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On the identification of discontinuous matrix diffusion in elliptic equation

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Abstract. The aim of this paper is to study the identification a discontinuous matrix diffusion parameter in the elliptic partial differential equation considered with mixed non-homogenous boundary conditions on a boundary of bounded open subset domain in two dimensional space. This parameter is taken as a matrix valued on bounded variation space. The observation can be partially or globally given in the domain into consideration. We reformulate the associated inverse problem to an optimization one, we prove the existence of solution and we study the discrete case by using finite element method and we expose a result of the convergence of the solution of the discrete problem to continuous one. We describe an optimization algorithm and the numerical results are discussed in the end.

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1. Introduction

In this paper we are concerned with the mathematical analysis and the numerical approximation for the identification of diffusion parameter when dealing with linear elliptic equation defined on a bounded open set $\Omega \subset \mathbb{R}^d$, $d = 1, 2$ whose boundary Γ is assumed to be Lipschitz and partitioned into two parts Γ_1 and Γ_2 such that $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$, with different boundary condition on each part. More precisely, we consider the following problem

$$\left\{ \begin{array}{l} \text{Given an observation } z \text{ of } u, \text{ find } (\mathcal{A}, u) \text{ such that} \\ \left\{ \begin{array}{rll} -\nabla \cdot (\mathcal{A}\nabla u) & = f & \text{in} \quad \Omega \\ u & = g & \text{on} \quad \Gamma_1 \\ \alpha u + \mathcal{A}\nabla u \cdot n & = h & \text{on} \quad \Gamma_2 \end{array} \right. \end{array} \right. \tag{1}$$

where n is the outer unit normal, f, g, h and α are the given functions.

The equation (1) is a model problem for many industrial applications particularly on underground water investigations [15, 16] or medical imaging [17, 18] and many other applications. The desire to identify parameters have motivated several approaches in the literature [2, 3, 4, 5, 6, 7, 8, 9]. The augmented Lagrangian method is frequently [8, 6] applied for parameter identifying problem. In [9] the aforementioned method is combined with the level-set method. Other methods are used, namely, the conjugate gradient method [7] and the fast approximate inference method based on expectation propagation for exploring the posterior probability distribution arising from the Bayesian formulation of inverse problems in [1]. The regularization [3, 10] and the theoretical analysis [5] of the inverses problems is the object of many others works.

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In this work, we will treat a general case of the function $\mathcal{A}$ where $\mathcal{A}$ is the matrix diffusion to be identified in an appropriate space. The originality of this paper is to extend the work of many authors for this type of inverse problem. For this, we are interesting to identify the discontinuous matrix diffusion $\mathcal{A} : \Omega \to \mathbb{R}^{d\times d}$ in some subspace of $L^1(\Omega, \mathbb{R}^{d\times d})$ described later. In order, to investigate even non-smooth matrix diffusion parameter we propose to take it in the space of functions of bounded variation noted $BV(\Omega)$ as the space of all functions in $L^1(\Omega, \mathbb{R}^{d\times d})$ whose distributional derivative belongs to the space of matrix valued measures:

$$BV(\Omega) = \left\{\mathcal{A} \in L^1(\Omega, \mathbb{R}^{d\times d}) : D\mathcal{A} \in \mathcal{M}(\Omega, \mathbb{R}^{d\times d})\right\} \tag{2}$$

equipped with the norm $\|\mathcal{A}\|_{BV(\Omega)} = \int_\Omega \|\mathcal{A}(x)\|_{\mathbb{R}^{d\times d}}\, dx + \int_\Omega |D\mathcal{A}|$,
where $\|B\|_{\mathbb{R}^{d\times d}} = trace(B^tB)$, for all matrix $B \in \mathbb{R}^{d\times d}$ and

$$\int_\Omega |D\mathcal{A}| = \sup\left\{\int_\Omega \mathcal{A}\nabla g dx; \quad g \in [\mathcal{C}_0^1(\Omega)]^{d\times d\times d},\ \text{and } |g| \le 1 \text{ in } \Omega\right\}.$$

So that, we elaborate the existence of solution to the inverse problem under consideration with weak hypothesis. Concerning the minimizing problem, we consider the cost functional as much general to cover a large class of observations.

The contributions of this paper are detailed in the sections that follows according to this plan, in the next section, we elaborate the continuous problem, by formulating the optimization problem. We also establish the optimality conditions and give some examples of observations to be considered. In the third section, we derive the discrete problem by finite element discretization and we prove its convergence to the continuous one. A description of the algorithm considered based on the nonlinear and linear conjugate gradient method and we present, in the fourth section, the numerical results to illustrate the convergence of algorithm.

2. The inverse continuous problem

Let K a nonempty closed convex subset of $BV(\Omega)$ defined as follows

$$K = \left\{\mathcal{A} \in BV(\Omega) : k_1|z|^2 \le \mathcal{A}(x)z\cdot z \le k_2|z|^2 \quad \forall x \in \Omega, \forall z \in \mathbb{R}^d\right\} \tag{3}$$

It is well known that if we take as a cost functional an integral of the form

$$\int_\Omega L\left(\mathcal{A}(x), u(x)\right) dx$$

then, in general, an optimal configuration does not exist. However, the addition of a regularization term of the form of $BV(\Omega)$ semi-norm of the parameter to be estimated, here $\mathcal{A}$, is enough to imply the existence of classical minimum, see for example [3, 10]. In other words, if we take as a cost the functional

$$J(\mathcal{A}, u) = \int_\Omega L(\mathcal{A}(x), u(x))dx + \beta \int_\Omega |D\mathcal{A}| \tag{4}$$

where $\beta > 0$. Hence, the inverse problem (1) can be formulated as follows

$$\begin{cases} \min\limits_{(\mathcal{A},u)\in K\times V} J(\mathcal{A}, u) \\ \displaystyle\int_\Omega \mathcal{A}(x)\nabla u(x)\nabla v(x)dx + \int_{\Gamma_2} \alpha uv d\sigma - \int_\Omega f(x)v(x)dx - \int_{\Gamma_2} hv d\sigma = 0,\ \forall v \in V_0. \end{cases} \tag{5}$$

Here V_0 is a subset of $H^1(\Omega)$ defined by $V_0 = \left\{v \in H^1(\Omega) : v_{|\Gamma_1} = 0\right\}$. The functional L can take different expressions depending on the type of observation. The observation

$z \in Z$ can be defined on the entire domain Ω or just on some regions of it. In the case when z is defined on Ω, a typical function L is of the form

$$L(\mathcal{A}, u) = \int_{\Omega} \mathcal{A}\nabla(u - z)\nabla(u - z)dx.$$

In the case when the observation z is defined only in a part $\Omega_z \subset \Omega$, the functional L can be formulated in the terms

$$L(\mathcal{A}, u) = \int_{\Omega_z} (u - z)^2 \, dx$$

and when the measurements are available only on a portion $\gamma \subset \Gamma$ of the boundary of Ω, the functional L can be defined by

$$L(\mathcal{A}, u) = \int_{\gamma} (u - z)^2 \, d\sigma.$$

In the general case, we suppose that our observation is defined in $\Omega_z \subset \bar{\Omega}$ and that $L : K \times \mathbb{R} \to \mathbb{R}$ is a C^1-Caratheodory function satisfying appropriate growth conditions that will be shown to be as

$$L(\mathcal{A}(x), u(x)) \geq \gamma_1(x) - \xi_1 |u(x)|^2 \quad \text{for almost } x \in \Omega_z \tag{6}$$

$$L(\mathcal{A}(x), u(x)) \leq \gamma_2(x) + \xi_2 |z(x)|^2 \quad \text{for almost } x \in \Omega_z \tag{7}$$

for some $\gamma_1, \gamma_2 \in L^1(\Omega_z)$ and $\xi_1, \xi_2 \geq 0$.
We introduce an operator $\mathcal{E}$ from $K \times V$ into V_0 defined for all $v \in V_0$ by

$$\begin{aligned}\int_{\Omega} \nabla\mathcal{E}(\mathcal{A}, u)\nabla v dx \quad &= \quad \int_{\Omega} \mathcal{A}(x)\nabla u(x)\nabla v(x)dx + \int_{\Gamma_2} \alpha u v d\sigma \\ &\quad - \int_{\Omega} f(x)v(x)dx - \int_{\Gamma_2} h v d\sigma.\end{aligned}$$

We start by studying the existence of a solution for the constraint optimization problem (5).

Theorem 2.1. *The problem* (5) *has at least one solution* $(\bar{\mathcal{A}}, \bar{u}) \in K \times V$.

Proof. Let $A_{ad} := \{(\mathcal{A}, u) \in K \times V : \mathcal{E}(\mathcal{A}, u) = 0\}$ be the admissible set of the constraint optimization problem (5). It is well known, see for example [20, 21], that the direct problem has at least one solution $u \in V$ for all $\mathcal{A} \in K$. This implies that A_{ad} is nonempty. From the assumption (6), there exists a minimizing sequence $\{(\mathcal{A}_n, u_n)\}_{n \geq 1} \in A_{ad}$ such that

$$\lim_{n \to \infty} J(\mathcal{A}_n, u_n) = \inf_{(\mathcal{A}, u) \in K \times V} J(\mathcal{A}, u). \tag{8}$$

For each $n > 0$ we have $J(\mathcal{A}_n, u_n) \leq C$, then by definition of J and the fact that $\mathcal{E}(\mathcal{A}_n, u_n) = 0$ we have

$$\|u_n\|_{H^1(\Omega)} \leq C \quad \textit{and} \quad \|\mathcal{A}_n\|_{BV(\Omega)} \leq C. \tag{9}$$

Therefore, taking a subsequence if necessary, we can assume that there exists a pair $(\mathcal{A}, u) \in BV(\Omega) \times V$ such that u_n converges weakly to u in V, and $\mathcal{A}_n$ converges strongly to $\mathcal{A}$ in $L^1(\Omega, \mathbb{R}^{d \times d})$. Since $(\mathcal{A}_n, u_n) \in A_{ad}$, we have

$$\int_{\Omega} \mathcal{A}_n(x)\nabla u_n(x)\nabla v(x)dx + \int_{\Gamma_2} \alpha u v d\sigma - \int_{\Gamma_2} h v d\sigma - \int_{\Omega} f(x)v(x)dx = 0, \ \forall v \in V_0 \tag{10}$$

therefore,

$$\begin{aligned}
&\left|\int_\Omega \mathcal{A}_n(x)\nabla u_n(x)\nabla v(x)dx - \int_\Omega \mathcal{A}(x)\nabla u(x)\nabla v(x)dx\right| \\
&\quad \leq \left|\int_\Omega (\mathcal{A}_n(x)-\mathcal{A}(x))\nabla u_n(x)\nabla v(x)dx\right| + \left|\int_\Omega \mathcal{A}(x)\nabla(u_n(x)-u(x))\nabla v(x)dx\right| \\
&\quad \leq \left(\int_\Omega |\mathcal{A}_n(x)-\mathcal{A}(x)||\nabla u_n(x)|^2dx\right)^{\frac{1}{2}} \left(\int_\Omega |\mathcal{A}_n(x)-\mathcal{A}(x)||\nabla v(x)|^2dx\right)^{\frac{1}{2}} \\
&\qquad + \left|\int_\Omega \mathcal{A}(x)\nabla(u_n(x)-u(x))\nabla v(x)dx\right|
\end{aligned}$$

using the fact that K is bounded in $BV(\Omega)$, $\|u_n\|_{H^1(\Omega)} \leq C$ and the Lebesgue dominant convergence, we obtain

$$\int_\Omega \mathcal{A}_n(x)\nabla u_n(x)\nabla v(x)dx \text{ converges to } \int_\Omega \mathcal{A}(x)\nabla u(x)\nabla v(x)dx \qquad \text{for all } v \text{ in } V_0.$$

This convergence implies that $\mathcal{E}(\mathcal{A},u)=0$. Thus $(\mathcal{A},u)$ is a feasible solution of (5). Let us prove that it is a solution. Since u_n converges to u strongly in $L^2(\Omega)$ we can take a subsequence in such a way that $u_n(x)$ converges to $u(x)$ for almost all $x\in\Omega$. We set $l_n(x)=L(\mathcal{A}_n(x),u_n(x))+\xi_1|u_n|^2-\gamma_1(x)$ and $l(x)=L(\mathcal{A}(x),u(x))+\xi_1|u|^2-\gamma_1(x)$. Then $l_n(x)$ converges to $l(x)$ almost everywhere and $l_n\geq 0$ for $n\geq 1$. Therefore, by applying Fatou's Lemma and the lower semi-continuity of the BV-norm, we get

$$\begin{aligned}
J(\mathcal{A},u) &= \int_\Omega l(x)dx - \int_\Omega \left(\xi_1|u|^2-\gamma_1(x)\right)dx + \beta\int_\Omega |D\mathcal{A}|dx \\
&\leq \liminf_{n\to\infty}\left(\int_\Omega l_n(x)dx - \int_\Omega \left(\xi_1|u_n|^2-\gamma_1(x)\right)dx + \beta\int_\Omega |D\mathcal{A}_n|dx\right) \\
&\leq \liminf_{n\to\infty}\left(\int_\Omega L(\mathcal{A}_n,u_n)dx + \beta\int_\Omega |D\mathcal{A}_n|dx\right) \\
&\leq \liminf_{n\to\infty} J(\mathcal{A}_n,u_n).
\end{aligned}$$

Then $(\mathcal{A},u)=(\bar{\mathcal{A}},\bar{u})$ is the solution of the constraint optimization problem (5). □

Next, we derive the optimality condition for the constraint optimization problem (5). Thus we introduce the augmented Lagrangian functional defined for any given constant $r\geq 0$ by :

$$\begin{array}{llll}
\mathcal{L}_r: & K\times V\times V & \to & \mathbb{R} \\
& (\mathcal{A},u,\lambda) & \mapsto & J(\mathcal{A},u) + \displaystyle\int_\Omega \nabla\lambda(x)\nabla\mathcal{E}(\mathcal{A}(x),u(x))dx \\
& & & +\dfrac{r}{2}\left\|\nabla\mathcal{E}(\mathcal{A}(x),u(x))\right\|^2_{L^2(\Omega)}
\end{array} \tag{11}$$

The following theorem shows that the constraint optimization problem (5) is equivalent to the saddle-point problem associated with the Lagrangian functional $\mathcal{L}_r$.

Theorem 2.2. *If $(\bar{\mathcal{A}},\bar{u})\in K\times V$ is a solution of the constraint optimization problem (5), then there exists $\bar{\lambda}\in V_0$ such that $(\bar{\mathcal{A}},\bar{u},\bar{\lambda})\in K\times V\times V_0$ is saddle-point of the Lagrangian $\mathcal{L}_0(r=0)$.*

Proof. Let $(\bar{\mathcal{A}},\bar{u})\in K\times V$ a solution of the constraint optimization problem (5). We start this proof by verifying that the functional

$$\begin{array}{llll}
\mathcal{E}'(\bar{\mathcal{A}},\bar{u}): & K\times V & \to & V_0 \\
& (\mathcal{A},u) & \mapsto & (-\Delta)^{-1}\nabla\cdot\left(\bar{\mathcal{A}}\nabla u+\mathcal{A}\nabla\bar{u}\right)
\end{array}$$

is surjective. For any $\mathcal{A} \in K$ and $w \in V_0$, there exists $u \in V$ satisfying the equation $-\nabla \cdot \left(\bar{A}\nabla u\right) = \Delta w + \nabla \cdot (\mathcal{A}\nabla \bar{u})$ in $H^{-1}(\Omega)$ and $(-\Delta)^{-1}\left(\nabla \cdot \left(\bar{A}\nabla u + \mathcal{A}\nabla \bar{u}\right)\right) = w$ in V_0. Then, $\mathcal{E}'(\bar{\mathcal{A}}, \bar{u})$ is surjective. As the functional $(u, v) \mapsto \int_\Omega \nabla u \nabla v dx$ is an isomorphism in $V \times V$, the surjectivity of $\mathcal{E}'(\bar{\mathcal{A}}, \bar{u})$ implies that the functional $b(\mathcal{A}, u, \lambda) := \displaystyle\int_\Omega \nabla \left\langle \mathcal{E}'(\bar{\mathcal{A}}, \bar{u}), (\mathcal{A}, u)\right\rangle \nabla\lambda dx$ is surjective. Moreover, there exists [19], $k > 0$ such that

$$\sup_{u\in V} \frac{b(\mathcal{A}, u, \lambda)}{\|u\|_{H^1(\Omega)}} \geq k \|\lambda\|_{H^1(\Omega)} \quad \forall \lambda \in V_0. \tag{12}$$

In the following, we demonstrate the existence of $\bar{\lambda} \in V_0$ such that

$$\left\langle \nabla J(\bar{\mathcal{A}}, \bar{u}); (\mathcal{A}, u)\right\rangle + b(\mathcal{A}, u, \bar{\lambda}) = 0 \quad \forall (\mathcal{A}, u) \in K \times V \tag{13}$$

We take $F(u) = \left\langle \nabla J(\bar{\mathcal{A}}, \bar{u}), (\mathcal{A}, u)\right\rangle$, $B(u, \lambda) = b(\mathcal{A}, u, \lambda)$ and we define

$$\mathcal{V} = \{u \in V : B(u, \lambda) = 0 \quad \forall \lambda \in V_0\}.$$

Thus, for every $u \in \mathcal{V}$, we have $F(u) = 0$. Then, the equation (13) is equivalent to find $\lambda \in V_0$ such that

$$F(u) = -B(u, \lambda) \quad \forall u \in \mathcal{V}^\perp \tag{14}$$

For $\lambda \in V_0$ fixed, the linear form $u \mapsto B(u, \lambda)$ is continuous in $\mathcal{V}^\perp$. From the Riesz representation, there exists a unique $T_\lambda \in \mathcal{V}^\perp$ such that

$$\begin{aligned} &B(u, \lambda) = (T_\lambda, u) \quad \forall u \in \mathcal{V}^\perp \\ &\|T_\lambda\|_V = \|b(\cdot, \cdot, \lambda)\|_{(\mathcal{V}^\perp)'} = \sup_{u\in\mathcal{V}^\perp} \frac{b(A, u, \lambda)}{\|u\|_{H^1(\Omega)}} \end{aligned} \tag{15}$$

this defines a continuous linear operator T on V.
If $Im(T) := \{T_\lambda; \quad \lambda \in V_0\} = \mathcal{V}^\perp$, then, we have $F \in (\mathcal{V}^\perp)'$. From the Riesz representation, there exists $v \in \mathcal{V}^\perp$ such that $F(u) = (v, u) \quad \forall u \in \mathcal{V}^\perp$. As $Im(T) = \mathcal{V}^\perp$, there exists $\lambda \in V_0$ such that $T_\lambda = -v$ and we have

$$B(u, \lambda) = (T_\lambda, u) = -(v, u) = -F(u) \quad \forall u \in \mathcal{V}^\perp.$$

Remains to show that $Im(T) = \mathcal{V}^\perp$. First, we demonstrate that $Im(T)$ is closed in $\mathcal{V}^\perp$. Let $\{\lambda_n\}_{n\geq 1} \in V$ such that $T_{\lambda_n} \to v \in \mathcal{V}^\perp$. The sequence $\{T_{\lambda_n}\}_{n>0}$ is a sequence of Cauchy in $\mathcal{V}^\perp$. Using the condition (12), we get that $\{\lambda_n\}_n$ is also a sequence of Cauchy and $\lambda_n \to \lambda$ in V_0. From the continuity of T, we have $T_\lambda = v$. Thus, $Im(T)$ is closed in $\mathcal{V}^\perp$. Obviously, we have $Im(T)^\perp = \{0\}$, then we conclude that $Im(T) = \mathcal{V}^\perp$. □

It is clear that any saddle-point of $\mathcal{L}_0$ is a saddle-point of $\mathcal{L}_r, r > 0$, and that if $(\bar{\mathcal{A}}, \bar{u}, \bar{\lambda}) \in K \times V \times V_0$ is saddle-point of the augmented Lagrangian $\mathcal{L}_r$ then $(\bar{\mathcal{A}}, \bar{u}) \in K \times V$ is a solution of the constraint optimization problem (5). Then, the consequence of the previous theorem is the equivalence between the constraint optimization problem and the saddle-point problem.

3. The inverse discrete problem

The aim of this section is to formulate the discrete problem by using a finite element discretization, in the first step. In the second step, the aim is to establish the estimations of error for the proposed formulation. Let $\{\tau_h\}_{h>0}$ be a family of regular triangulations of the domain Ω. Denote by P_h the standard piecewise liner finite element space over the triangulation τ_h:

$$P_h = \left\{ a_h \in \mathcal{C}^0(\overline{\Omega}),\ :\ \forall T \in \tau_h, \quad a_{h_{|_T}} \in \mathbb{P}^1(T) \right\}$$

and $V_h = P_h \cap V$, $\quad V_h^0 = P_h \cap V_0$, $\quad K_h = \{\mathcal{A}_h \in K : (\mathcal{A}_{i,j})_{1\le i,j\le d} \in P_h\}$. For any $(\mathcal{A}_h, u_h) \in K_h \times V_h$ we define $\mathcal{E}_h(\mathcal{A}_h, u_h)$ the discrete version of the operator $\mathcal{E}(\mathcal{A}, u)$ as a solution of the discrete variational problem

$$\begin{aligned} \int_\Omega \nabla \mathcal{E}_h(\mathcal{A}_h, u_h)\nabla v_h dx \quad = \quad & \int_\Omega \mathcal{A}_h(x)\nabla u_h(x)\nabla v_h(x)dx + \int_{\Gamma_2} \alpha u_h v_h d\sigma \\ & - \int_\Omega f(x)v_h(x)dx - \int_{\Gamma_2} h v_h d\sigma, \quad \forall v_h \in V_h^0. \end{aligned} \tag{16}$$

In implementations, we approximate the regularization functional by $\sqrt{|\nabla \mathcal{A}_h|^2 + \varepsilon h^2}$ and we always take ε small to avoid dividing zero numbers. Then, the discrete augmented Lagrangian $\mathcal{L}_h^r$ is defined by

$$\begin{aligned} \mathcal{L}_h^r : \quad K_h \times V_h \times V_h^0 \quad & \to \quad \mathbb{R} \\ (\mathcal{A}_h, u_h, \lambda_h) \quad & \mapsto \quad J_h(\mathcal{A}_h, u_h) + \int_\Omega \nabla \lambda_h(x) \nabla \mathcal{E}_h(\mathcal{A}_h(x), u_h(x))dx \\ & \qquad + \tfrac{r}{2} \left\| \nabla \mathcal{E}_h(\mathcal{A}_h, u_h) \right\|_{L^2(\Omega)}^2 \end{aligned} \tag{17}$$

with, $J_h(\mathcal{A}_h, u_h) = \displaystyle\int_\Omega L(\mathcal{A}_h, u_h)dx + \beta \int_\Omega \sqrt{|\nabla A_h|^2 + \varepsilon h^2}dx.$

As in the proof for the continuous saddle-point problem of the previous section, we can prove the existence of the Lagrangian multiplier $\bar{\lambda}_h \in V_h$ satisfying

$$\mathcal{L}_h^r(\bar{\mathcal{A}}_h, \bar{u}_h, \lambda_h) \le \mathcal{L}_h^r(\mathcal{A}_h, \bar{u}_h, \bar{\lambda}_h) \le \mathcal{L}_h^r(\mathcal{A}_h, u_h, \bar{\lambda}_h), \quad \forall (\mathcal{A}_h, u_h, \lambda_h) \in K_h \times V_h \times V_h^0.$$

In the next theorem, we show that the saddle-point of the discrete problem converges to that of the continuous one.

Theorem 3.1. *If $\{(\bar{\mathcal{A}}_h, \bar{u}_h, \bar{\lambda}_h)\}_{h>0} \in K_h \times V_h \times V_h^0$ is a sequence of the saddle-point of the discrete augmented Lagrangian $\mathcal{L}_h^r$. Then, there exists a subsequence that converges strongly in $L^1(\Omega, \mathbb{R}^{d\times d}) \times L^2(\Omega) \times L^2(\Omega)$ to some saddle-point $\{(\bar{\mathcal{A}}, \bar{u}, \bar{\lambda})\} \in K_h \times V_h \times V_h^0$ of the augmented Lagrangian $\mathcal{L}_r$.*

Proof. Let $(\bar{\mathcal{A}}_h, \bar{u}_h, \bar{\lambda}_h) \in K_h \times V_h \times V_h^0$ be the saddle-point of $\mathcal{L}_h^r$. Then, $\forall (\mathcal{A}_h, u_h) \in K_h \times V_h$ we have

$$J_h(\bar{\mathcal{A}}_h, \bar{u}_h) \le J_h(\mathcal{A}_h, u_h) + \int_\Omega \nabla \bar{\lambda}_h \nabla \mathcal{E}(\mathcal{A}_h, u_h)dx + \frac{r}{2} \left\| \nabla \mathcal{E}(\mathcal{A}_h, u_h) \right\|_{L^2(\Omega)}^2 \tag{18}$$

By letting $\mathcal{A}_h = kId$, and $u_h \in V_h$ the unique solution of the equation

$$\int_\Omega \nabla u_h(x) \nabla v_h(x)dx + \int_{\Gamma_2} \frac{\alpha}{k} u_h v_h d\sigma = \int_\Omega \frac{1}{k} f(x)v_h(x)dx + \int_{\Gamma_2} \frac{1}{k} h v_h d\sigma \quad \forall v_h \in V_h^0$$

We deduce from (18) that $\|\bar{\mathcal{A}}_h\|_{BV(\Omega)} + \|\bar{u}_h\|_{H^1(\Omega)} \le C$. Taking $\mathcal{A}_h = \bar{\mathcal{A}}_h$ in (18), we get $\forall u_h \in V_h$

$$\begin{aligned}
&\int_\Omega \gamma_1(x)dx - \xi_1 \int_\Omega \bar{u}_h^2 dx \leq J(\bar{\mathcal{A}}_h, u_h) + \int_\Omega \bar{\mathcal{A}}_h \nabla u_h(x) \nabla \bar{\lambda}_h(x) dx \\
&\quad - \int_\Omega f(x)\bar{\lambda}_h(x)dx + \int_{\Gamma_2} \alpha u_h \bar{\lambda}_h d\sigma - \int_{\Gamma_2} h\bar{\lambda}_h d\sigma + \frac{r}{2} \left\| \nabla \mathcal{E}_h(\bar{\mathcal{A}}_h, u_h) \right\|^2_{L^2(\Omega)} \\
&\leq \int_\Omega \bar{\mathcal{A}}_h \nabla u_h(x) \nabla \bar{\lambda}_h(x) dx + C_1 \left\| \nabla \bar{\lambda}_h \right\|^2_{L^2(\Omega)} \\
&\quad + \frac{C}{C_1} \left(\|f\|^2_{H^{-1}(\Omega)} + \|h\|^2_{L^2(\Gamma_2)} \right) + C \left(\|\nabla u_h\|^2_{L^2(\Omega)} + \|u_h\|^2_{L^2(\Gamma_2)} + \xi_2 \|z\|^2_Z \right)
\end{aligned}$$

with $C_1 > 0$. By taking $u_h = -\varepsilon\bar{\lambda}_h + \tilde{g} \in V_h$ with $\tilde{g} \in H^1(\Omega)$ is an extension of g in $H^1(\Omega)$, for some constant $\varepsilon > 0$ and $C_1 = \frac{\xi_1 \varepsilon}{2}$, we write

$$\begin{aligned}
\int_\Omega \gamma_1(x)dx + \frac{\xi_1\varepsilon}{2} \left\| \nabla \bar{\lambda}_h \right\|^2_{L^2(\Omega)} \quad \leq \quad & C \Big(\varepsilon^2 \left\| \nabla \bar{\lambda}_h \right\|^2_{L^2(\Omega)} + \varepsilon^2 \|\nabla \tilde{g}\|^2_{L^2(\Omega)} + \tfrac{1}{\varepsilon} \|f\|^2_{H^{-1}(\Omega)} \\
& + \frac{1}{\varepsilon} \|h\|^2_{L^2(\Gamma_2)} + \|z\|^2_Z + \|\bar{u}_h\|^2_{L^2(\Omega)} \Big)
\end{aligned}$$

An adequate choice of ε and the boundedness of $\{\bar{u}_h\}_h$ give $\left\| \nabla \bar{\lambda}_h \right\|^2_{L^2(\Omega)} \leq C$. Therefore, the sequence $\{(\bar{\mathcal{A}}_h, \bar{u}_h, \bar{\lambda}_h)\}_{h>0} \in K_h \times V_h \times V_h$ has a subsequence, still denoted by $\{(\bar{\mathcal{A}}_h, \bar{u}_h, \bar{\lambda}_h)\}_{h>0}$, such that

$$\bar{u}_h \rightharpoonup u \quad \text{in} \quad V, \quad \bar{\mathcal{A}}_h \to \mathcal{A} \quad \text{in} \quad L^1(\Omega) \quad \text{and} \quad \bar{\lambda}_h \rightharpoonup \lambda \quad \text{in} \quad V_0 \tag{19}$$

with $(\mathcal{A}, \bar{u}, \bar{\lambda}) \in K \times V \times V_0$.
Using same argument as in the proof of Theorem 2.1, we establish that $\mathcal{E}_h(\bar{\mathcal{A}}_h, \bar{u}_h)$ converges weakly to $\mathcal{E}(\bar{\mathcal{A}}, \bar{u})$ in V_0. Thus, $\mathcal{L}_r(\bar{\mathcal{A}}, \bar{u}, \lambda) \leq \mathcal{L}_r(\bar{\mathcal{A}}, \bar{u}, \bar{\lambda})$, for all λ in V_0.
Now we take in (18) $(\mathcal{A}_h, u_h) = (I_h\tilde{\mathcal{A}}_\varepsilon, R_h u) \in K_h \times V_h$ with,

$$\left(\tilde{\mathcal{A}}_\varepsilon\right)_{1\leq i,j\leq d} = \begin{cases} k_1 & if \quad (\mathcal{A}_\varepsilon)_{1\leq i,j\leq d} < k_1 \\ (\mathcal{A}_\varepsilon)_{1\leq i,j\leq d} & if \quad k_1 \leq (\mathcal{A}_\varepsilon)_{1\leq i,j\leq d} \leq k_2 \\ k_2 & if \quad (\mathcal{A}_\varepsilon)_{1\leq i,j\leq d} > k_2 \end{cases}$$

where, $\mathcal{A}_\varepsilon \in \mathcal{C}^\infty(\bar{\Omega}, \mathbb{R}^{d\times d})$ is given by the density of $\mathcal{C}^\infty(\bar{\Omega}, \mathbb{R}^{d\times d})$ in $W^{1,1}(\Omega, \mathbb{R}^{d\times d})$ and the approximation property of functions with BV (see, [12, 11]) and satisfies

$$\int_\Omega \|\mathcal{A}_\varepsilon(x) - \mathcal{A}(x)\|_{\mathbb{R}^{d\times d}} dx < \varepsilon \quad , \quad \left| \int_\Omega \|\nabla \mathcal{A}_\varepsilon\|_{\mathbb{R}^{d\times d}} dx - \int_\Omega |D\mathcal{A}| \right| < \varepsilon$$

The operators $I_h : \mathcal{C}(\bar{\Omega}) \to V_h$ and $R_h : V \to V_h^0$ are, respectively, the standard nodal value interpolation and the projection operator.
From the relation (18), we get

$$\begin{aligned}
J_h(\bar{\mathcal{A}}_h, \bar{u}_h) \leq J_h(I_h\tilde{\mathcal{A}}_\varepsilon, R_h u) + \int_\Omega \nabla \bar{\lambda}_h \nabla \mathcal{E}_h((I_h\tilde{\mathcal{A}}_\varepsilon, R_h u) dx \\
+ \frac{r}{2} \left\| \nabla \mathcal{E}_h((I_h\tilde{\mathcal{A}}_\varepsilon, R_h u) \right\|^2_{L^2(\Omega)}
\end{aligned} \tag{20}$$

then, we can argue as in the proof of the Theorem 2.1, by taking

$$l_h(x) = L(\bar{\mathcal{A}}_h, \bar{u}_h(x)) + \xi_1|\bar{u}_h|^2 - \gamma_1(x) \text{ and } l(x) = L(\bar{\mathcal{A}}(x), \bar{u}(x)) + \xi_1|\bar{u}|^2 - \gamma_1(x)$$

and using the lower semi-continuity of the BV-norm (see, [12, 11]), to obtain

$$J(\bar{\mathcal{A}}, \bar{u}) \leq \liminf_{h\to 0} J_h(\bar{\mathcal{A}}_h, \bar{u}_h) \tag{21}$$

Using the convergences $\lim_{h\to 0} I_h\tilde{\mathcal{A}}_\varepsilon = \tilde{\mathcal{A}}_\varepsilon$ in $W^{1,1}(\Omega)$ and $\lim_{h\to 0} R_h u = u$ in V we obtain as in the proof of theorem (2.1), $\lim_{h\to 0} \mathcal{E}_h(I_h\tilde{\mathcal{A}}_\varepsilon, R_h u) = \mathcal{E}(\tilde{\mathcal{A}}_\varepsilon, u)$, in V_0. Therefore, tending $h \to 0$ in (20) and using (21) we obtain

$$J(\bar{\mathcal{A}}, \bar{u}) \le J(\tilde{\mathcal{A}}_\varepsilon, u) + \int_\Omega \nabla\bar{\lambda}\nabla\mathcal{E}(\tilde{\mathcal{A}}_\varepsilon, u)dx + \frac{r}{2}\left\|\nabla\mathcal{E}(\tilde{\mathcal{A}}_\varepsilon, u)\right\|^2_{L^2(\Omega)} \tag{22}$$

From the definition of $\tilde{\mathcal{A}}_\varepsilon \in K$ we have $\tilde{\mathcal{A}}_\varepsilon \to \mathcal{A}$ in $L^1(\Omega, \mathbb{R}^{d\times d})$. This implies that $\lim_{\varepsilon\to 0} \mathcal{E}_h(\tilde{\mathcal{A}}_\varepsilon, u) = \mathcal{E}(\mathcal{A}, u)$ in V_0 and $\int_\Omega |\nabla\tilde{\mathcal{A}}_\varepsilon|dx \le \int_\Omega |D\mathcal{A}|dx + \varepsilon$. We finally get $J(\bar{\mathcal{A}}, \bar{u}) \le \mathcal{L}_r(\mathcal{A}, u, \bar{\lambda})$ for any $(\mathcal{A}, u) \in K \times V$, by tending $\varepsilon \to 0$ in (22) and using the fact that L is Caratheodory with respect to the first variable. $\square$

4. Algorithm and numerical results

This section is dedicated to some numerical implementation issues. To find the saddle-points of the discrete augmented Lagrangian $\mathcal{L}_h^r$ described in last section, we use a variant of Uzawa algorithm. The proposed algorithm is summarized in the following.

Algorithm

Choose $\lambda_h^0 \in V_h^0$, $\mathcal{A}_h^0 \in K_h$ and $r, \rho > 0$.

For $n \ge 0$, assume that λ_h^n and $\mathcal{A}_h^n$ are known, compute $\lambda_h^{n+1}, \mathcal{A}_h^{n+1}$, u_h^{n+1} as follows:

(1) Set $k = 0$, $\mathcal{A}^{n,0} = \mathcal{A}^{n-1}$.

(2) For $k \ge 0$ assuming that λ_h^n and $\mathcal{A}_h^{n,k}$ are known, compute $u_h^{n,k+1}$ and $\mathcal{A}_h^{n,k+1}$ as follows:
  - (a) Find $u_h^{n,k+1} \in V_h$ such that $u_h^{n,k+1} = \arg\min_{v_h\in V_h} \mathcal{L}_h^r(\mathcal{A}_h^{n,k}, v_h, \lambda^n)$.
  - (b) Find $\mathcal{A}_h^{n,k+1} \in K_h$ such that $\mathcal{A}_h^{n,k+1} = \arg\min_{\mathcal{B}_h\in K_h} \mathcal{L}_h^r(\mathcal{B}_h, u_h^{n,k}, \lambda^n)$.
  - (c) If $\|\mathcal{A}_h^{n,k+1} - \mathcal{A}_h^{n,k}\|_2 \le tol$, take $u_h^{n+1} = u_h^{n,k+1}$ and $\mathcal{A}_h^{n+1} = \mathcal{A}_h^{n,k+1}$; else do $k+1 \to k$ and return to (a).

(3) Update the multiplier as $\lambda_h^{n+1} = \lambda_h^n + \rho r\mathcal{E}_h(\mathcal{A}_h^{n+1}, u_h^{n+1})$.
Do $n+1 \to n$ and return to 1.

For fixed $\mathcal{A}_h$ and λ_h, the Lagrangian $\mathcal{L}_h^r$ is linear with respect to u_h then to solve the problem described in step (a) of proposed algorithm, we use, at each iteration, linear gradient conjugate method. However, the regularization term is nonlinear with respect to $\mathcal{A}_h$, and we treat numerically the step (b) by the use nonlinear conjugate gradient algorithm combined with Wolfe algorithm for the search of line direction.

In order to illustrate the performance of the proposed numerical method, we solve our inverse problem when the analytical solution can be given by

$$u(x, y) = \sin(\pi x)\sin(\pi y)$$

in two dimensional domain $\Omega =]0, 1[\times]0, 1[$. The cost functional is given by $J(\mathcal{A}, u) = J_0(\mathcal{A}, u) + \beta\int_\Omega |D\mathcal{A}|$ with $J_0(\mathcal{A}, u) = \int_\Omega \mathcal{A}\nabla(u-z)\nabla(u-z)dx$.
The source term f, g, h are constructed from the given analytical solution u and the coefficient $\mathcal{A}$ to be identified.

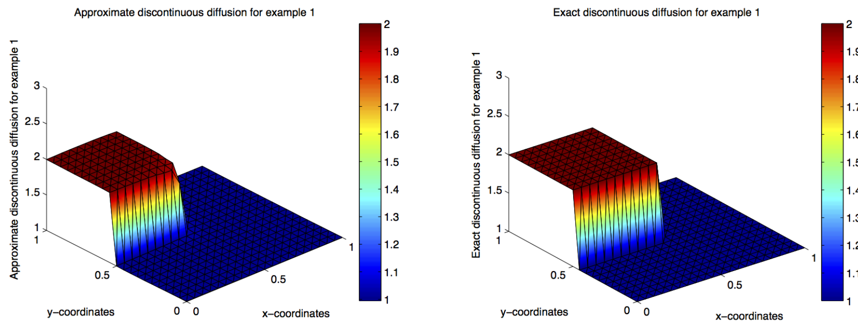

FIGURE 1. The exact solution (right) and the numerically identified solution (left).

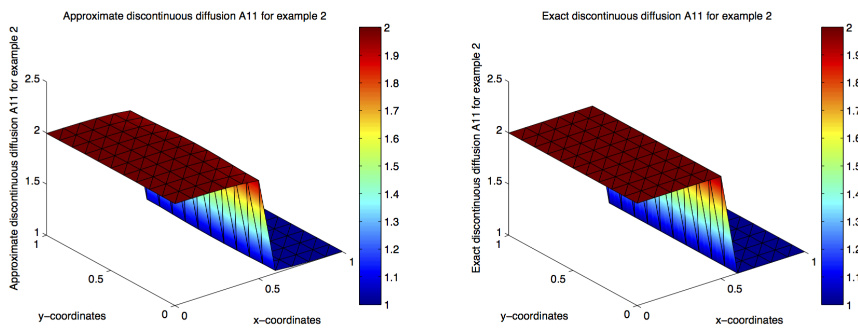

FIGURE 2. The exact solution (right) and the numerically identified solution (left) of $\mathcal{A}_{11}$.

Example 4.1. In this example we consider the case when the discontinuous parameter coefficient is

$$\mathcal{A}(x,y) = \begin{cases} 2 & \text{if} \quad 0 \le x, y \le 0.5 \\ 1 & \text{if} \quad 0.5 < x, y \le 1 \end{cases}$$

In Figure 1 we have plotted the analytical and the calculated discontinuous parameter. From this figure it can be seen that the final parameter coincides with the exact parameter.

Example 4.2. In the second example, we discuss the case when the discontinuous matrix is a diagonal matrix coefficient. The components are given by

$$\mathcal{A}_{11}(x,y) = \begin{cases} 2 & \text{if} \quad y \le 0.5 \\ 1 & \text{if} \quad y > 0.5 \end{cases} \quad ; \quad \mathcal{A}_{22}(x,y) = \begin{cases} 3 & \text{if} \quad y \le 0.5 \\ 2 & \text{if} \quad y > 0.5 \end{cases}$$

We have plotted in Figures 2, 3 the analytical and the calculated discontinuous diagonal matrix. We observe that the discrepancy between the calculated and exact discontinuous diagonal matrix is about 1.4% at convergence.

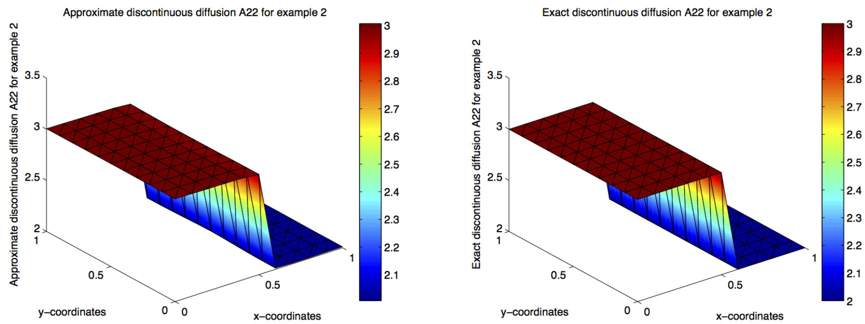

FIGURE 3. The exact solution (right) and the numerically identified solution (left) of $\mathcal{A}_{22}$.

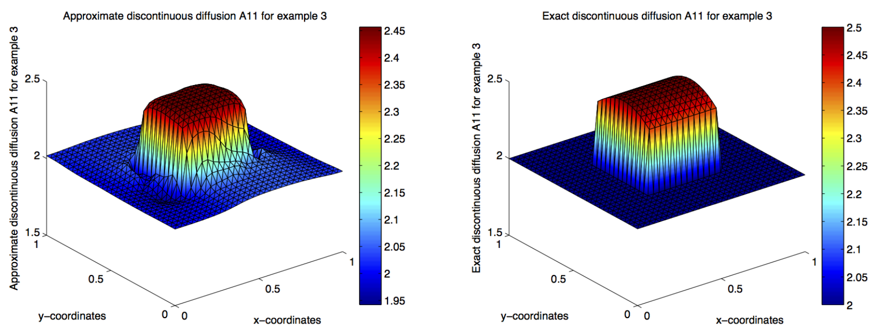

FIGURE 4. small The exact solution (right) and the numerically identified solution (left) of $\mathcal{A}_{11}$.

Example 4.3. We end the examples by the identification of a discontinuous symmetric matrix coefficient. For this, we take

$$\mathcal{A}_{11}(x,y) = \begin{cases} 2 + 0.5\sin(\pi x) & \text{if } |x-0.5| \le 0.2 \text{ and } |y-0.5| \le 0.2 \\ 2 & \text{otherwise} \end{cases};$$

$$\mathcal{A}_{22}(x,y) = \begin{cases} 3 + 0.3\sin(\pi y) & \text{if } |x-0.5| \le 0.2 \text{ and } |y-0.5| \le 0.2 \\ 3 & \text{otherwise} \end{cases};$$

$$\mathcal{A}_{12}(x,y) = \begin{cases} 1.25 & \text{if } (x-0.5)^2 + (y-0.5)^2 \le 0.3^2 \\ 1 & \text{otherwise} \end{cases}$$

Figures 4 - 5 show the approximate and the exact solution. We observe that the discrepancy between the calculated and exact discontinuous symmetric matrix is about 6.3% at final iteration.

5. Conclusion

The discontinuous diffusion matrix is considered as an element of bounded variation space, The mathematical analysis is presented taking into account different cases of obtained observations. Additional regularization term is introduced to ensure existence of an optimal diffusion parameter. We have proposed an algorithm based on uzawa method. Several examples have been implemented and have underlined

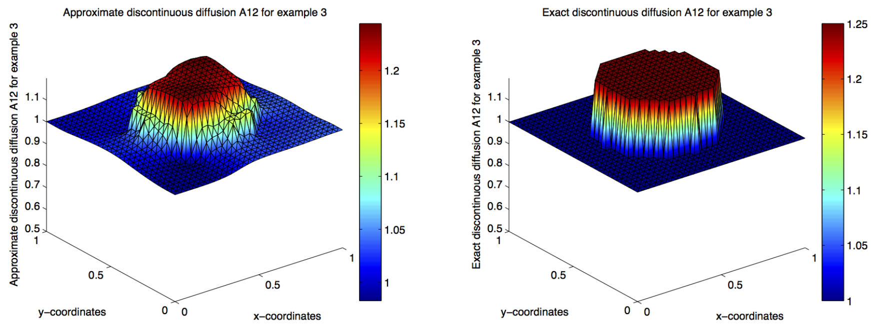

FIGURE 5. The exact solution (right) and the numerically identified solution (left) of $\mathcal{A}_{12}$.

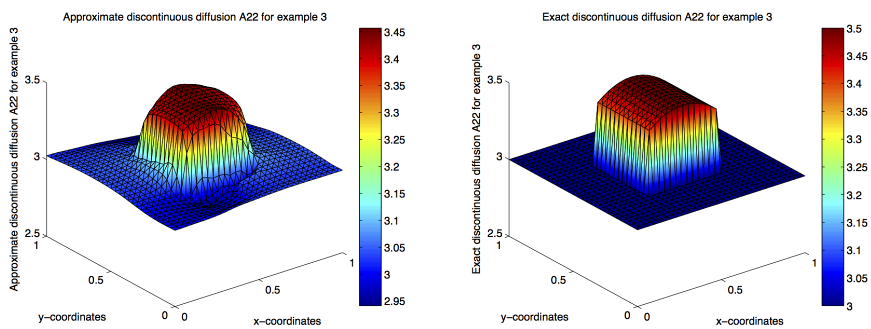

FIGURE 6. The exact solution (right) and the numerically identified solution (left) of $\mathcal{A}_{22}$.

the good approximation of the discontinuous exact diffusion matrix with some given observation.

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