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Stabilization of variable coefficients Euler-Bernoulli beam equation with a tip mass controlled by combined feedback forces

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Abstract. In this paper, we consider stability of a vibrating beam system clamped at one end, controlled by combined forces, with a mass attached at the other end. By adopting the Riesz basis approach, it is shown that the closed-loop system is a Riesz spectral system. Consequently, the exponential stability, spectrum-determined growth condition, and optimal decay rate are obtained. A numerical simulation of the spectrum is also presented.

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1. Introduction

In this paper, we consider stability of a vibrating beam system clamped at one end, controlled by combined forces, with a tip mass attached at the other end. This system can be described by the following Euler-Bernoulli beam equation:

$$\begin{cases} \rho(x)y_{tt}(x,t) + (EI(x)y_{xx}(x,t))_{xx} = 0, & 0 < x < 1,\ t > 0, \\ y(0,t) = y_x(0,t) = 0, & t > 0, \\ EI(1)y_{xx}(1,t) = -\gamma y_{xt}(1,t), & t > 0, \\ (my_{tt} - (EI(.)y_{xx})_x)(1,t) = (-\alpha y_t + \beta(EI(.)y_{xx})_{xt})(1,t) & t > 0, \\ y(x,0) = y_0(x)\ ,\ y_t(x,0) = y_1(x) & 0 < x < 1, \end{cases} \tag{1}$$

where y is the amplitude of the vibration, m is the tip mass attached to the free end of the beam, x, t stand respectively for the position and time, $\rho(x)$ is the mass density of the beam and $EI(x)$ is its flexural rigidity, α, β, γ are constants feedback gains. Our problem is to prove that the solutions of the resulting closed-loop system decay uniformly to zero and the optimal decay rate can be determined by the spectrum of the closed-loop system.

The constant coefficient version of (1), $\rho = EI = 1$ has been investigated in [7], say, where the former shows that all of the generalized eigenfunctions of (1) form a Riesz basis for the state Hilbert space and the exponential stability is obtained from the spectrum of the system. Also, when $\gamma = 0$, Conrad and Morgül in [2] show the exponential stability of the system by the energy multiplier method for any $\alpha, \beta > 0$. In the case where $m = \alpha\beta$, their study leads to show that a set of generalized eigenfunctions of system (1) forms a Riesz basis for the state Hilbert space, and that the spectrum-determined growth condition holds, both for almost $\alpha > 0$. B.Z. Guo [3] improved this result for any $m, \alpha, \beta > 0$ using an abstract result about the Riesz basis property of discrete operators in general Hilbert spaces [4].

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The rest of this paper is organized as follows. In section 2, we convert system (1) into an evolution equation in an appropriate Hilbert space, and then prove that the evolutionary system is associated to a C_0 semigroup of linear operator whose generator has compact resolvent. Hence the problem is well posed. Some asymptotic expressions of eigenvalues and eigenfunctions are also presented. Section 3 is devoted to prove the Riesz basis property and the exponential stability of the system. Finally, we use the finite difference scheme with the QZ method to study numerically the spectrum of the system.

Throughout this paper, we assume that

$$\begin{cases} (EI(.),\rho(.))\in[C^4(0,1)]^2,\ p=\displaystyle\int_0^1\eta(s)ds, \eta(s)=\left(\frac{\rho(s)}{EI(s)}\right)^{1/4}, \\ EI,\rho,\beta,\gamma,m>0,\alpha\geq 0. \end{cases} \tag{2}$$

2. Eigenvalue problem

We start our investigation by formulating the problem in the following Hilbert space: $\mathbb{H}=\mathbb{V}\times L^2(0,1)\times\mathbb{C}, \mathbb{V}=\{f\in H^2(0,1)/f(0)=f'(0)=0\}$, with the inner product defined as: $\forall(F=(f_1,g_1,\zeta_1),G=(f_2,g_2,\zeta_2))\in\mathbb{H}^2$

$$(F,G)_{\mathbb{H}}=\int_0^1(\rho(x)g_1(x)\overline{g_2(x)}+EI(x)f_1''(x)\overline{f_2''(x)})dx+K\zeta_1\zeta_2. \tag{3}$$

where $K=\dfrac{\beta^2}{m+\alpha\beta}>0$.

Define a linear operator $\mathbb{A}:D(\mathbb{A})\subset\mathbb{H}\to\mathbb{H}$ as

$$\begin{aligned} D(\mathbb{A}) &= \{(f,g,\zeta)\in(H^4(0,1)\cap\mathbb{V})\times\mathbb{V}\times\mathbb{C}/EI(1)f''(1) \\ &= -\gamma g'(1),\zeta=-(EI(1)f'')'(1)+m\beta^{-1}g(1)\} \end{aligned} \tag{4}$$

$$\mathbb{A}(f,g,\zeta)=(g,-1/\rho(.)(EI(.)f'')'',-\beta^{-1}\zeta-\beta^{-1}(\alpha-m\beta^{-1})g(1)), \tag{5}$$

with the initial condition $Y_0=(y_0,y_1,-(EI(.)y_0'')'(1)+m\beta^{-1}y_1(1))$, the system (1) can be written as an evolutionary equation in $\mathbb{H}$

$$\begin{cases} \frac{dY(t)}{dt}=\mathbb{A}Y(t), \\ Y(t)=(y(.,t),y_t(.,t),-(EI(.)y_{xx})_x(1,t)+m\beta^{-1}y_t(1,t)),\quad Y(0)=Y_0. \end{cases} \tag{6}$$

Lemma 2.1. *(i) $\mathbb{A}^{-1}$ exists and is compact on $\mathbb{H}$. Hence the spectrum $\sigma(\mathbb{A})$ of $\mathbb{A}$ consists of isolated eigenvalues only: $\sigma(\mathbb{A})=\sigma_p(\mathbb{A})$, where $\sigma_p(\mathbb{A})$ denotes the set of eigenvalues of $\mathbb{A}$. Moreover, each eigenfunction corresponding to $\lambda\in\sigma(\mathbb{A}),\ \lambda\neq-\beta^{-1}$ is of the form*

$$\overrightarrow{\Phi}=(\lambda^{-1}\Phi,\Phi,-\frac{\beta^{-1}(\alpha-m\beta^{-1})}{\lambda+\beta^{-1}}\Phi(1)),$$

where $\Phi\neq 0$ satisfies

$$\begin{cases} \lambda^2\rho(x)\Phi(x)+(EI(.)\Phi'')''(x)=0, & 0<x<1, \\ \Phi(0)=\Phi'(0)=0, \\ EI(1)\Phi''(1)=-\gamma\lambda\Phi'(1), \\ (1+\lambda\beta)(EI(.)\Phi'')'(1)=(\alpha\lambda+\lambda^2 m)\Phi(1) \end{cases} \tag{7}$$

(ii) for any $\lambda\in\sigma(\mathbb{A}), Re(\lambda)\leq 0$.

Proof. (i) A direct calculation shows that

$$\mathbb{A}^{-1}(f,g,\zeta)=(f_1,g_1,\zeta_1),\ \ \forall(f,g,\zeta)\in\mathbb{H},$$

where

$$\begin{cases} g_1=f,\\ \zeta_1=-\beta\zeta-(\alpha-m\beta^{-1})f(1),\\ f_1(x)=\displaystyle\int_0^x\int_0^y drdy\left[\frac{(\beta\zeta+\alpha f(1))(r-1)-\gamma f'(1)}{EI(r)}+\frac{1}{EI(r)}\int_1^r\int_s^1\rho(t)g(t)dtds\right]\end{cases}$$

Since $|\zeta_1|\le|\beta|\zeta+\left|\alpha-m\beta^{-1}\right|\|f\|_{\mathbb{H}^2}$, it follows that

$$\left\|\mathbb{A}^{-1}(f,g,\zeta)\right\|_{\mathbb{H}^4\times\mathbb{H}^2\times\mathbb{C}}\le M\,\|(f,g,\zeta)\|_{\mathbb{H}}$$

for some constant $M>0$. By the Sobolev embedding theorem [1], $\mathbb{A}^{-1}$ is compact on $\mathbb{H}$.

(ii) For any $Y=(f,g,\zeta)\in D(\mathbb{A})$,

$$\begin{aligned} Re(\mathbb{A}Y,Y)_{\mathbb{H}} &= -\frac{K}{\beta}\left|EI(1)f'''(1)\right|^2-\frac{K\alpha m}{\beta^2}\left|g(1)\right|^2-\gamma\left|g'(1)\right|^2\\ &\quad+(\beta^{-2}K(\alpha\beta+m)-1)Re(\overline{EI(1)f'''(1)}g(1)). \end{aligned}\tag{8}$$

Note that given the particular choice of K, the fourth term in the formula above zero, which proves that the operator $\mathbb{A}$ is dissipative. Other conclusions are obvious, and the details are omitted. □

Let us now study (7). Rewrite (7) to be the standard form of a linear differential operator with generalized homogeneous boundary conditions

$$\begin{cases} \Phi^{(4)}(x)+((2EI'\Phi'''+EI''\Phi'')/EI)(x)+\lambda^2\eta^4(x)\Phi(x)=0,\ 0<x<1,\\ \Phi(0)=\Phi'(0)=0,\\ EI(1)\Phi''(1)=-\gamma\lambda\Phi'(1),\\ (1+\lambda\beta)(EI(.)\Phi'')'(1)=(\alpha\lambda+\lambda^2m)\Phi(1)\end{cases}\tag{9}$$

Let $\lambda\in\sigma(\mathbb{A})$, First, the dominant term "$\Phi^{(4)}(x)+\lambda^2\eta^4(x)\Phi(x)$" of (9), is transformed to become a uniform form by space scaling. In fact, we make the space scaling transformation

$$\Phi(x)=f(z),\ z=z(x)=\frac{1}{p}\int_0^x\eta(s)ds\tag{10}$$

then f satisfies the following system

$$\begin{cases} f^{(4)}(z)+a(z)f'''(z)+b(z)f''(z)+c(z)f'(z)+\lambda^2p^4f(z)=0,\\ f(0)=f'(0)=0,\\ f''(1)=(a_0\gamma\lambda+b_0)f'(1),\\ (\lambda+\beta^{-1})(f^{(3)}(1)+c_0f''(1)+d_0f'(1))=e_0\beta^{-1}\lambda(\alpha+m\lambda)f(1)\end{cases}\tag{11}$$

where $a(z),b(z)$ and $c(z)$ are the smooth functions defined by

$$\begin{cases} a(z)=2p\eta^{-2}(3\eta'+\eta EI'/EI)(x),\\ b(z)=p^2(\eta^{-3}(3\eta^{-1}\eta'^2+4\eta''+(6\eta'EI'+\eta EI'')/EI))(x),\\ c(z)=p^3(\eta^{-1}+\eta^{-4}(2\eta''EI'+\eta'EI'')/EI)(x),\end{cases}\tag{12}$$

and a_0,b_0,c_0,d_0 and e_0 are constants given by

$$\begin{cases} a_0=-p\eta^{-1}(1)/EI(1),\ b_0=-p\eta''(1)\eta^{-2}(1),\\ c_0=p^{-1}\eta^{-1}(1)(3\eta^{-1}(1)\eta'(1)+EI'(1)/EI(1)),\\ d_0=p^2\eta^{-3}(1)(\eta''(1)+EI'(1)\eta'/EI(1)),\\ e_0=p^3\eta^{-3}(1)/EI(1).\end{cases}\tag{13}$$

Second, in order to cancel the term $f'''(z)$ in (11), as was done in [6], we make the invertible transformation

$$f(z) = e^{-\frac{1}{4}\int_0^z a(s)ds} g(z) \tag{14}$$

then g satisfies the following system

$$\begin{cases} g^{(4)}(z) + a_1(z)g''(z) + a_2(z)g'(z) + a_3(z)g(z) + \lambda^2 p^4 g(z) = 0 \\ g(0) = g'(0) = 0, g''(1) + c_{11}g'(1) + c_{12}g(1) + a_0\lambda\gamma g'(1) + c_{13}\gamma\lambda g(1) = 0, \\ (\lambda + \beta^{-1})(g^{(3)}(1) + c_{21}g''(1) + c_{22}g'(1) + c_{23}g(1)) = \lambda\beta^{-1}e_0(\alpha + m\lambda)g(1), \end{cases} \tag{15}$$

where $a_1(z), a_2(z)$ and $a_3(z)$ are the smooth functions defined by

$$\begin{cases} a_1(z) = -(3a'(z)/2) - (3a^2(z)/8) + b(z), \\ a_2(z) = (a^3(z)/8) - a''(z) - (a(z)b(z)/2) + c(z), \\ a_3(z) = (3a'^2(z)/16) - (a'''(z)/4) - (3a^4(z)/256) + (3a^2(z)a'(z)/32) \\ \qquad +b(z)((a^2(z)/16) - (a'(z)/4)) - (a(z)c(z)/4) \end{cases} \tag{16}$$

and $c_{11}, c_{12}, c_{13}, c_{21}, c_{22}$ and c_{23} are constants defined by

$$\begin{cases} c_{11} = (a(1)/2) + b_0, \\ c_{12} = (a'(1)/4) - (a^2/16)(1) - (ab_0/4)(1), \\ c_{13} = -(a/4)(1) + c_0, \\ c_{21} = -(3a/4)(1) + c_0, \\ c_{22} = (3a^2/16)(1) - (3a'/4)(1) - (ac_0/2)(1) + d_0, \\ c_{23} = -(a''(1)/4) + (3aa'(1)/16) - (a^2(1)/64 + c_0(-(a'(1)/4) \\ \qquad\qquad +(a^2(1)/16)) - (a(1)d_0/4), \end{cases} \tag{17}$$

Now, we proceed as in section 4, chapter 2 of [6] to estimate asymptotically the solution of (15). Since system (1) is dissipative, all eigenvalues are located on the left half complex plane. Due to the conjugate property of the eigenvalues, we may consider only $\lambda = \dfrac{\tau^2}{p^2}$ with $\dfrac{\pi}{2} \le \arg\lambda \le \pi$, then $\dfrac{\pi}{4} \le \arg\tau \le \dfrac{\pi}{2}$.

Let us choose $\omega_j, j = 1, 2, 3, 4$ as follows

$$\omega_1 = \frac{-1+i}{\sqrt{2}}, \omega_2 = \frac{1+i}{\sqrt{2}}, \omega_3 = -\omega_2,\ \omega_4 = -\omega_1.$$

Consequently, we have for $\tau \in S = \{\tau \mid \ \dfrac{\pi}{4} \le \arg\tau \le \dfrac{\pi}{2}\}$

$$\begin{aligned} Re(\tau\omega_1) &= -\mid\tau\mid\sin(\arg\tau + \frac{\pi}{4}) \le -\frac{\sqrt{2}}{2}\mid\tau\mid < 0, \\ Re(\tau\omega_2) &= \mid\tau\mid\cos(\arg\tau + \frac{\pi}{4}) \le 0. \end{aligned} \tag{18}$$

The following lemma comes from Theorem 2.4 in section 4, chapter 2 of Naimark [6].

Lemma 2.2. *For* $\mid\tau\mid$ *large enough and* $\tau \in S$, *there are four linearly independent solutions* g_k , $k = 1, 2, 3, 4$, *to* $g^{(4)}(z)+a_1(z)g''(z)+a_2(z)g'(z)+a_3(z)g(z)+\tau^4 g(z) = 0$, *such that*

$$\begin{cases} g_k(z) = e^{\tau\omega_k z}(1 + O(\frac{1}{\tau})), & g'_k(z) = \tau\omega_k e^{\tau\omega_k z}(1 + O(\frac{1}{\tau})), \\ g''_k(z) = (\tau\omega_k)^2 e^{\tau\omega_k z}(1 + O(\frac{1}{\tau})), & g'''_k(z) = (\tau\omega_k)^3 e^{\tau\omega_k z}(1 + O(\frac{1}{\tau})). \end{cases} \tag{19}$$

Let g be a solution of the system (15), then there exist 4 constants $d_k, k = 1, 2, 3, 4$ such that

$$g(z) = d_1g_1(z) + d_2g_2(z) + d_3g_3(z) + d_4g_4(z), \tag{20}$$

where $g_k,\ k=1,2,3,4$ are defined by Lemma 2.2. By boundary conditions, $d_i, i = 1,2,3,4$ are solutions to the following boundary system of linear algebraic equations

$$\left\{\begin{array}{l} d_1g_1(0)+d_2g_2(0)+d_3g_3(0)+d_4g_4(0)=0,\\ d_1g_1'(0)+d_2g_2'(0)+d_3g_3'(0)+d_4g_4'(0)=0,\\ \sum_{k=1}^4 d_k[g_k''(1)+c_{11}g_k'(1)+c_{12}g_k(1)+a_0\gamma(\tau^2/p^2)g_k'(1)+c_{13}\gamma(\tau^2/p^2)g_k(1)]=0,\\ \sum_{k=1}^4 d_k[(\tau^2/p^2)+\beta^{-1})[g_k'''(1)+c_{21}g_k''(1)+c_{22}g_k'(1)+c_{23}g_k(1)]\\ -(\tau^2/p^2)\beta^{-1}e_0(\alpha+m(\tau^2/p^2))g_k(1)]=0. \end{array}\right. \tag{21}$$

From (18) and (19), for any $k=1,2,3,4$

$$\left\{\begin{array}{l} g_k(0)=1+O(\frac{1}{\tau}),\ g_k'(0)=\tau\omega_k(1+O(\frac{1}{\tau})),\\ \{g_k''(1)+c_{11}g_k'(1)+c_{12}g_k(1)+a_0\gamma(\tau^2/p^2)g_k'(1)+c_{13}\gamma(\tau^2/p^2)g_k(1)\}\\ \qquad =a_0\gamma(\tau^3/p^2)\omega_k e^{\tau\omega_k}(1+O(\frac{1}{\tau})),\\ \{g_k'''(1)-((\tau^2/p^2)+\beta^{-1})\{g_k'''(1)+c_{12}g_k''(1)+c_{22}g_k'(1)+c_{23}g_k(1)\}\\ \qquad -(\tau^2/p^2)\beta^{-1}e_0(\alpha+m(\tau^2/p^2))g_k(1)=(\tau^5/p^2)\omega_k^3e^{\tau\omega_k}(1+O(\frac{1}{\tau})), \end{array}\right. \tag{22}$$

The system (15) has a nonzero solution if and only if τ satisfies the characteristic equation

$$\det\begin{bmatrix} [1] & [1] & [1] & [1]\\ \tau\omega_1[1] & \tau\omega_2[1] & \tau\omega_3[1] & \tau\omega_4[1]\\ \tau^3\omega_1e^{\tau\omega_1}[1] & \tau^3\omega_2e^{\tau\omega_2}[1] & \tau^3\omega_3e^{\tau\omega_3}[1] & \tau^3\omega_4e^{\tau\omega_4}[1]\\ \tau^5\omega_1^3e^{\tau\omega_1}[1] & \tau^5\omega_2^3e^{\tau\omega_2}[1] & \tau^5\omega_3^3e^{\tau\omega_3}[1] & \tau^5\omega_4^3e^{\tau\omega_4}[1] \end{bmatrix}=0 \tag{23}$$

where $[1]=1+O(\tau^{-1})$.

Since $\omega_4=-\omega_1$ and $\omega_3=-\omega_2$, then the above equation is equivalent to

$$\det\begin{bmatrix} [1] & [1] & e^{\tau\omega_2}[1] & e^{\tau\omega_1}[1]\\ \omega_1[1] & \omega_2[1] & -\omega_2e^{\tau\omega_2}[1] & -\omega_1e^{\tau\omega_1}[1]\\ \omega_1e^{\tau\omega_1}[1] & \omega_2e^{\tau\omega_2}[1] & -\omega_2[1] & -\omega_1[1]\\ \omega_1e^{\tau\omega_1}[1] & \omega_2e^{\tau\omega_2}[1] & -\omega_2^3[1] & -\omega_1^3[1] \end{bmatrix}=0 \tag{24}$$

Since $\mid e^{\tau\omega_2}\mid = e^{|\tau|\frac{1}{\sqrt{2}}(\cos(\arg\tau)-\sin(\arg\tau))}\leq 1,\ e^{\tau\omega_1}=O(e^{-\frac{1}{\sqrt{2}}|\tau|})$ when $\mid\tau\mid\to\infty$, we may rewrite (24) as

$$\det\begin{bmatrix} 1 & 1 & e^{\tau\omega_2} & 0\\ \omega_1 & \omega_2 & -\omega_2e^{\tau\omega_2} & 0\\ 0 & \omega_2e^{\tau\omega_2} & -\omega_2 & -\omega_1\\ 0 & \omega_2^3e^{\tau\omega_2} & -\omega_2^3 & -\omega_1^3 \end{bmatrix}+O(\tau^{-1})=0 \tag{25}$$

then

$$-e^{2\tau\omega_2}(\omega_1^2-\omega_2^2)(\omega_1\omega_2^2+\omega_1^2\omega_2)+(\omega_1^2-\omega_2^2)(\omega_1\omega_2^2-\omega_1^2\omega_2)+O(\tau^{-1})=0$$

which results in

$$e^{2\tau\omega_2}=\frac{\omega_2-\omega_1}{\omega_1+\omega_2}+O(\tau^{-1})=-i+O(\tau^{-1}) \tag{26}$$

Since the matrix in (25) has rank 3 for each sufficiently large τ_n there is only one linearly solution to (15) for $\tau=\tau_n$. Hence each λ_n is geometrically simple for n sufficiently large. By solving (26), we obtain the following lemma by the same argument of those of section 4, chapter 2 of Naimark [6].

Lemma 2.3. *(i) There is a family of eigenvalues $(\lambda_n, \overline{\lambda_n})$ of $\mathbb{A}$ which satisfies*

$$\lambda_n = \frac{\tau_n^2}{p^2},\ \tau_n = \frac{1}{\sqrt{2}}(n-\frac{1}{4})\pi(i+1) + O(n^{-1}) \text{ when } n \to +\infty, \tag{27}$$

(ii) For n sufficiently large λ_n is geometrically simple.

Lemma 2.4. *Let λ_n and τ_n be defined as in Lemma 2.3. Then the unique (up to a scalar) associated solution g_n to (15) has the following asymptotic expansion*

$$g_n(z) = -2(1+i)[\sin(n-\frac{1}{4})\pi z - \cos(n-\frac{1}{4})\pi z + e^{-(n-\frac{1}{4})\pi z}] + O(\frac{1}{n}) \tag{28}$$

$$\tau_n^{-2} g_n''(z) = 2(i-1)[\cos(n-\frac{1}{4})\pi z - \sin(n-\frac{1}{4})\pi z + e^{-(n-\frac{1}{4})\pi z}] + O(\frac{1}{n}), \tag{29}$$

Moreover

$$\tau_n^{-1} g_n'(z) = -\frac{4}{\sqrt{2}}[\cos(n-\frac{1}{4})\pi z - \sin(n-\frac{1}{4})\pi z - e^{-(n-\frac{1}{4})\pi z}] + O(\frac{1}{n}). \tag{30}$$

Proof. From Lemma 2.2. as well as simple facts of linear algebra, the eigenfunction g_n corresponding to the eigenvalue $\lambda_n = \dfrac{\tau_n^2}{p^2}$ is given by

$$g_n(z) = \det\begin{bmatrix} [1] & [1] & e^{\tau_n w_2}[1] & e^{\tau_n w_1}[1] \\ e^{\tau_n\omega_1 z}[1] & e^{\tau_n\omega_2 z}[1] & e^{\tau_n\omega_2(1-z)}[1] & e^{\tau_n\omega_1(1-z)}[1] \\ \omega_1 e^{\tau_n\omega_1}[1] & \omega_2 e^{\tau_n\omega_2}[1] & -\omega_2[1] & -\omega_1[1] \\ \omega_1^3 e^{\tau_n\omega_1}[1] & \omega_2^3 e^{\tau_n\omega_2}[1] & -\omega_2^3[1] & -\omega_1^3[1] \end{bmatrix} \tag{31}$$

It follows from (18) that

$$g_n(z) = \det\begin{bmatrix} 1 & 1 & e^{\tau_n\omega_2} & 0 \\ e^{\tau_n\omega_1 z} & e^{\tau_n\omega_2 z} & e^{\tau_n\omega_2(1-z)} & e^{\tau_n\omega_1(1-z)} \\ 0 & \omega_2 e^{\tau_n\omega_2} & -\omega_2 & -\omega_1 \\ 0 & \omega_2^3 e^{\tau_n\omega_2} & -\omega_2^3 & -\omega_1^3 \end{bmatrix} + O(\frac{1}{\tau_n}) \tag{32}$$

After a simple calculation, we find that

$$\begin{aligned} g_n(z) &= \omega_1\omega_2(\omega_1^2-\omega_2^2)[e^{\tau_n\omega_2 z} + e^{\tau_n\omega_2}e^{\tau_n\omega_2(1-z)} - e^{\tau_n\omega_1 z} - e^{\tau_n\omega_1 z}e^{2\tau_n\omega_2}] + O(\frac{1}{\tau_n}) \\ &= -2(1+i)[\sin(n-\frac{1}{4})\pi z - \cos(n-\frac{1}{4})\pi z + e^{-(n-\frac{1}{4})\pi z}] + O(\frac{1}{n}). \end{aligned}$$

Similarly

$$g_n^{(k)}(z) = \tag{33}$$

$$\tau_n^k \det\begin{bmatrix} [1] & [1] & e^{\tau_n\omega_2}[1] & e^{\tau_n\omega_1}[1] \\ \omega_1^k e^{\tau_n\omega_1 z}[1] & \omega_2^k e^{\tau_n\omega_2 z}[1] & (-\omega_2)^k e^{\tau_n\omega_2(1-z)}[1] & (-\omega_1)^k e^{\tau_n\omega_1(1-z)}[1] \\ \omega_1 e^{\tau_n\omega_1}[1] & \omega_2 e^{\tau_n\omega_2}[1] & -\omega_2[1] & -\omega_1[1] \\ \omega_1^3 e^{\tau_n\omega_1}[1] & \omega_2^3 e^{\tau_n\omega_2}[1] & -\omega_2^3[1] & -\omega_1^3[1] \end{bmatrix}$$

where $k = 1, 2$. Equations (29) and (30) can be proved similarly. The lemma is proved. □

Noting that $\tau_n^{-2} f_n'(z) = O(n^{-1})$. From the transformations (10) and (14), we obtain the asymptotic expression of eigenfunctions which are as follows

Lemma 2.5. *Let λ_n and τ_n defined by (27). There is a solution of (7) corresponding to λ_n which has the following asymptotic property:*

$$e^{1/4\int_0^z a(s)ds}\Phi_n(x) = -2(1+i)[\sin(n-\frac{1}{4})\pi z - \cos(n-\frac{1}{4})\pi z + e^{-(n-\frac{1}{4})\pi z}] + O(\frac{1}{n}) \quad (34)$$

$$\lambda_n^{-1}\Phi_n''(x) = 2\Lambda(x)(i-1)[\cos(n-\frac{1}{4})\pi z - \sin(n-\frac{1}{4})\pi z + e^{-(n-\frac{1}{4})\pi z}] + O(\frac{1}{n}) \quad (35)$$

where $\Lambda(x) = e^{-\frac{1}{4}\int_0^z a(s)ds}\eta^2(x)$,

3. Riesz basis property and exponential stability

In order to apply B.Z. Guo theorem [3] to the operator $\mathbb{A}$, we need a reference basis. For system (1), this is accomplished by collecting approximately normalized eigenfunctions of the following system

$$\begin{cases} \rho(x)y_{tt}(x,t) + (EI(x)y_{xx}(x,t))_{xx} = 0, & 0 < x < 1,\ t > 0, \\ y(0,t) = y_x(0,t) = (EI(.)y_{xx})_{xt}(1,t) = y_{xt}(1,t) = 0, & t > 0, \end{cases} \quad (36)$$

Naturally, we consider the well-posed conservative system as follows:

$$\begin{cases} \rho(x)y_{tt}(x,t) + (EI(x)y_{xx}(x,t))_{xx} = 0, & 0 < x < 1,\ t > 0, \\ y(0,t) = y_x(0,t) = (EI(.)y_{xx})_x(1,t) = y_{xt}(1,t) = 0, & \\ y(x,0) = y_0(x)\ ,\ \ y_t(x,0) = y_1(x), & t > 0, \end{cases} \quad (37)$$

which has the same eigenvalues as the system (1). In order to get the same space as the system (1) i.e., $\mathbb{H}$, we complete the conservative system by ordinary differential equation, so we construct the auxiliary system described by the following equations:

$$\begin{cases} \rho(x)y_{tt}(x,t) + (EI(x)y_{xx}(x,t))_{xx} = 0, & 0 < x < 1,\ t > 0, \\ y(0,t) = y_x(0,t) = (EI(.)y_{xx})_x(1,t) = y_{xt}(1,t) = \dot{\zeta}(t) = 0 & t > 0, \\ y(x,0) = y_0(x)\ \ ,\ y_t(x,0) = y_1(x)\ , & 0 < x < 1, \end{cases} \quad (38)$$

Alternatively, we can formulate the auxiliary system above as a problem of evolution in $\mathbb{H}$ as follows: $\dfrac{dY(t)}{dt} = \mathbb{A}_0 Y(t)$, where the operator $\mathbb{A}_0 : D(\mathbb{A}_0) \subset \mathbb{H} \to \mathbb{H}$ is defined by

$$\begin{cases} \mathbb{A}_0(f,g,\zeta) = (g, -1/\rho(.)(EI(.)f'')'', 0) \\ \forall (f,g,\zeta) \in D(\mathbb{A}_0), D(\mathbb{A}_0) = \{(f,g,\zeta) \in \mathbb{X}/(EI(.)f'')'(1) = 0, g'(1) = 0\}. \end{cases} \quad (39)$$

where $\mathbb{X} = (H^4(0,1) \cap \mathbb{V}) \times \mathbb{V} \times \mathbb{C}$. We easily show that $\mathbb{A}_0$ is nothing but the operator $\mathbb{A}$ with $\alpha = m = \beta^{-1} = \gamma^{-1} = 0$. $\mathbb{A}_0$ is skew-adjoint with compact resolvent.
It is seen that all the analysis in the previous sections for the operator $\mathbb{A}$ are still true for the operator $\mathbb{A}_0$. Therefore, the following result is obtained.

Lemma 3.1. *Each eigenvalue υ_{n_0} of $\mathbb{A}_0$ with sufficiently large module is geometrically simple and hence algebraically simple. The eigenfunctions $\overrightarrow{\Psi_{n_0}} = (\upsilon_{n_0}^{-1}\Psi_{n_0}, \Psi_{n_0}, 0) \bigcup$ {their conjugates} of υ_{n_0} have the following asymptotic expressions*

$$e^{1/4\int_0^z a(s)ds}\Psi_{n_0}(x) = -2(1+i)[\sin(n-\frac{1}{4})\pi z - \cos(n-\frac{1}{4})\pi z + e^{-(n-\frac{1}{4})\pi z}] + O(\frac{1}{n}) \quad (40)$$

$$\upsilon_{n_0}^{-1}\Psi_{n_0}''(x) = 2\Lambda(x)(i-1)[\cos(n-\frac{1}{4})\pi z - \sin(n-\frac{1}{4})\pi z + e^{-(n-\frac{1}{4})\pi z}] + O(\frac{1}{n}) \quad (41)$$

Where all $(\upsilon_{n_0}, \overline{\upsilon_{n_0}})$, but possibly a finite number of other eigenvalues, are composed of all the eigenvalues of $\mathbb{A}_0$. The eigenfunctions $\overrightarrow{\Psi_{n_0}} = (\upsilon_{n_0}^{-1}\Psi_{n_0}, \Psi_{n_0}, 0)$ are normalized approximately. From a well known result in functional analysis, we know that the eigenfunctions of $\mathbb{A}_0$ form an orthogonal basis for $\mathbb{H}$, particularly, all $\overrightarrow{\Psi_{n_0}}$ and their conjugates form an (orthogonal) Riesz basis for $\mathbb{H}$.

Then there exists a positive integer large enough N such that,

$$\sum_{n=N+1}^{+\infty} \left\| \overrightarrow{\Phi}_n - \overrightarrow{\Psi}_{n_0} \right\|_{\mathbb{H}}^2 = \sum_{n=N+1}^{+\infty} O(n^{-2}) < +\infty, \tag{42}$$

The same result is verified for their conjugates.

We can now apply Theorem of B.Z. Guo [3] to obtain the main results of the present paper.

Theorem 3.2. *Let the operator* $\mathbb{A}$ *defined by* (4), (5). *Then*

(i) *There is a sequence of generalized eigenfunctions of* $\mathbb{A}$ *which forms a Riesz basis for the state Hilbert space* $\mathbb{H}$.

(ii) *The eigenvalues* $(\lambda_n, \overline{\lambda_n})$ *of* $\mathbb{A}$ *have the asymptotic behavior* (27).

(iii) *All* $\lambda \in \sigma(\mathbb{A})$ *with sufficiently large modulus are algebraically simple. Therefore,* $\mathbb{A}$ *generates a* C_0 *semi-group. Moreover, for the semigroup* $e^{\mathbb{A}t}$ *generated by* $\mathbb{A}$*, the spectrum-determined growth condition holds:* $\omega(\mathbb{A}) = S(\mathbb{A})$*, where*

$\omega(\mathbb{A}) = \lim_{t\to+\infty} \frac{1}{t} \| e^{\mathbb{A}t} \|$ *is the growth order of* $e^{\mathbb{A}t}$ *and* $S(\mathbb{A}) = \sup\{Re(\lambda)/\ \lambda \in \sigma(\mathbb{A})\}$ *is the spectral bound of* $\mathbb{A}$.

Now, we are in a position to show the exponential stability. Since the spectrum determined growth condition holds: $S(\mathbb{A}) = \omega(\mathbb{A})$. System (1) is exponentially stable if and only if there is $\omega_0 > 0$ such that $Re(\lambda) < -\omega_0\ \ \forall \lambda \in \sigma(\mathbb{A})$.

Lemma 3.3. *Let* λ_n *be defined by* (27)*, then there is an* $\omega_0 > 0$ *such that*

$$\lim_{n\to+\infty} Re(\lambda_n) = -\omega_0 < 0.$$

Proof. Let (λ_n, Φ_n) in (7) where Φ_n is defined by (34). Multiplying (7) by $\overline{\Phi}_n$ and integrating by parts from 0 to 1 with respect to x, we obtain

$$\begin{aligned}
\lambda_n^2 [\int_0^1 \rho(x) \mid \Phi_n(x) \mid^2 dx + \frac{m \mid \Phi_n(1) \mid^2}{\mid 1 + \lambda_n\beta \mid^2}] \\
+ \lambda_n [\frac{(\mid \lambda_n \mid^2 m\beta + \alpha)}{\mid 1 + \lambda_n\beta \mid^2} \mid \Phi_n(1) \mid^2 + \gamma \mid \Phi_n'(1) \mid^2] \\
+ \frac{\alpha\beta \mid \lambda_n \mid^2 \mid \Phi_n(1) \mid^2}{\mid 1 + \lambda_n\beta \mid^2} + \int_0^1 EI(x) \mid \Phi_n''(x) \mid^2 dx = 0,
\end{aligned}$$

Since $Im\lambda_n \neq 0$ for sufficiently large n, we have from the above equation that

$$\begin{aligned}
2\, Re\lambda_n [\int_0^1 \rho(x) \mid \Phi_n(x) \mid^2 dx + \frac{m \mid \Phi_n(1) \mid^2}{\mid 1 + \lambda_n\beta \mid^2}] \\
= -[\frac{(\mid \lambda_n \mid^2 m\beta + \alpha)}{\mid 1 + \lambda_n\beta \mid^2} \mid \Phi_n(1) \mid^2 + \gamma \mid \Phi_n'(1) \mid^2] \\
= -[\frac{(\mid \lambda_n \mid^2 m\beta + \alpha)}{\mid 1 + \lambda_n\beta \mid^2} \mid \Phi_n(1) \mid^2 + \frac{EI(1)^2 \mid \lambda_n^{-1} \Phi_n''(1) \mid^2}{\gamma}]
\end{aligned}$$

From (34) and (35), we have

$$\lim_{n\to+\infty} EI(1)^2 \mid \lambda_n^{-1}\Phi_n''(1) \mid^2 dx = 16\rho(1)EI(1)\ e^{-\frac{1}{2}\int_0^1 a(s)\ ds}dx,$$

$$\lim_{n\to+\infty} \frac{\mid \lambda_n\Phi_n(1) \mid^2}{\mid 1+\lambda_n\beta \mid^2} = \frac{16}{\beta^2}e^{-\frac{1}{2}\int_0^1 a(s)\ ds}dx,\ \lim_{n\to+\infty} \frac{\mid \Phi_n(1) \mid^2}{\mid 1+\lambda_n\beta \mid^2} = 0$$

and by Riemann Lebesgue Lemma, we have

$$\lim_{n\to+\infty} \int_0^1 \rho(x) \mid \Phi_n(x) \mid^2 dx = 8\int_0^1 \rho(x)\ e^{-\frac{1}{2}\int_0^z a(s)\ ds}dx,$$

Hence

$$\lim_{n\to+\infty} Re\lambda_n = -\frac{(\frac{m}{\beta}+\frac{\rho(1)EI(1)}{\gamma})e^{-\frac{1}{2}\int_0^1 a(s)\ ds}dx}{\int_0^1 \rho(x)\ e^{-\frac{1}{2}\int_0^z a(s)\ ds}dx} < 0.$$

The result follows. □

Theorem 3.4. *The semigroup $e^{\mathbb{A}t}$ is exponentially stable for any $m, \beta, \gamma > 0, \alpha \geq 0$.*

Proof. By Lemma 3.2. and $\omega(\mathbb{A}) = S(\mathbb{A})$ we need only to show that $Re(\lambda) < 0$, for any $\lambda \in \sigma(\mathbb{A})$. First $\mathbb{A}$ is dissipative, then $Re(\lambda) \leq 0$, for any $\lambda \in \sigma(\mathbb{A})$. So Now, if $\mathbb{A}Y = \lambda Y, Y = (\Psi, \Phi, \zeta)$ and $Re(\lambda) = 0$, then $\Phi = \lambda\Psi$, $\Psi'''(1) = \Phi(1) = 0$, we deduce from (7) that

$$\begin{cases} \lambda^2\rho(x)\Phi(x) + (EI(.)\Phi'')''(x) = 0, \quad 0 < x < 1, \\ \Phi(0) = \Phi'(0) = \Phi(1) = \Phi'(1) = \Phi''(1) = \Phi'''(1) = 0 \end{cases}$$

the above equation has a zero solution only [4]. Hence $\Phi = 0$, thus $\Psi = \zeta = 0$. Therefore, $Re(\lambda) < 0$, $\forall\lambda \in \sigma(\mathbb{A})$. The proof is complete. □

4. Numerical simulation

In this section, we use finite difference method to study numerically the spectrum of the operator $\mathbb{A}$, then we apply QZ method [5], to approach the calculation of the spectrum of the eigenvalue problem (7). Finally, we study the influence of parameters of feedback control on the convergence rate of the energy.

The chosen beam is characterized by the following figures (NKSA units)

$$\rho(x) = (x+1)^2\ (Kg/m),\ EI(x) = (x+1)^4\ (Kg \times m^3/s^2),$$

Let $n \in IN^*$, $h = \dfrac{1}{n}, n = 100$ and $x_i = ih,\ i = 0, 1, ..., n$. We use central finite difference method [8], then a simple calculation gives

$$\begin{cases} a_i\Phi_{i-2} + b_i\Phi_{i-1} + (c_i + \lambda^2h^4)\Phi_i + d_i\Phi_{i+1} + e_i\Phi_{i+2} = 0,\ i = 2, ..., n-2 \\ \Phi_0 = 0,\ 4\Phi_1 - \Phi_2 = 0, \\ \Phi_{n-2}(2EI_n + \lambda h\gamma) - 4\Phi_{n-1}(EI_n + \lambda h\gamma) + \Phi_n(2EI_n + 3\lambda h\gamma) = 0, \\ (1+\lambda\beta)(EI_n\Phi_{n-3} + 3(hEI_n' + EI_n)\Phi_{n-2} - 3(2EI_n'h + EI_n)\Phi_{n-1}) \\ +(-3h^3m\lambda^2 + \lambda(-3h^3\alpha + \beta + 3h\beta EI_n') + 3hEI_n' + EI_n)\Phi_n = 0 \end{cases} \tag{43}$$

where $a_i,\ b_i, c_i,\ d_i$ and $e_i,\ i = 2...n-2$ are defined by

$$\begin{aligned} a_i &= \frac{EI_{i-1}}{\rho_i},\ b_i = -2\frac{EI_{i-1} + EI_i}{\rho_i}, \\ c_i &= \frac{EI_{i-1} + 4EI_i + EI_{i+1}}{\rho_i},\ d_i = -2\frac{EI_i + EI_{i+1}}{\rho_i},\ e_i = \frac{EI_{i+1}}{\rho_i} \end{aligned} \tag{44}$$

We write the system (43) under the matrix form:

$$\left\{\begin{array}{l}\lambda^2 PY + \lambda QY + RY = 0,\\ Y = (y_1, ..., y_n)^T.\end{array}\right. \tag{45}$$

where P, Q and R are $n \times n$ matrices. It is easy to verify that we can rewrite the system in the following form

$$\mathcal{A}\mathcal{U} = \lambda\mathcal{B}\,\mathcal{U}, \mathcal{U} = (Z, Y)^T \mathcal{A} = \begin{pmatrix} P & 0 \\ 0 & -R \end{pmatrix},\ \mathcal{B} = \begin{pmatrix} 0 & P \\ P & Q \end{pmatrix}. \tag{46}$$

The eigenvalues are calculated easily by Matlab in PC by using QZ algorithm.
1. $\beta = 10, \alpha = 20, \gamma = 1$. We change the value of m. We conclude that spectrum don't change as m is increased. Which is reassuring, since in this case the feedback is independent of the mass.(Figure 1 left.)

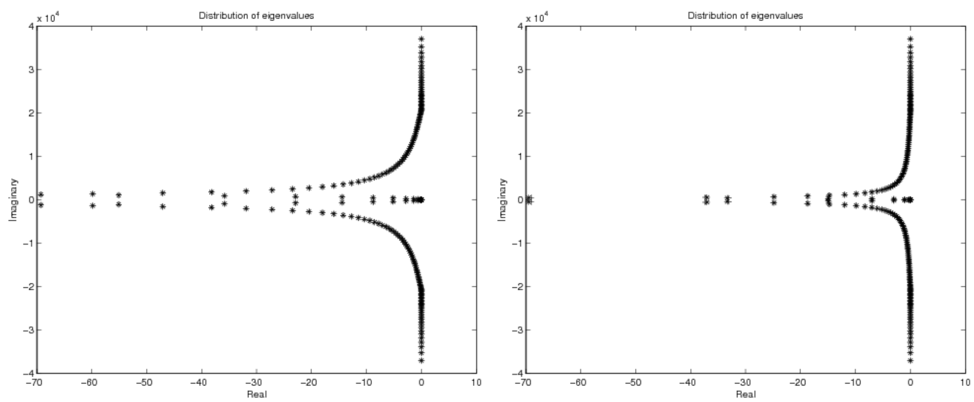

FIGURE 1. Distribution of eigenvalues $m = 1, 5$(left) and $\beta = 1, 5$(right).

2. $m = 5, \alpha = 10, \gamma = 1$. We change the value of β. We conclude that spectrum don't change as β, is increased. (Figure 1 right.)
3. $m = 10, \beta = 5, \gamma = 1$. We change the value of α. We conclude that spectrum don't change as, α is increased. (Figure 2 left.)

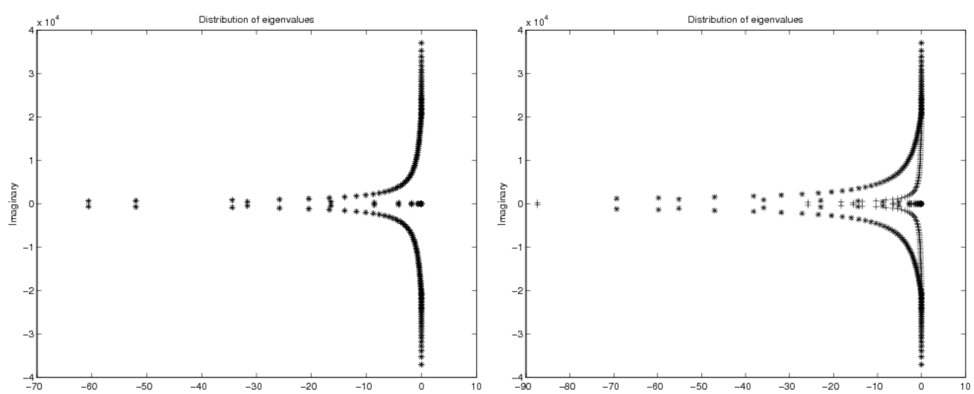

FIGURE 2. Distribution of eigenvalues $\alpha = 1, 5$ (left) and $\gamma = 1, 5$(right).

4. $m = 10, \beta = 5, \alpha = 1$. We change the value of γ. We conclude that spectrum seems to move to the right as γ is increased. (Figure 2 right.)

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