Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 42(1), 2015, Pages 158–166
ISSN: 1223-6934

Iterative method for computing a Schur form of symplectic matrix

A. Mesbahi, A.H. Bentbib, A. Kanber, and F. Ellaggoune

Abstract. We present in this paper a constructive iterative method to compute an orthogonal and symplectic Schur form. This structured Schur form reduction is used to compute eigenvalues and invariant subspaces of symplectic matrices.

Key words and phrases. Symplectic matrix, Symplectic reflector, ortho-SR decomposition, Schur decomposition.

1. Introduction

The computation of the eigenvalues and eigenvectors of matrices play important roles in many applications in the physical sciences. For example, they play a prominent role in image processing applications. Measurement of image sharpness can be done using the concept of eigenvalues [6]. In this paper we are interesting in a structured eigenvalue problem. We propose a practical method to compute eigenvalues and invariant subspaces for symplectic matrices witch are of particular importance in applications such as optimal control [7, 8]. Symplectic matrices are widely used in SR factorization witch is a principal step in structure-preserving methods. It is a long-standing open problem to compute the eigenvalues and the Schur form in particular the one how preserved the structure [3, 4]. In the case of an eigenvalue problem of structured matrices, the preservation of this structure may help exploit the symmetry of the problem and to improve the accuracy and efficiency calculations of invariant subspaces and eigenvalues [5]. Our proposed method here is based on an orthogonal and symplectic SR factorization based on orthogonal and symplectic reflectors. Orthogonality is used to preserve the stability and Symplecticity is used to preserve the structure. In paragraph 3, we give and prove an orthogonal and symplectic Schur form theorem for real symplectic matrix. An algorithm that compute this structured Schur form reduction is given and numerical experimental results are presented to illustrate the effectiveness of our approach.

2. Terminology, notation and some basic facts

In this section we collect several well-known properties of structured matrices, for future reference.

An ubiquitous matrix in this work is the skew-symmetric matrix
$J_{2n} = \begin{pmatrix} O_n & I_n \\ -I_n & O_n \end{pmatrix}$, where I_n and O_n are the $n \times n$ identity and zero matrix, respectively. Note that $J_{2n}^{-1} = J_{2n}^T = -J_{2n}$. In the following, we will drop the

This paper has been presented at Congrès MOCASIM, Marrakech, 19-22 November 2014.

subscript n and $2n$ whenever the dimension of corresponding matrix is clear from its context. The J-transpose of any $2n$-by-$2p$ matrix M is defined by $M^J = J_{2p}^T M^T J_{2n} \in \mathbb{R}^{2p\times 2n}$. Any matrix $S \in \mathbb{R}^{2n\times 2p}$ satisfying the property $S^T J_{2n} S = J_{2p}$ ($S^J S = I_{2p}$) is called symplectic matrix. This property is also called J-orthogonality. The symplectic similarity transformations preserve structured matrix.

Remark 2.1. An augmented matrix

$$S = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & P_{11} & 0 & P_{12} \\ 0 & 0 & I & 0 \\ 0 & P_{21} & 0 & P_{22} \end{pmatrix}$$

is symplectic if and only if the matrix $P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$ is symplectic.

2.1. Symplectic reflectors. Setting $E_i = [e_i \; e_{n+i}] \in \mathbb{R}^{2n\times 2}$ for $i = 1, \cdots, n$ where e_i is the i-th element of the canonical basis of $\mathbb{R}^{2n}$. We then obtain

$$E_i^J = E_i^T \text{ and } E_i^J E_j = \delta_{ij} I_2$$

where $\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$

We recall the symplectic reflector on $\mathbb{R}^{2n\times 2}$ (see [1, 2]), which is defined in parallel with elementary reflectors.

Proposition 2.1. *Let U, V be two $2n$-by-2 real matrices satisfying $U^J U = V^J V = I_2$. If the 2-by-2 matrix $C = I_2 + V^J U$ is nonsingular, then, the transformation $S = (U+V)C^{-1}(U+V)^J - I_{2n}$ is symplectic and takes U to V. It's called a symplectic reflector. Additionally, if $U^J = U^T$ and $V^J = V^T$, then S is orthogonal and symplectic.*

Proposition 2.2. *Let $u \in \mathbb{R}^{2n}$ be a nonzero $2n$-component real vector. The reflector*

$$S = (U + \sqrt{\alpha}E_1)(\alpha I_2 + \sqrt{\alpha}E_1^J U)^{-1}(U + \sqrt{\alpha}E_1)^J - I_{2n}$$

where $U = [u \;\; -Ju]$, is orthogonal and symplectic and verify $Su = \sqrt{\alpha}e_1$ where $\alpha = u^T u = \|u\|_2^2$.

Proof. Since $U^J U = \alpha I_2$ with $\alpha = u^T u = \|u\|_2^2 > 0$, then a simple calculation gives the result. □

2.2. Symplectic matrices.

Proposition 2.3. *If $M \in \mathbb{R}^{2n\times 2n}$ is symplectic then M^T is symplectic.*

Proof. Since

$$\begin{aligned} MJM^T &= MJ(M^T JM)M^{-1}J^{-1} \\ &= MJJM^{-1}J^{-1} \\ &= -I_n J^{-1} \\ &= J \\ M^T JM = J \Rightarrow MJM^T &= MJ(M^T JM)M^{-1}J^{-1} = \\ &= MJJM^{-1}J^{-1} = -I_n J^{-1} = J. \end{aligned}$$

This proves that M^T is symplectic. □

Proposition 2.4. *If* $M = \begin{pmatrix} A & C \\ B & D \end{pmatrix} \in \mathbb{R}^{2n\times 2n}$ *where* $A, B, C, D \in \mathbb{R}^{n\times n}$*, then* M *is symplectic if and only if the matrices* A, B, C, D *verify :*

$$\begin{cases} A^TB \text{ and } C^TD \text{ are symmetric,} \\ A^TD - B^TC = I_n. \end{cases}$$

Proof. We have

$$M^TJM = \begin{pmatrix} B^TA - A^TB & B^TC - A^TD \\ D^TA - C^TB & D^TC - C^TD \end{pmatrix}.$$

Then

$$\begin{aligned} M \text{ is symplectic} \quad & \Leftrightarrow \begin{cases} B^TA = A^TB \text{ and } D^TC = C^TD \\ A^TD - B^TC = D^TA - C^TB = I_n \end{cases} \\ & \Leftrightarrow \begin{cases} A^TB \text{ and } C^TD \text{ are symmetric,} \\ A^TD - B^TC = I_n. \end{cases} \end{aligned}$$

□

Theorem 2.5. *A matrix* $M \in \mathbb{R}^{2n\times 2n}$ *is orthogonal and symplectic if and only if* M *is in the following form*

$$M = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$$

where $A, B \in \mathbb{R}^{n\times n}$ *verify*

$$\begin{cases} A^TB \text{ is symmetric,} \\ A^TA + B^TB = I_n \end{cases}$$

Proof. If $M = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$ is orthogonal and symplectic, where $A, B, C, D \in \mathbb{R}^{n\times n}$, then

$$J = MJM^T = MJM^{-1}.$$

That proves $JM = MJ$. Then $C = -B$ and $D = A$. By the result of Proposition 2.4, $A, B \in \mathbb{R}^{n\times n}$ verify

$$\begin{cases} A^TB \text{ is symmetric,} \\ A^TA + B^TB = I_n \end{cases}$$

□

2.3. Ortho-*SR* decomposition. By using ortho-symplectic reflectors defined above, we decompose a symplectic real $2n$-by-$2n$ matrix A on the form $A = SR$ where $S \in \mathbb{R}^{2n\times 2n}$ is orthogonal and symplectic and R is symplectic and is in the following form

$$R \;=\; \begin{array}{c} \\ n \\ \\ n \end{array} \left(\begin{array}{cccc|cccc} \multicolumn{4}{c|}{\overleftrightarrow{\quad n \quad}} & \multicolumn{4}{c}{\overleftrightarrow{\quad n \quad}} \\ \star & \star & \cdots & \star & * & * & \cdots & * \\ 0 & * & \ddots & \vdots & * & * & \ddots & \vdots \\ \vdots & \ddots & \ddots & * & \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & * & * & \cdots & * & * \\ \hline 0 & 0 & \cdots & 0 & * & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots & * & * & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & * & \cdots & * & * \end{array}\right). \tag{1}$$

Algorithm : Ortho-symplectic SR-decomposition

Hereafter, Matlab notations are used
Input : $A \in \mathbb{R}^{2n\times 2n}$ a symplectic matrix
Output: A $2n \times 2n$ orthogonal and symplectic matrix S and a symplectic matrix R in the form (1) such that $A = SR$
For k=1 to n **do**
  $u := A(:,k); E_k = [e_k, -Je_k]; I = eye(2n);$
  $I_k = blkdiag(-eye(k), eye(n-k), -eye(k), eye(n-k))$
  For j=1 to k-1 **do**
    $u(j) = 0; u(n+j) = 0$
  end For
  If ($\|u\| \neq 0$) **then**
    $u = \dfrac{u}{\|u\|}$
    $U = (u, -J\ u)$
    $C = I - 2E_k^J\ U$
    If ($\det C \neq 0$) **then**
      $T = (U+E)C^{-1}(U+E_k)^J - I_k$
    else
      $C = I + 2E^J\ U$
      $T = (U+E)C^{-1}(U+E_k)^J + I$
    end If
    $S := ST; A := TA$
  end If
end For

3. Ortho-symplectic Schur form

Lemma 3.1. *If A is a symplectic matrix and $u \in \mathbb{C}^{2n}$ an eigenvector corresponding to an eigenvalue λ of A. Suppose that λ is outside the unit circle, then there exists an unitary and symplectic reflector S such that $Su = \alpha e_1$.*

Proof. Let u be an eigenvector corresponding to an eigenvalue λ of A such that $|\lambda| \neq 1$ and $\|u\| = 1$.

$U = [u, -J\overline{u}] \in \mathbb{C}^{2n\times 2}$ is unitary and symplectic ($U^H U = U^J U = I_2$). Indeed,

$$U^J U = J^T U^H J U = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u^H J u & u^H u \\ -u^H u & u^H J u \end{pmatrix} = \begin{pmatrix} \|u\|^2 & u^H J u \\ -u^H J u & \|u\|^2 \end{pmatrix}.$$

The matrix A is symplectic, then $A^H J A = J$ and,

$$u^H J u = \frac{1}{|\lambda|^2}(Au)^H J A u = \frac{1}{|\lambda|^2} u^H J u.$$

Since $|\lambda| \neq 1$, then $u^H J u = 0$. Let set $N = E_1^J U$. If -1 is not eigenvalue of the 2-by-2 matrix N, then the reflector $S = (U + E_1)(I_2 + E_1^J U)^{-1}(U + E_1)^J - I_{2n}$ is unitary and symplectic and verify $SU = \alpha E_1$. Then $Su = \alpha e_1$(here $\alpha = 1$). If -1 is eigenvalue of N, then 1 is not eigenvalue of $E_1^J U$. Indeed, we have $E_1^J U = \begin{pmatrix} e_1^T u & -e_1^T J\overline{u} \\ e_{n+1}^T u & -e_{n+1}^T J\overline{u} \end{pmatrix} = \begin{pmatrix} u_1 & -\overline{u_{n+1}} \\ u_{n+1} & \overline{u_1} \end{pmatrix}$ then $\det(E_1^J U) = |u_1|^2 + |u_{n+1}|^2 \geq 0$. The reflector $S = (U - E_1)(I_2 - E_1^J U)^{-1}(U - E_1)^J - I_{2n}$ is unitary and symplectic and verify $SU = \alpha E_1$ and then $Su = \alpha e_1$ (here $\alpha = -1$). $\square$

Theorem 3.2. *Let A be a $2n$-by-$2n$ real symplectic matrix such that all eigenvalues of A are outside the unit circle. Then there exists an unitary and symplectic matrix Q such that $A = QRQ^T$ where R is symplectic and is in the following form*

$$R = \left(\begin{array}{cccc|cccc}
\star & \star & \cdots & \star & * & * & \cdots & * \\
0 & * & \ddots & \vdots & * & * & \ddots & \vdots \\
\vdots & \ddots & \ddots & * & \vdots & \ddots & \ddots & * \\
0 & \cdots & 0 & * & * & \cdots & * & * \\
\hline
0 & 0 & \cdots & 0 & * & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots & * & * & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & 0 & * & \cdots & * & *
\end{array}\right).$$

Proof. The proof can be seen by using the principe of induction on n.

Let λ be an eigenvalue of A and u a normalized corresponding eigenvector. $U = [u, -Ju] \in \mathbb{R}^{2n\times 2}$ is unitary and symplectic ($U^H U = U^J U = I$). From Lemma 3.1 there exists an unitary and symplectic reflector S_1 such that $S_1 u = \alpha e_1$ and then

$$S_1 A S_1^T e_1 = \lambda e_1.$$

The matrix $R_1 = S_1 A S_1^H$ is symplectic and it is in the following form

$$S_1 A S_1^H = R_1 = \left(\begin{array}{cccc|cccc}
\lambda & * & \cdots & * & * & * & \cdots & * \\
0 & * & \ddots & \vdots & * & * & \ddots & \vdots \\
\vdots & \vdots & \ddots & * & \vdots & \ddots & \ddots & * \\
0 & * & \cdots & * & * & \cdots & * & * \\
\hline
0 & 0 & \cdots & 0 & \frac{1}{\lambda} & 0 & \cdots & 0 \\
0 & * & \ddots & \vdots & * & * & \cdots & * \\
\vdots & \vdots & \ddots & * & \vdots & \ddots & \ddots & * \\
0 & * & \cdots & * & * & \cdots & * & *
\end{array}\right)$$

$$= \left(\begin{array}{cc|cc}
\lambda & v^T & x & w^T \\
0 & A & X & B \\
\hline
0 & 0 & y & 0 \\
0 & C & Y & D
\end{array}\right)$$

The matrix $R_2 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is also symplectic. Indeed,

$$R_1 = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}, \; A', B', C', D' \in \mathbb{R}^{n \times n}$$

We have

$$R_1{}^T J R_1 = \begin{pmatrix} C'^T A' - A'^T C' & C'^T B' - A'^T D' \\ D'^T A' - B'^T C' & D'^T B' - B'^T D' \end{pmatrix}.$$

And

$$R_2{}^T J R_2 = \begin{pmatrix} C^T A - A^T C & C^T B - A^T D \\ D^T A - B^T C & D^T B - B^T D \end{pmatrix}.$$

By block multiplication, we have

$$C'^T A' - A'^T C' = \begin{pmatrix} 0 & O \\ 0 & C^T A - A^T C \end{pmatrix}, C'^T B' - A'^T D' = \begin{pmatrix} -1 & 0 \\ X & C^T B - A^T D \end{pmatrix}.$$

$$D'^T B' - B'^T D' = \begin{pmatrix} x & X \\ X & D^T B - A^T D \end{pmatrix}, D'^T A' - B'^T C' = \begin{pmatrix} 1 & X \\ X & D^T A - B^T C \end{pmatrix}.$$

R_1 is symplectic then $R_1{}^T J R_1 = J$ and

$$\begin{aligned} C'^T A' - A'^T C' &= 0, & C'^T B' - A'^T D' &= -I, \\ D'^T A' - B'^T C' &= I \;\textit{and} & D'^T B' - B'^T D' &= 0. \end{aligned}$$

Consequently

$$\begin{aligned} C^T A - A^T C &= 0, & C^T B - A^T D &= -I, \\ D^T A - B^T C &= I \;\textit{and} & D^T B - B^T D &= 0 \end{aligned}$$

That proves that R_2 is symplectic (see, Proposition 2.4).
By induction hypothesis there exist unitary and symplectic reflector, $\widetilde{S_2}$, such that $\widetilde{S_2} R_2 \widetilde{S_2}^T = \widetilde{R_2}$,

$$\widetilde{R_2} = \left(\begin{array}{c|c} \begin{array}{c} \overset{n-1}{\longleftrightarrow} \\ n-1\left\{\begin{array}{cccc} * & * & \cdots & * \\ 0 & * & \ddots & \vdots \\ \vdots & \vdots & \ddots & * \\ 0 & 0 & \ldots & 0 \end{array}\right. \end{array} & \begin{array}{c} \overset{n-1}{\longleftrightarrow} \\ \begin{array}{cccc} * & * & \cdots & * \\ * & * & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ * & \cdots & * & * \end{array} \end{array} \\ \hline \begin{array}{c} \overset{n-1}{\longleftrightarrow} \\ n-1\left\{\begin{array}{cccc} 0 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \ldots & 0 \end{array}\right. \end{array} & \begin{array}{cccc} * & 0 & \cdots & 0 \\ * & * & \ldots & 0 \\ \vdots & \ddots & \ddots & 0 \\ * & \cdots & * & * \end{array} \end{array}\right)$$

Let

$$\widetilde{S_2} = \begin{array}{cc} & \begin{array}{cc} \overset{n-1}{\longleftrightarrow} \qquad\qquad & \qquad\qquad \overset{n-1}{\longleftrightarrow} \end{array} \\ \begin{array}{c} n-1 \\ \\ \\ \\ n-1 \end{array} & \left(\begin{array}{cccc|cccc} s^{(1)}_{1,1} & \cdots & & s^{(1)}_{1,n-1} & s^{(1)}_{1,n} & \cdots & & s^{(1)}_{1,2n-1} \\ s^{(1)}_{2,1} & \cdots & & s^{(1)}_{2,n-1} & s^{(1)}_{2,n} & \cdots & & s^{(1)}_{2,2n-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ s^{(1)}_{n-1,1} & \cdots & & s^{(1)}_{n-1,n-1} & s^{(1)}_{n-1,n} & \cdots & & s^{(1)}_{n-1,2n-1} \\ \hline s^{(1)}_{n,1} & \cdots & & s^{(1)}_{n,n-1} & s^{(1)}_{n,n} & \cdots & & s^{(1)}_{n,2n-1} \\ s^{(1)}_{n+1,1} & \cdots & & s^{(1)}_{n+1,n-1} & s^{(1)}_{n+1,n} & \cdots & & s^{(1)}_{n+1,2n-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ s^{(1)}_{2n-1,1} & \cdots & & s^{(1)}_{2n-1,n-1} & s^{(1)}_{2n-1,n} & \cdots & & s^{(1)}_{2n-1,2n-1} \end{array}\right) \end{array}$$

We take

$$S_2 = \begin{array}{cc} & \begin{array}{cc} \overset{n}{\longleftrightarrow} \qquad\qquad & \qquad\qquad \overset{n}{\longleftrightarrow} \end{array} \\ \begin{array}{c} n \\ \\ \\ \\ \\ n \end{array} & \left(\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & s^{(1)}_{1,1} & \cdots & s^{(1)}_{1,n-1} & 0 & s^{(1)}_{1,n} & \cdots & s^{(1)}_{1,2n-1} \\ 0 & s^{(1)}_{2,1} & \cdots & s^{(1)}_{2,n-1} & 0 & s^{(1)}_{2,n} & \cdots & s^{(1)}_{2,2n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & s^{(1)}_{n-1,1} & \cdots & s^{(1)}_{n-1,n-1} & 0 & s^{(1)}_{n-1,n} & \cdots & s^{(1)}_{n-1,2n-1} \\ \hline 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & s^{(1)}_{n,1} & \cdots & s^{(1)}_{n,n-1} & 0 & s^{(1)}_{n,n} & \cdots & s^{(1)}_{n,2n-1} \\ 0 & s^{(1)}_{n+1,1} & \cdots & s^{(1)}_{n+1,n-1} & 0 & s^{(1)}_{n+1,n} & \cdots & s^{(1)}_{n+1,2n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & s^{(1)}_{2n-1,1} & \cdots & s^{(1)}_{2n-1,n-1} & 0 & s^{(1)}_{2n-1,n} & \cdots & s^{(1)}_{2n-1,2n-1} \end{array}\right) \end{array}$$

Then S_2 is unitary and symplectic and verify $S_2 S_1 A S_2^H S_1^H = R$ were R is symplectic and in the following form

$$R = \begin{array}{cc} & \begin{array}{cc} \overset{n}{\longleftrightarrow} \qquad & \qquad \overset{n}{\longleftrightarrow} \end{array} \\ \begin{array}{c} n \\ \\ \\ \\ n \end{array} & \left(\begin{array}{cccc|cccc} \star & \star & \cdots & \star & * & * & \cdots & * \\ 0 & * & \ddots & \vdots & * & * & \ddots & \vdots \\ \vdots & \ddots & \ddots & * & \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & * & * & \cdots & * & * \\ \hline 0 & 0 & \cdots & 0 & * & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots & * & * & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & * & \cdots & * & * \end{array}\right) \end{array}$$

$\square$

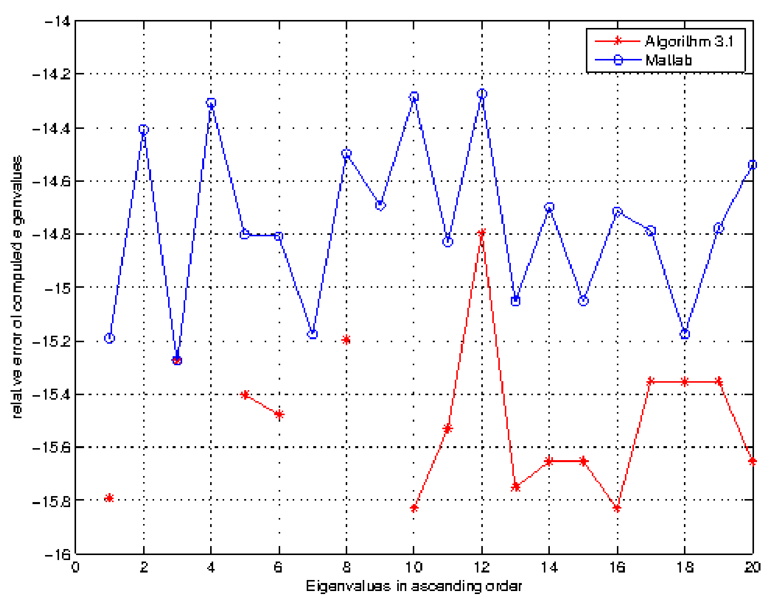
3.1. Algorithm 3.1. Algorithm : Iterative method to compute Schur form of symplectic matrix

Input : $A \in \mathbb{R}^{2n\times 2n}$ Symplectic , $X_0 \in \mathbb{R}^{2n\times n}$
Output: Ortho-symplectic Schur form of A
$V = X_0$
Repeat
  $W = AV$
  $W = SR$ **ortho-symplectic** SR **decomposition**
  $V = S[:, 1:n]$
  $RR = R[1:n, 1:n]$
until (Convergence)

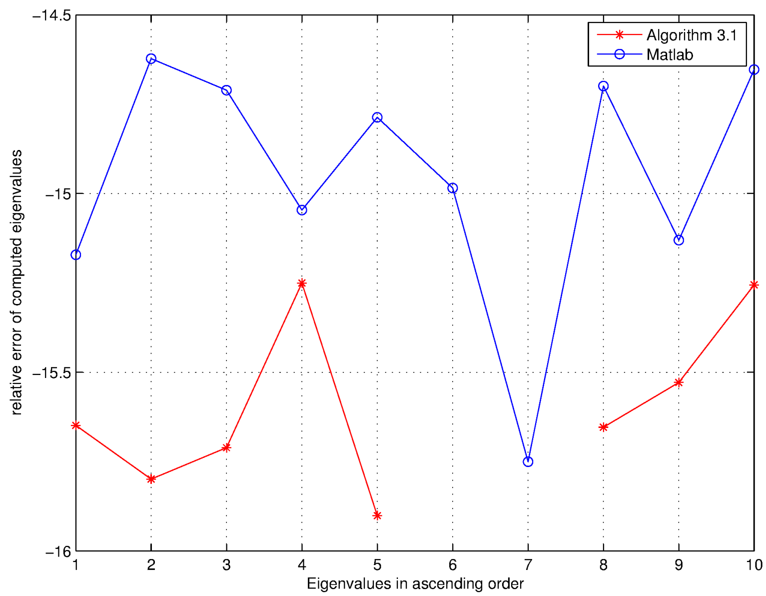
3.2. Numerical examples. We compared and tested the numerical results obtained by Algorithm 3.1 with Matlab *eig* function. Our numerical experiments were carried out with Matlab (R2009a) and run it on a Core Duo Pentium processor. The symplectic matrix A is obtained from the matrix $\left(\begin{array}{c|c} M & M \\ \hline 0 & M^{-T}\end{array}\right)$ where $M = diag(v)$ and by using symplectic similarity transformations randomly generated by symplectic reflectors as :

$$A = S\left(\begin{array}{c|c} M & M \\ \hline 0 & M^{-T}\end{array}\right)S^J.$$

Example 3.1. In this example $n = 20$ and v is constructed as follows.
For $k = 1$ to $fix(n/2)$ (Here Matlab notation is used)
$v(k) = k + 1$ *and* $v(\frac{n}{2} + k) = k^{-1}$



Example 3.2. In this example $n = 10$ and v is constructed as follows.
For $k = 1$ to $fix(n/2)$ (Here Matlab notation is used)
$v(k) = \frac{1}{k+1}$ *and* $v(\frac{n}{2} + k) = (\sqrt{\frac{k}{1000}})^{-1}$



References

[1] S. Agoujil and A. H. Bentbib, On the reduction of Hamilotonian matrices to a Hamiltonian Jordan canonical form, *Int. Jour. Math. Stat. (IJMS)* **4** (2009), 12–37.

[2] S. Agoujil and A. H. Bentbib, New symplectic transformation on $\mathbb{C}^{2n\times 2}$: Symplectic reflectors, *Int. Jour. of Tomography and Statistics (IJTS)* **11** (2009), 99–117.

[3] P. Benner, V. Mehrmann, and H. Xu, A new method for computing the stable invariant subspace of a real Hamiltonian matrix, *J. Comput. Appl. Math.* **86** (1997), no. 1, 17–43.

[4] P. Benner, V. Mehrmann, and H. Xu, A numerically stable, structure preserving method for computing the eigenvalues of real Hamiltonian or symplectic pencils, *Numer. Math.* **78** (1998), 329–358.

[5] M. Dosso and M. Sadkane, A spectral trichotomy method for symplectic matrices, *Numer. Algor.* **52** (2009), 187–212.

[6] H. Gaidhane, V. Hote, and V.Singh, A New Approach for Estimation of Eigenvalues of Images, *IJCA (0975 - 8887)* **26** (2011), no. 9, 1–6.

[7] I. Gohberg, P. Lancaster, and L. Rodman, *Indefinite Linear Algebra and Applications*, Birkhauser Verlag, Basel, 2005.

[8] P. Lancaster and L. Rodman, *Algebraic Riccati Equations*, Clarendon Press, Oxford, 1995.

(A. Mesbahi, F. Ellaggoune) Department of Mathematics and Informatics, University of Souk Ahras, Algeria
E-mail address: amerbrahim@gmail.com

(A.H. Bentbib, A. Kanber) Department of Mathematics, Cadi Ayyad University, Faculty of Science and Technology, Marrakech 42000, Morocco
E-mail address: a.bentbib@uca.ma, kanber@uca.ma