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# Continuous spectrum of a fourth order eigenvalue problem with variable exponent under Neumann boundary conditions

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Abstract. In this paper we will study the existence of solutions for the nonhomogeneous elliptic equation with variable exponent $\Delta^2_{p(x)}u = \lambda|u|^{q(x)-2}u$, in a smooth bounded domain, under Neumann boundary conditions.

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## 1. Introduction

We are concerned here with the eigenvalue problem:

$$\begin{cases} \Delta^2_{p(x)}u = \lambda|u|^{q(x)-2}u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial}{\partial \nu}(|\Delta u|^{p(x)-2}\Delta u) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N \geq 1$, $\Delta^2_{p(x)}u = \Delta(|\Delta u|^{p(x)-2}\Delta u)$, is the $p(x)$-biharmonic operator, $\lambda \geq 0$, $p, q$ are continuous functions on $\overline{\Omega}$.

The aim of this work is to study the existence of solutions for the nonhomogeneous eigenvalue problem (1), by considering different situations concerning the growth rates involved in the above quoted problem, we will prove the existence of a continuous family of elgenvalues.

In recent years, the study of differential equations and variational problems with $p(x)$-growth conditions is an interesting topic, which arises from nonlinear electrorheological fluids and other phenomena related to image processing, elasticity and the flow in porous media. In this context we refer to [9], [10], [5], [13], [11], [12].

This work is motivated by recent results in mathematical modeling of non Newtonian fluids and elastic mechanics, in particular, the electrorheological fluids (Smart fluids). This important class of fluids is characterized by change of viscosity, which is not easy to manipulate and depends on the electric field. These fluids, which are known under the name ER fluids, have many applications in electric mechanics, fluid dynamics etc...

In the case where $p(x) = q(x)$, the authors in [13] investigated the eigenvalues of the $p(x)-$biharmonic with Navier boudary conditions. And in [15] they considered the problem

$$\begin{cases} \Delta^2_{p(x)}u = \lambda|u|^{q(x)-2}u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

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where $p, q$ are continuous functions on $\overline{\Omega}$. Using the mountain pass lemma and Ekeland variational principle, they prove the existence of a continuous family of eigenvalues. Motivated by this work, we will study the existence of solutions for the non-homogeneous elliptic eigenvalue problem

$$\begin{cases} \Delta^2_{p(x)} u = \lambda |u|^{q(x)-2} u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial}{\partial \nu}(|\Delta u|^{p(x)-2}\Delta u) = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

in the space $X = \{u \in W^{2,p(x)}(\Omega) : \ \frac{\partial u}{\partial \nu} = 0\}$.

## 2. Preliminaries

In order to deal with $p(x)-$biharmonic operator problems, we need some results on spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$ and some properties of $p(x)-$biharmonic operator, which we will use later.

Define the generalized Lebesgue space by:

$$L^{p(x)}(\Omega) = \left\{ u : \ \Omega \longrightarrow \mathbb{R}, \ \text{measurable and } \int_\Omega |u(x)|^{p(x)} dx < \infty \right\},$$

where $p \in C_+(\overline{\Omega})$ and

$$C_+(\overline{\Omega}) = \left\{ h \in C(\overline{\Omega}) : \ h(x) > 1, \ \forall x \in \overline{\Omega} \right\}.$$

Denote

$$p^+ = \max_{x \in \overline{\Omega}} p(x), \quad p^- = \min_{x \in \overline{\Omega}} p(x),$$

and for all $x \in \overline{\Omega}$ and $k \geq 1$

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N, \end{cases}$$

and

$$p^*_k(x) = \begin{cases} \frac{Np(x)}{N-kp(x)} & \text{if } kp(x) < N, \\ +\infty & \text{if } kp(x) \geq N. \end{cases}$$

One introduces in $L^{p(x)}(\Omega)$ the following norm

$$|u|_{p(x)} = \inf \left\{ \mu > 0; \ \int_\Omega \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \leq 1 \right\},$$

and the space $(L^{p(x)}(\Omega), |.|_{p(x)})$ is a Banach.

**Proposition 2.1.** [21] *The space $(L^{p(x)}(\Omega), |.|_{p(x)})$ is separable, uniformly convex, reflexive and its conjugate space is $L^{q(x)}(\Omega)$ where $q(x)$ is the conjugate function of $p(x)$ i.e*

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1, \quad \forall x \in \Omega.$$

For all $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$ the Hölder's type inequality

$$\left| \int_\Omega uv dx \right| \leq \left( \frac{1}{p^-} + \frac{1}{q^-} \right) |u|_{p(x)} |v|_{q(x)}$$

holds true.
The Sobolev space with variable exponent $W^{k,p(x)}(\Omega)$ is defined by

$$W^{k,p(x)}(\Omega)=\{u\in L^{p(x)}(\Omega):\ \ D^{\alpha}u\in L^{p(x)}(\Omega),\ \ |\alpha|\leq k\},$$

where

$$D^{\alpha}u=\frac{\partial^{|\alpha|}u}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}...\partial x_N^{\alpha_N}},$$

is the derivation in distribution sense, with $\alpha=(\alpha_1,\alpha_2,...,\alpha_N)$ is a multi-index and $|\alpha|=\displaystyle\sum_{i=1}^{i=N}\alpha_i$.

The space $W^{k,p(x)}(\Omega)$, equipped with the norm

$$\|u\|_{k,p(x)}=\sum_{|\alpha|\leq k}|D^{\alpha}u|_{p(x)},$$

also becomes a Banach, separable and reflexive space. For more details, we refer to [22], [21],[11], [25].

**Remark 2.1.** [26] The norm $\|u\|_{2,p(x)}$ is equivalent to the norm $\|u\|=|\Delta u|_{p(x)}$ and $(W^{2,p(x)}(\Omega);\|.\|)$ is a Banach, separable and reflexive space.

Through this paper, we will consider the following space

$$X=\{u\in W^{2,p(x)}(\Omega):\ \ \frac{\partial u}{\partial\nu}=0\}.$$

which is considered by F.Mouradi and all in [16]. They have proved that $X$ is a nonempty, well defined and closed subspace of $W^{2,p(x)}(\Omega)$. For this they have showed the following boundary trace embedding theorem for variable exponent Sobolev spaces.

**Theorem 2.2.** [16] *Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ with $C^2$ boundary. If $2p(x)\geq N\geq 2$ for all $x\in\overline{\Omega}$, then for all $q\in C_+(\Omega)$ there is a continuous boundary trace embedding*

$$W^{2,p(x)}(\Omega)\hookrightarrow L^{q(x)}(\partial\Omega), \tag{4}$$

*and*

$$W^{2,p(x)}(\Omega)\hookrightarrow W^{1,p(x)}(\partial\Omega), \tag{5}$$

*Proof.*(2.1) We choose $p,q\in C_+(\overline{\Omega})$ such that for all $x\in\overline{\Omega}$, $2p(x)\geq N$.

There exists the following continuous embedding

$$W^{2,p(x)}(\Omega)\hookrightarrow W^{2,p^-}(\Omega), \tag{6}$$

and

$$L^{q^+}(\Omega)\hookrightarrow L^{q(x)}(\partial\Omega). \tag{7}$$

By using the classical boundary trace embedding theorem, since $2p^-\geq N$ and $q^+\geq 1$, there exists the continuous embedding

$$W^{2,p^-}(\Omega)\hookrightarrow L^{q^+}(\partial\Omega). \tag{8}$$

And by combining (6), (7), (8) we deduce that $W^{2,p(x)}(\Omega)$ is continuously embedded into $L^{q(x)}(\partial\Omega)$.

(2.2) Since $2p^- > N$ and $p^+ > 1$, we have the continuous embedding (see [24])

$$W^{2,p^-}(\Omega) \hookrightarrow W^{1,p^+}(\partial\Omega). \tag{9}$$

Moreover

$$W^{1,p^+}(\partial\Omega) \hookrightarrow W^{1,p(x)}(\Omega). \tag{10}$$

Then from (6), (9) and (10) we deduct the result.

□

**Proposition 2.3.** [16] *If $2p(x) \geq N$ for all $x \in \overline{\Omega}$, then the set*

$$X = \{u \in W^{2,p(x)}(\Omega) \ \ \frac{\partial u}{\partial \nu}|_{\partial\Omega} = 0\}$$

*is a closed subspace of* $W^{2,p(x)}(\Omega)$

*Proof.* Consider the operator

$$\begin{array}{rccc} D: & W^{2,p(x)}(\Omega) & \longrightarrow & L^{p(x)}(\partial\Omega) \\ & u & \longmapsto & \frac{\partial u}{\partial \nu}|_{\partial\Omega}. \end{array}$$

We prove that $D$ is continuous from $(W^{2,p(x)}(\Omega), \|.\|)$ to $(L^{p(x)}(\Omega), |.|_{L^{p(x)}(\partial\Omega)})$.
For this, we prove the continuity of the operator

$$\begin{array}{rccc} \nabla: & W^{2,p(x)}(\Omega) & \longrightarrow & (L^{p(x)}(\partial\Omega))^N \\ & u & \longmapsto & (\nabla u)|_{\partial\Omega}, \end{array}$$

from $(W^{2,p(x)}(\Omega), \|.\|)$ to $((L^{p(x)}(\partial\Omega))^N, \|.\|_{p(x),N})$, with $\|\overrightarrow{n}\|_{p(x),N} = \displaystyle\sum_{i=1}^{i=N} |n_i|_{p(x)}$.
Let $(u_n)_n \subset W^{2,p(x)}(\Omega)$ be a sequence such that $n_n \longrightarrow u$ in $W^{2,p(x)}(\Omega)$. Using the second assertion of theorem (2.2), we have $u_n \longrightarrow u$ in $W^{1,p(x)}(\partial\Omega)$, what implies that $\nabla u_n \longrightarrow \nabla u$ in $(L^{p(x)}(\partial\Omega))^N$, and then $\nabla$ is continuous.

Moreover, $D = T \circ \nabla$ with $T$ is the linear function defined as

$$\begin{array}{rccc} T: & (L^{p(x)}(\partial\Omega))^N & \longrightarrow & L^{p(x)}(\partial\Omega) \\ & \overrightarrow{n} & \longmapsto & \overrightarrow{n}.\overrightarrow{v}, \end{array}$$

where $\overrightarrow{v}(x) = (\alpha_1(x), \alpha_2(x), ..., \alpha_N(x))$ is the outer unit normal vector and $\sum_{i=1}^{i=N} |\alpha_i(x)|^2 = 1$ for all $x \in \partial\Omega$.
The operator $T$ is continuous, indeed, for $\overrightarrow{n} \in (L^{p(x)}(\partial\Omega))^N$, we have

$$|\overrightarrow{n}.\overrightarrow{v}|_{p(x)} = |\sum_{i=1}^{i=N} n_i\alpha_i| \leq \sum_{i=1}^{i=N} |n_i\alpha_i|_{p(x)}.$$

On the other hand, we have $\displaystyle\sum_{i=1}^{i=N} |\alpha_i(x)|^2 = 1$, then $|\alpha_i(x)| \leq 1$ for all $x \in \partial\Omega$, $i \in \{1, 2, ..., N\}$.
Consequently, we deduct that

$$|\overrightarrow{n}.\overrightarrow{v}|_{L^{p(x)}(\partial\Omega)} \leq \sum_{i=1}^{i=N} |n_i|_{p(x)} = \|\overrightarrow{n}\|_{p(x),N},$$

which assert that $T$ is continuous and then $D$ is also continuous. Finally, since $X = D^{-1}(\{0\})$, it result that $X$ is closed in $W^{2,p(x)}(\Omega)$. Hence, the proof of the proposition is completed. □

**Remark 2.2.** $(X; \|.\|)$ is a Banach, separable and reflexive space.

**Proposition 2.4.** *If we put*

$$I(u) = \int_\Omega |\Delta u|^{p(x)} dx,$$

*then for all $u \in X$ the following relations hold true*

(i) $\|u\| < 1 \ (= 1; > 1) \Longleftrightarrow I(u) < 1 \ (= 1; > 1)$,
(ii) $\|u\| \leq 1 \Longrightarrow \|u\|^{p^+} \leq I(u) \leq \|u\|^{p^-}$,
(iii) $\|u\| \geq 1 \Longrightarrow \|u\|^{p^-} \leq I(u) \leq \|u\|^{p^+}$,
*for all $u_n \in X$, we have*
(iv) $\|u_n\| \longrightarrow 0 \Longleftrightarrow I(u_n) \longrightarrow 0$,
(v) $\|u_n\| \longrightarrow \infty \Longleftrightarrow I(u_n) \longrightarrow \infty$

A pair $(u, \lambda) \in X \times \mathbb{R}$ is a weak solution of (3) provided that

$$\int_\Omega |\Delta u|^{p(x)-2} \Delta u \Delta v dx = \lambda \int_\Omega |u|^{q(x)-2} uv dx, \quad \forall v \in X.$$

In the case where $u$ is nontrivial, such a pair $(u, \lambda)$ is called an eigenpair, $\lambda$ is an eigenvalue and $u$ is called an associated eigenfunction.

**Proposition 2.5.** *If $u \in X$ is a weak solution of* (3) *and $u \in C^4(\overline{\Omega})$ then $u$ is a classical solution of* (3).

*Proof.* Let $u \in C^4(\overline{\Omega})$ be a weak solution of problem (3) then for every $\varphi \in X$, we have

$$\int_\Omega |\Delta u|^{p(x)-2} \Delta u \Delta \varphi dx = \lambda \int_\Omega |u|^{q(x)-2} u \varphi dx. \tag{11}$$

By applying Green formula, we have:

$$\int_\Omega \Delta(|\Delta u|^{p(x)-2} \Delta u) \varphi dx = -\int_\Omega \nabla(|\Delta u|^{p(x)-2} \Delta u).\nabla \varphi dx + \int_{\partial\Omega} \varphi \frac{\partial}{\partial \nu}(|\Delta u|^{p(x)-2} \Delta u) dx, \tag{12}$$

and

$$\int_\Omega |\Delta u|^{p(x)-2} \Delta u \Delta \varphi dx = -\int_\Omega \nabla(|\Delta u|^{p(x)-2} \Delta u).\nabla \varphi dx + \int_{\partial\Omega} |\Delta u|^{p(x)-2} \Delta u \frac{\partial}{\partial \nu}(\varphi) dx, \tag{13}$$

then we have

$$\int_\Omega \Delta(|\Delta u|^{p(x)-2} \Delta u) \Delta \varphi dx = \int_\Omega |\Delta u|^{p(x)-2} \Delta u \Delta \varphi dx,$$

the result follows. □

We will use the following lemma proved by Szulkin [20].

**Lemma 2.6.** *Let $E$ be a real Banach space and $A$, $B$ be symmetric subsets of $E \setminus \{0\}$ which are closed in $E$. Then:*

(a) *If there exists an odd continuous mapping $f : A \longrightarrow B$, then $\gamma(A) \leq \gamma(B)$.*
(b) *If $A \subset B$, then $\gamma(A) \leq \gamma(B)$.*
(c) $\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$.
(d) *If $\gamma(B) < +\infty$, $\gamma(\overline{A - B}) \geq \gamma(A) - \gamma(B)$.*
(e) *If $A$ is compact, then $\gamma(A) < +\infty$ and there exists a neighborhood $N$ of $A$, $N$ a symmetric subset of $E \setminus \{0\}$, closed in $E$ such that $\gamma(N) = \gamma(A)$*

(f) *If $N$ is a symmetric and bounded neighborhood of the origin in $\mathbb{R}^k$ and if $A$ is homeomorphic to the boundary of $N$ by an odd homeomorphism, then $\gamma(A) = k$.*
(g) *If $E_0$ is a subspace of $E$ of codimension $k$ and if $\gamma(A) > k$, then $A \cap E_0 = \emptyset$.*

In what follows, we have to need the following proposition which is an extension of Sobolev embedding theorems to the Sobolev spaces with variable exponent.

**Proposition 2.7.** *Let $p \in C_+(\overline{\Omega})$ such that $2p(x) > N$ for all $x \in \overline{\Omega}$, then*
(1) *there exists a continuous and compact embedding of $W^{2,p(x)}(\Omega)$ into $L^{q(x)}(\Omega)$, for all $q \in C_+(\Omega)$.*
(2) *there exists a continuous embedding of $W^{2,p(x)}(\Omega)$ into $C(\overline{\Omega})$.*

*Proof.* (1) we can refer to [5].
(2) For each $x \in \overline{\Omega}$, we have $2p(x) > N$.
Then, there exists a neighborhood $U_x \subset \overline{\Omega}$ such that

$$2p^-(U_x) > N,$$

where $p^-(U_x) = \inf_{y \in U_x} p(y)$.
Hence, we get a family open covering $\{U_x\}_{x\in\overline{\Omega}}$ for the compact set $\overline{\Omega}$. For a subcovering $\{U_i\}_{i=1,\dots,r}$, one considers $m_i$ such that

$$0 \le m_i < 2 - \frac{N}{p_i^-} < m_i + 1.$$

Thanks to the theorem 7.26 [14], there exists a continuous embedding

$$W^{2,p_i^-}(U_i) \hookrightarrow C^{m_i,\alpha_i}(\overline{U_i}), \tag{14}$$

where $\alpha_i = 2 - \frac{N}{p_-} - m_i$.
On the othere hand, for all $i \in \{1, 2, ..., r\}$, it easy to see that

$$W^{2,p(x)}(U_i) \subset W^{2,p_i^-}(U_i). \tag{15}$$

and

$$C^{m_i,\alpha_i}(\overline{U_i}) \subset C(\overline{U_i}). \tag{16}$$

From (14), (15) and (16), it follows that

$$W^{2,p(x)}(U_i) \subset C(\overline{U_i}),$$

for all $U_i, i = 1, 2, ..., r$. This assert that the embedding

$$W^{2,p(x)}(\Omega) \hookrightarrow C(\overline{\Omega}),$$

is continuous.

□

The Euler-Lagrange functional associated with (1) is defined as $\Phi_\lambda : X \to \mathbb{R}$,

$$\Phi_\lambda(u) = \int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)}\, dx - \lambda \int_\Omega \frac{1}{q(x)} |u|^{q(x)} dx.$$

Standard arguments imply that $\Phi_\lambda \in C^1(X, \mathbb{R})$ and

$$\langle \Phi_\lambda'(u), v\rangle = \int_\Omega |\Delta u|^{p(x)-2} \Delta u \Delta v\, dx - \lambda \int_\Omega |u|^{q(x)-2} uv dx,$$

for all $u, v \in X$. Thus the weak solutions of (1) coincide with the critical points of $\Phi_\lambda$. If such a weak solution exists and is nontrivial, then the corresponding $\lambda$ is an eigenvalue of problem (1).

Next, we write $\Phi'_\lambda$ as

$$\Phi'_\lambda = A - \lambda B,$$

where $A, B : X \to X'$ are defined by

$$\langle A(u), v\rangle = \int_\Omega |\Delta u|^{p(x)-2}\Delta u \Delta v\, dx,$$

$$\langle B(u), v\rangle = \int_\Omega |u|^{q(x)-2} uv\, dx.$$

We have

**Proposition 2.8.** [2, Proposition 2.5]

(i) *$B$ is completely continuous, namely, $u_n \rightharpoonup u$ in $X$ implies $B'(u_n) \to B'(u)$ in $X'$.*

(ii) *$A$ satisfies condition $(S^+)$, namely, $u_n \rightharpoonup u$, in $X$ and $\limsup\langle A(u_n), u_n - u\rangle \leq 0$, imply $u_n \to u$ in $X$.*

**Remark 2.3.** Noting that $\Phi'_\lambda$ is still of type $(S^+)$. Hence, any bounded (PS) sequence of $\Phi_\lambda$ in the reflexive Banach space $X$ has a convergent subsequence,

## 3. Main results and proofs

In what follows, we assume that the functions $p, q \in C_+(\overline{\Omega})$.

**Theorem 3.1.** *If*

$$q^+ < p^-, \tag{17}$$

*then any $\lambda > 0$ is an eigenvalue for problem* (1)*. Moreover, for any $\lambda > 0$ there exists a sequence $(u_n)$ of nontrivial weak solutions for problem* (1) *such that $u_n \to 0$ in $X$.*

We want to apply the symmetric mountain pass lemma in [7].

**Theorem 3.2.** *(Symmetric mountain pass lemma) Let $E$ be an infinite dimensional Banach space and $I \in C^1(E, R)$ satisfy the following two assumptions:*

(A1) *$I(u)$ is even, bounded from below, $I(0) = 0$ and $I(u)$ satisfies the Palais-Smale condition (PS), namely, any sequence $u_n$ in $E$ such that $I(u_n)$ is bounded and $I'(u_n) \to 0$ in $E$ as $n \to \infty$ has a convergent subsequence.*

(A2) *For each $k \in \mathbb{N}$, there exists an $A_k \in \Gamma_k$ such that $\sup_{u\in A_k} I(u) < 0$.*

*Then, $I(u)$ admits a sequence of critical points $u_k$ such that*

$$I(u_k) < 0, u_k \neq 0 \text{ and } \lim_k u_k = 0,$$

*where $\Gamma_k$ denote the family of closed symmetric subsets $A$ of $E$ such that $0 \notin A$ and $\gamma(A) \geq k$ with $\gamma(A)$ is the genus of $A$, i.e.,*

$$\gamma(K) = \inf\{k \in \mathbb{N} : \exists h : K \to \mathbb{R}^k\backslash\{0\} \text{ such that } h \text{ is continuous and odd }\}.$$

We start with two auxiliary results.

**Lemma 3.3.** *The functional $\Phi_\lambda$ is even, bounded from below and satisfies the (PS) condition; $\Phi_\lambda(0) = 0$.*

*Proof.* It is clear that $\Phi_\lambda$ is even and $\Phi_\lambda(0) = 0$. Since $q^+ < p^-$ and $X$ is continuously embedded both in $L^{q^\pm}(\Omega)$, there exist two positive constants $d_1, d_2 > 0$ such that

$$\int_\Omega |u|^{q^+} dx \leq d_1\|u\|^{q^+}, \quad \int_\Omega |u|^{q^-} dx \leq d_2\|u\|^{q^-}, \quad \forall u \in X.$$

According to the fact that

$$|u(x)|^{q(x)} \leq |u(x)|^{q^+} + |u(x)|^{q^-}, \quad \forall x \in \overline{\Omega}, \tag{18}$$

for all $u \in X$, we have

$$\begin{aligned}\Phi_\lambda(u) &\geq \frac{1}{p^+}\int_\Omega |\Delta u|^{p(x)} - \frac{\lambda d_1}{q^-}\|u\|^{q^+} - \frac{\lambda d_2}{q^-}\|u\|^{q^-}\\ &\geq \frac{1}{p^+}\alpha(\|u\|) - \frac{\lambda d_1}{q^-}\|u\|^{q^+} - \frac{\lambda d_2}{q^-}\|u\|^{q^-},\end{aligned}$$

where $\alpha : [0, +\infty[ \to \mathbb{R}$ is defined by

$$\alpha(t) = \begin{cases} t^{p^+}, & \text{if } t \leq 1,\\ t^{p^-}, & \text{if } t > 1.\end{cases} \tag{19}$$

As $q^+ < p^-$, $\Phi_\lambda$ is bounded from below and coercive because, that is, $\Phi_\lambda(u) \to \infty$ as $\|u\| \to \infty$.

It remains to show that the functional $\Phi_\lambda$ satisfies the (PS) condition to complete the proof. Let $(u_n) \subset X$ be a (PS) sequence of $\Phi_\lambda$ in $X$; that is,

$$\Phi_\lambda(u_n) \text{ is bounded and } \Phi'_\lambda(u_n) \to 0 \text{ in } X'. \tag{20}$$

Then, by the coercivity of $\Phi_\lambda$, the sequence $(u_n)$ is bounded in $X$. By the reflexivity of $X$, for a subsequence still denoted $(u_n)$, we have

$$u_n \rightharpoonup u \quad \text{in } X.$$

Since $q^+ < p^-$, it follows from theorem 3.2 that $u_n \rightharpoonup u$ in $L^{q(x)}(\Omega)$. Using the properties of Nemytskii operator $N_q(x)$ defined by

$$N_{q(x)}(v)(x) = \begin{cases} |v(x)|^{q(x)-2}v(x) & \text{if } v(x) \neq 0,\\ 0 & \text{otherwise,}\end{cases}$$

we deduce that

$$\langle B(u_n), u_n - u\rangle = \int_\Omega |u_n(x)|^{q(x)-2}u_n(x)(u_n(x) - u)\,dx \to 0. \tag{21}$$

In view of (20) and (21), we obtain

$$\Phi'_\lambda(u_n) + \lambda\langle B(u_n), u_n - u\rangle = \langle A(u_n), u_n - u\rangle \to 0 \quad \text{as } n \to \infty.$$

According to the fact that $A$ satisfies condition $(S^+)$, we have $u_n \to u$ in $X$. The proof is complete. □

**Lemma 3.4.** *For each $n \in \mathbb{N}^*$, there exists an $H_n \in \Gamma_n$ such that*

$$\sup_{u \in H_n} \Phi_\lambda(u) < 0.$$

*Proof.* Let $v_1, v_2, \ldots, v_n \in C_0^\infty(\Omega)$ such that $\operatorname{supp}(v_i) \cap \operatorname{supp}(v_j) = \emptyset$ if $i \neq j$ and $\operatorname{meas}(\operatorname{supp}(v_j)) > 0$ for $i, j \in \{1, 2, \ldots, n\}$. Take $F_n = \operatorname{span}\{v_1, v_2, \ldots, v_n\}$, it is clear that $\dim F_n = n$ and

$$\int_\Omega |v(x)|^{q(x)}dx > 0 \quad \text{for all } v \in F_n \setminus \{0\}.$$

Denote $S = \{v \in X : \ \|v\| = 1\}$ and $H_n(t) = t(S \cap F_n)$ for $0 < t \leq 1$. Obviously, $\gamma(H_n(t)) = n$, for all $t \in ]0, 1]$.

Now, we show that, for any $n \in \mathbb{N}^*$, there exist $t_n \in ]0, 1]$ such that

$$\sup_{u \in H_n(t_n)} \Phi_\lambda(u) < 0.$$

Indeed, for $0 < t \leq 1$, we have

$$\begin{aligned}\sup_{u\in H_n(t)} \Phi_\lambda(u) &\leq \sup_{v\in S\cap F_n} \Phi_\lambda(tv)\\ &= \sup_{v\in S\cap F_n} \Big\{ \int_\Omega \frac{t^{p(x)}}{p(x)}|\Delta v(x)|^{p(x)}dx - \lambda \int_\Omega \frac{t^{q(x)}}{q(x)}|v(x)|^{q(x)}\,dx\Big\}\\ &\leq \sup_{v\in S\cap F_n} \Big\{\frac{t^{p^-}}{p^-}\int_\Omega |\Delta v(x)|^{p(x)}dx - \frac{\lambda t^{q^+}}{q^+}\int_\Omega |v(x)|^{q(x)}\,dx\Big\}\\ &= \sup_{v\in S\cap F_n} \Big\{t^{p^-}\Big(\frac{1}{p^-} - \frac{\lambda}{q^+}\frac{1}{t^{p^- - q^+}}\int_\Omega |v(x)|^{q(x)}\,dx\Big)\Big\}.\end{aligned}$$

Since $m := \min_{v\in S\cap F_n} \int_\Omega |v(x)|^{q(x)}\,dx > 0$, we may choose $t_n \in ]0,1]$ which is small enough such that

$$\frac{1}{p^-} - \frac{\lambda}{q^+}\frac{1}{t_n^{p^- - q^+}} m < 0.$$

This completes the proof. $\square$

*Proof of Theorem 3.1.* By Lemmas 3.3, 3.4 and Theorem 3.2, $\Phi_\lambda$ admits a sequence of nontrivial weak solutions $(u_n)_n$ such that for any $n$, we have

$$u_n \neq 0, \quad \Phi'_\lambda(u_n) = 0, \quad \Phi_\lambda(u_n) \leq 0, \quad \lim_n u_n = 0. \tag{22}$$

$\square$

**Theorem 3.5.** *If*

$$q^- < p^- \quad \textit{and} \quad q^+ < p_2^*(x) \quad \textit{for all } x \in \overline{\Omega}, \tag{23}$$

*then there exists $\lambda^* > 0$ such that any $\lambda \in (0,\lambda^*)$ is an eigenvalue for problem* (1).

For applying Ekeland's variational principle. We start with two auxiliary results.

**Lemma 3.6.** *There exists $\lambda^* > 0$ such that for any $\lambda \in (0,\lambda^*)$ there exist $\rho, a > 0$ such that $\Phi_\lambda(u) \geq a > 0$ for any $u \in X$ with $\|u\| = \rho$.*

*Proof.* Since $q(x) < p_2^*(x)$ for all $x \in \overline{\Omega}$, it follows that $X$ is continuously embedded in $L^{q(x)}(\Omega)$. So, there exists a positive constant $c_1$ such that

$$|u|_{q(x)} \leq c_1\|u\|, \quad \text{for all } u \in X. \tag{24}$$

Let us fix $\rho \in ]0,1[$ such that $\rho < \frac{1}{c_1}$. Then relation (24) implies $|u|_{q(x)} < 1$, for all $u \in X$ with $\|u\| = \rho$. Thus,

$$\int_\Omega |u|^{q(x)}dx \leq |u|_{q(x)}^{q^-}, \quad \text{for all } u \in X \text{ with } \|u\| = \rho. \tag{25}$$

Combining (24) and (25), we obtain

$$\int_\Omega |u|^{q(x)}dx \leq c_1^{q^-}\|u\|^{q^-}, \quad \text{for all } u \in X \text{ with } \|u\| = \rho. \tag{26}$$

Hence, from (26) we deduce that for any $u \in X$ with $\|u\|_k = \rho$, we have

$$\begin{aligned}\Phi_\lambda(u) &\geq \frac{1}{p^+}\int_\Omega |\Delta u|^{p(x)}dx - \frac{\lambda}{q^-}\int_\Omega |u|^{q(x)}dx \\ &\geq \frac{1}{p^+}\|u\|^{p^+} - \frac{\lambda}{q^-}c_1^{q^-}\|u\|^{q^-} \\ &= \frac{1}{p^+}\rho^{p^+} - \frac{\lambda}{q^-}c_1^{q^-}\rho^{q^-} \\ &= \rho^{q^-}(\frac{1}{p^+}\rho^{p^+-q^-} - \frac{\lambda}{q^-}c_1^{q^-}).\end{aligned}$$

Putting

$$\lambda_* = \frac{\rho^{p^+-q^-}}{2p^+}\frac{q^-}{c_1^{q^-}}, \tag{27}$$

for any $u \in X$ with $\|u\| = \rho$, there exist $a = \rho^{p^+}/(2p^+)$ such that

$$\Phi_\lambda(u) \geq a > 0.$$

This completes the proof. □

**Lemma 3.7.** *There exists $\psi \in X$ such that $\psi \geq 0$, $\psi \neq 0$ and $\Phi_\lambda(t\psi) < 0$, for $t > 0$ small enough.*

*Proof.* Since $q^- < p^-$, there exist $\varepsilon_0 > 0$ such that

$$q^- + \varepsilon_0 < p^-.$$

Since $q \in C(\overline{\Omega})$, there exist an open set $\Omega_0 \subset \Omega$ such that

$$|q(x) - q^-| < \varepsilon_0, \quad \text{for all } \ x \in \Omega_0.$$

Thus, we deduce

$$q(x) \leq q^- + \varepsilon_0 < p^-, \quad \text{for all } x \in \Omega_0. \tag{28}$$

Take $\psi \in C_0^\infty(\Omega)$ such that $\overline{\Omega}_0 \subset \operatorname{supp}\psi$, $\psi(x) = 1$ for $x \in \overline{\Omega}_0$ and $0 \leq \psi \leq 1$ in $\Omega$. Without loss of generality, we may assume $\|\psi\| = 1$, that is

$$\int_\Omega |\Delta\psi|^{p(x)}\, dx = 1. \tag{29}$$

By using (28), (29) and the fact

$$\int_{\Omega_0} |\psi|^{q(x)}dx = \operatorname{meas}(\Omega_0)$$

for all $t \in ]0,1[$, we obtain

$$\begin{aligned}\Phi_\lambda(t\psi) &= \int_\Omega \frac{t^{p(x)}}{p(x)}|\Delta\psi|^{p(x)}\, dx - \lambda\int_\Omega \frac{t^{q(x)}}{q(x)}|\psi|^{q(x)}dx \\ &\leq \frac{t^{p^-}}{p^-}\int_\Omega |\Delta\psi|^{p(x)}\, dx - \frac{\lambda}{q^+}\int_\Omega t^{q(x)}|\psi|^{q(x)}dx \\ &\leq \frac{t^{p^-}}{p^-} - \frac{\lambda}{q^+}\int_{\Omega_0} t^{q(x)}|\psi|^{q(x)}dx \\ &\leq \frac{t^{p^-}}{p^-} - \frac{\lambda t^{q^-+\varepsilon_0}}{q^+}\operatorname{meas}(\Omega_0).\end{aligned}$$

Then, for any $t < \delta^{\frac{1}{p^- - q^- - \varepsilon_0}}$, with $0 < \delta < \min\{1, \lambda p^- \operatorname{meas}(\Omega_0)/q^+\}$, we conclude that

$$\Phi_\lambda(t\psi) < 0.$$

The proof is complete. □

*Proof of Theorem 3.5.* By Lemma 3.6, we have

$$\inf_{\partial B_\rho(0)} \Phi_\lambda > 0. \tag{30}$$

On the other hand, from Lemma 3.7, there exists $\psi \in X$ such that $\Phi_\lambda(t\psi) < 0$ for $t > 0$ small enough. Using (26), it follows that

$$\Phi_\lambda(u) \geq \frac{1}{p^+}\|u\|^{p^+} - \frac{\lambda}{q^-} c_1^{q^-} \|u\|^{q^-} \quad \text{for } u \in B_\rho(0).$$

Thus,

$$-\infty < \underline{c}_\lambda := \inf_{\overline{B_\rho(o)}} \Phi_\lambda < 0,$$

Let

$$0 < \varepsilon < \inf_{\partial B_\rho(0)} \Phi_\lambda - \inf_{\overline{B_\rho(0)}} \Phi_\lambda.$$

Then, by applying Ekeland's variational principle to the functional

$$\Phi_\lambda : \overline{B_\rho(0)} \to \mathbb{R},$$

there exist $u_\varepsilon \in \overline{B_\rho(0)}$ such that

$$\Phi_\lambda(u_\varepsilon) \leq \inf_{\overline{B_\rho(0)}} \Phi_\lambda + \varepsilon,$$

$$\Phi_\lambda(u_\varepsilon) < \Phi_\lambda(u) + \varepsilon\|u - u_\varepsilon\| \;\; \text{for } u \neq u_\varepsilon.$$

Since $\Phi_\lambda(u_\varepsilon) < \inf_{\overline{B_\rho(0)}} \Phi_\lambda + \varepsilon < \inf_{\partial B_\rho(0)} \Phi_\lambda$, we deduce $u_\varepsilon \in B_\rho(0)$.

Now, define $I_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$ by

$$I_\lambda(u) = \Phi_\lambda(u) + \varepsilon\|u - u_\varepsilon\|.$$

It is clear that $u_\varepsilon$ is an minimum of $I_\lambda$. Therefore, for $t > 0$ and $v \in B_1(0)$, we have

$$\frac{I_\lambda(u_\varepsilon + tv) - I_\lambda(u_\varepsilon)}{t} \geq 0$$

for $t > 0$ small enough and $v \in B_1(0)$; that is,

$$\frac{\Phi_\lambda(u_\varepsilon + tv) - \Phi_\lambda(u_\varepsilon)}{t} + \varepsilon\|v\| \geq 0$$

for $t$ positive and small enough, and $v \in B_1(0)$. As $t \to 0$, we obtain

$$\langle \Phi'_\lambda(u_\varepsilon), v\rangle + \varepsilon\|v\| \geq 0 \quad \text{for all } v \in B_1(0).$$

Hence, $\|\Phi'_\lambda(u_\varepsilon)\|_{X'} \leq \varepsilon$. We deduce that there exists a sequence $(u_n)_n \subset B_\rho(0)$ such that

$$\Phi_\lambda(u_n) \to \underline{c}_\lambda \quad \text{and} \quad \Phi'_\lambda(u_n) \to 0. \tag{31}$$

It is clear that $(u_n)$ is bounded in $X$. By a standard arguments and the fact $A$ is type of $(S^+)$, for a subsequence we obtain $u_n \to u$ in $X$ as $n \to +\infty$. Thus, by (31) we have

$$\Phi_\lambda(u) = \underline{c}_\lambda < 0 \quad \text{and} \quad \Phi'_\lambda(u) = 0 \quad \text{as } n \to \infty. \tag{32}$$

The proof is complete. □

**Theorem 3.8.** *If*

$$p^+ < q^- \leq q^+ < p_2^*(x) \quad \textit{for all } x \in \overline{\Omega}, \tag{33}$$

*then for any* $\lambda > 0$, *problem* (1) *possesses a nontrivial weak solution.*

We want to construct a mountain geometry, and first need two lemmas.

**Lemma 3.9.** *There exist* $\eta, b > 0$ *such that* $\Phi_\lambda(u) \geq b$, *for* $u \in X$ *with* $\|u\| = \eta$.

*Proof.* Since $q^+ < p_2^*$, in view the Theorem 3.2, there exist $d_1, d_2 > 0$ such that

$$|u|_{q^+} \leq d_1\|u\| \quad \text{and} \quad |u|_{q^+} \leq d_2\|u\|.$$

Thus, from (18) we obtain

$$\begin{aligned}
\Phi_\lambda(u) &\geq \frac{1}{p^+}\int_\Omega |\Delta u(x)|^{p(x)}dx - \frac{\lambda}{q^-}\big[(d_1\|u\|)^{q^+} + (d_2\|u\|)^{q^-}\big]\\
&\geq \frac{1}{p^+}\alpha(\|u\|) - \frac{\lambda d_1^{q^+}}{q^-}\|u\|^{q^+} - \frac{\lambda d_2^{q^-}}{q^-}\|u\|^{q^-}\\
&= \begin{cases} (\frac{1}{p^+} - \frac{d_1^{q^+}}{q^-}\|u\|^{q^+-p^+} - \frac{\lambda d_2^{q^-}}{q^-}\|u\|^{q^- - p^+})\|u\|^{p^+} & \text{if } \|u\| \leq 1,\\ (\frac{1}{p^+} - \frac{d_1^{q^+}}{q^-}\|u\|^{q^+-p^-} - \frac{\lambda d_2^{q^-}}{q^-}\|u\|^{q^- - p^-})\|u\|^{p^-} & \text{if } \|u\| > 1.\end{cases}
\end{aligned}$$

Since $p^+ < q^- \leq q^+$, the functional $g : [0,1] \to \mathbb{R}$ defined by

$$g(s) = \frac{1}{p^+} - \frac{d_1^{q^+}}{q^-}s^{q^+-p^+} - \frac{\lambda d_2^{q^-}}{q^-}s^{q^--p^+}$$

is positive on neighborhood of the origin. So, the result of Lemma 3.9 follows. □

**Lemma 3.10.** *There exists* $e \in X$ *with* $\|e\| \geq \eta$ *such that* $\Phi_\lambda(e) < 0$, *where* $\eta$ *is given in Lemma 3.9.*

*Proof.* Choose $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$ and $\varphi \neq 0$. For $t > 1$, we have

$$\Phi_\lambda(t\varphi) \leq \frac{t^{p^+}}{p^-}\int_\Omega \big|\Delta\varphi(x)|^{p(x)}dx - \frac{\lambda t^{q^-}}{q^+}\int_\Omega |\varphi(x)|^{q(x)}dx.$$

Then, since $p^+ < q^-$, we deduce that

$$\lim_{t\to\infty}\Phi_\lambda(t\varphi) = -\infty.$$

Therefore, for $t > 1$ large enough, there is $e = t\varphi$ such that $\|e\| \geq \eta$ and $\Phi_\lambda(e) < 0$. This completes the proof. □

**Lemma 3.11.** *The functional* $\Phi_\lambda$ *satisfies the condition* $(PS)$.

*Proof.* Let $(u_n) \subset X$ be a sequence such that $d := \sup_n \Phi_\lambda(u_n) < \infty$ and $\Phi_\lambda'(u_n) \to 0$ in $X'$. By contradiction suppose that

$$\|u_n\| \to +\infty \text{ as } n \to \infty \quad \text{and} \quad \|u_n\| > 1 \quad \text{for any } n.$$

Thus,

$$\begin{aligned} d+1+\|u_n\| &\geq \Phi_\lambda(u_n) - \frac{1}{q^-}\langle \Phi_\lambda'(u_n), u_n\rangle \\ &= \int_\Omega \frac{1}{p(x)}|\Delta u_n|^{p(x)}dx - \frac{\lambda}{q^-}\int_\Omega |\Delta u_n|^{p(x)}dx + \lambda\int_\Omega (\frac{1}{q^-} - \frac{1}{q(x)})|u_n|^{q(x)}\,dx \\ &\geq (\frac{1}{p^+} - \frac{1}{q^-})\int_\Omega |\Delta u_n|^{p(x)}dx \\ &\geq (\frac{1}{p^+} - \frac{1}{q^-})\|u_n\|^{p^-}. \end{aligned}$$

This contradicts the fact that $p^- > 1$. So, the sequence $(u_n)$ is bounded in $X$ and similar arguments as those used in the proof of Lemma 3.4 completes the proof. □

*Proof of theorem 3.8.* From Lemmas 3.9 and 3.10, we deduce

$$\max(\Phi_\lambda(0), \Phi_\lambda(e)) = \Phi_\lambda(0) < \inf_{\|u\|=\eta} \Phi_\lambda(u) =: \beta.$$

By Lemma 3.11 and the mountain pass theorem, we deduce the existence of critical points $u$ of $\Phi_\lambda$ associated of the critical value given by

$$c := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} \Phi_\lambda(\gamma(t)) \geq \beta, \tag{34}$$

where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0 \text{ and } \gamma(1) = e\}$. This completes the proof. □

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