 0} \int_0^T \int_{\Omega_\epsilon^f} \frac{\partial u_\epsilon^f(t,x)}{\partial t} \phi_\epsilon^f dxdt = \frac{\mid Y_f \mid}{\mid Y \mid} \int_0^T \int_\Omega \frac{\partial u_0(t,x)}{\partial t} \phi_0(t,x) dxdt,$$

and

$$\begin{aligned} &\lim_{\epsilon \to 0} \int_0^T \int_{\Omega_\epsilon^f} K_\epsilon^f \nabla u_\epsilon^f \nabla \phi_\epsilon^f dxdt \\ &\quad = \frac{1}{|Y|} \int_0^T \int_\Omega \int_{Y_f} K^f(t,x,y) \nabla_x u_0(t,x) (\nabla_x \phi_0(x,t) + \nabla_y \phi_1(t,x,y)) dydxdt \\ &\quad + \frac{1}{|Y|} \int_0^T \int_\Omega \int_{Y_f} K^f(t,x,y) \nabla_y u_1(t,x,y) (\nabla_x \phi_0(x,t) + \nabla_y \phi_1(t,x,y)) dydxdt. \end{aligned}$$

The two scale convergence of V_ϵ to $V_0(x,y)$ can be easily obtained by using the same way as in [2]. Since we have that ∇u_ϵ two scale converges to $\nabla_x u_0 + \nabla_y u_1$, then by using the two scale convergence results stated in [1], we show that

$$V_\epsilon \nabla u_\epsilon^f \rightharpoonup \frac{1}{\mid Y \mid} \int_{Y_f} V_0(x,y)(\nabla_x u_0 + \nabla_y u_1) dy, \ \ in \ \ D'((0,T) \times \Omega),$$

which means that for ϕ_0 in $D((0,T) \times \Omega)$, we get

$$\int_0^T \int_{\Omega_\epsilon^f} V_\epsilon \nabla u_\epsilon^f \phi_0^f(t,x) \to \frac{1}{\mid Y \mid} \int_0^T \int_\Omega \int_{Y_f} V_0(x,y)(\nabla_x u_0 + \nabla_y u_1) \phi_0(t,x) dydxdt.$$

From the Lemma 4.3 and previous results, the obtained limit problem is defined as follow

$$\begin{aligned} 0 &= \frac{\mid Y_f \mid}{\mid Y \mid} \int_0^T \int_\Omega \frac{\partial u_0(t,x)}{\partial t} \phi_0(t,x) dxdt + \frac{\mid Y_s \mid}{\mid Y \mid} \int_0^T \int_\Omega \frac{\partial u_0(t,x)}{\partial t} \phi_0(t,x) dxdt \\ &+ \frac{1}{\mid Y \mid} \int_0^T \int_\Omega \int_{Y_f} K^f(t,x,y)(\nabla_x u_0(t,x) + \nabla_y u_1(t,x,y))(\nabla_x \phi_0(x,t) + \nabla_y \phi_1(t,x,y)) dydxdt \\ &+ \frac{1}{\mid Y \mid} \int_0^T \int_\Omega \int_{Y_s} K^s(t,x,y)(\nabla_x u_0(t,x) + \nabla_y u_1(t,x,y))(\nabla_x \phi_0(x,t) + \nabla_y \phi_1(t,x,y)) dydxdt \\ &+ \frac{1}{\mid Y \mid} \int_0^T \int_\Omega \int_{Y_f} V_0(x,y)(\nabla_x u_0 + \nabla_y u_1) \phi_0(t,x) dydxdt + \frac{\mid \Gamma \mid}{\mid Y \mid} \int_0^T \int_\Omega \int_\Gamma F(u_0) \phi_0(t,x) d\sigma_y dxdt. \end{aligned}$$

It remains to take $\phi_0 = 0$ to obtain the cell problem (21) and to take $\phi_1 = 0$ to obtain the homogenized problem (17). □

The existence and uniqueness result of the problem (21) is obtained thanks to Lax-Milgram result, when the existence and uniqueness result of problem (17) can be showed in similar way as in [4].

References

[1] G. Allaire, Homogenization and two-scale convergence, *SIAM J. Math. Anal.* **23** (1992), no. 6, 1482–1518.

[2] G. Allaire, *Homogénéisation des Équations de Stokes et de Navier-Stokes*, Thèse de doctorat, Université Paris 6, 1989.

[3] G. Allaire, Z. Habibi, Homogenization of a conductive, convective and radiative heat transfer problem in a heterogeneous domain, *SIAM J.Math. Anal.* **45** (2013), no. 3, 1136-1178.

[4] M. Choulli, An inverse problem for a semilinear parabolic equation, *J. Inverse Problems* **10** (1994), 1123–1132.

[5] D. Cioranescu, J.S.J. Paulin, Homogenization in open sets with holes, *Journal of Math. Analysis and Appl.* **71** (1979), no. 2, 590–607.

[6] C. Conca, J.I. Díaz, C. Timofte, Effective chemical processes in porous media, *J. Math. Models Methods Appl. Sci.* **13** (2003), no. 10, 1437–1462.

[7] C. Conca, J.I. Díaz, C.Timofte, On the homogenization of a transmission problem arising in chemistry, *Rom. Rep. Phys.* **56**, (2004), no. 4, 613–622.

[8] P. Donato, A. Nabil, Homogenization of semilinear parabolic equations in perforated domains, *Chin. Ann. Math. Ser.* (**25**) (2004), no. 2, 143–156.

[9] K. El Ganaoui, *Homogénéisation de modéles de transferts thermiques et radiatifs: Application au coeur des réacteurs à caloporteur gaz*, Thèse de doctorat, École Polytechnique, 2006.

[10] P. Goel, J. Sneyd, A. Friedman, Homogenization of the cell cytoplasm: the calcium bidomain equations, *Multiscale Model. Simul.* **5** (2006), no. 4, 1045–1062.

[11] J.L. Lions, E. Magenes, *Problémes aux limites non homogénes (3 volumes)*, Dunod, Paris, 1968.

[12] S. Monsurró, Homogenization of a two component composite with interfacial thermal barrier, *Adv. Math. Sci. Appl.* **13** (2003), no. 1, 43–63.

[13] O.A. Oleinik, *On homogenization problems in Trends and application of pure mathematics in mechanics*, Springer, Berlin, 1984.

[14] D. O'Regan, Y.J. Cho, Y.Q. Chen, *Topological degree theory and applications*, Series in Mathematical Analysis and Applications **10**, Chapman & Hall/CRC Taylor & Francis Group, 2006 .

[15] E. Sanchez-Palencia, *Non-homogenous media and vibration theory*, Lecture Notes in Physics Series **127**, Spinger, Berlin, 1980.

[16] M.A. Peter, M. Böhm, Different choices of scaling in homogenization of diffusion and interfacial exchange in a porous medium, *Math. Methods Appl. Sci.* **31** (2008), no. 11, 1257–1282.

[17] E. Zeidler, *Non lininear functional analysis and its application. III Variational methods and optimization*, Springer, Berlin, 1984.

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