Nonresonance conditions for a p-biharmonic operator with weight

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ABSTRACT. This work is devoted to study two nonlinear problems of fourth order governed by the p-biharmonic operators in nonresonance cases. In the first problem we establish the nonresonance part of the Fredholm’s alternative, the second is a nonresonance problem relative to the first eigensurface for the spectrum of the operator $\Delta^2_p u + 2\beta \nabla ((|\Delta u|^{p-2}\Delta u) + |\beta|^2|\Delta u|^{p-2}\Delta u$, where $\beta \in \mathbb{R}^N$ under Navier boundary conditions.

2010 Mathematics Subject Classification. 35A15, 35J40, 35J60.
Key words and phrases. Third order spectrum, nonresonance conditions, p-biharmonic operator.

1. Introduction

We consider the following problem

$$\begin{cases}
\Delta^2_p u + 2\beta \nabla ((|\Delta u|^{p-2}\Delta u) + |\beta|^2|\Delta u|^{p-2}\Delta u = f(x, u, \Delta u) + h(x) \quad \text{in } \Omega, \\
u = \Delta u = 0 \quad \text{on } \partial \Omega,
\end{cases}$$

where $\Omega$ is a bounded smooth domain in $\mathbb{R}^N$ ($N \geq 1$), $\beta \in \mathbb{R}^N$, $\Delta^2_p$ denotes the p-biharmonic operator defined by $\Delta^2_p u = \Delta(|\Delta u|^{p-2}\Delta u)$, $h \in L^{p'}(\Omega)$, $\left(p' = \frac{p}{p-1}\right)$, and $m \in M = \{m \in L^\infty(\Omega) : \text{meas}\{x \in \Omega / m(x) > 0\} \neq 0\}$.

The investigation of existence of solutions for problems at nonresonance has drawn the attention of many authors, see for example [1, 8, 10, 13].

Recently, Ben Haddouch et al. [4, 5, 3], showed that the spectrum of problem

$$\begin{cases}
\Delta^2_p u + 2\beta \nabla ((|\Delta u|^{p-2}\Delta u) + |\beta|^2|\Delta u|^{p-2}\Delta u = \Gamma m(x)|u|^{p-2}u \quad \text{in } \Omega, \\
u = \Delta u = 0 \quad \text{on } \partial \Omega,
\end{cases}$$

contains at least one sequence of positive eigensurfaces $(\Gamma_n^{p} (\cdot, m))_n$ defined by

$$\Gamma_n^{p} (\beta, m) = \inf_{K \in B_n} \sup_{u \in K} \int_{\Omega} e^{\beta \cdot x} |\Delta u|^{p} dx,$$

and

$$\Gamma_n^{p} (\beta, m) \rightarrow +\infty \quad \text{as} \quad n \rightarrow +\infty,$$

where

$$B_n = \{K \subset N_\beta : K \text{ is compact, symmetric and } \gamma(K) \geq n\}$$

and

$$N_\beta = \{u \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) : \int_{\Omega} me^{\beta \cdot x} |u|^{p} dx = 1\}.$$
Since \( m \in C(\Omega) \) and \( m \geq 0 \), the authors proved that the first eigensurface \( \Gamma_{1}^{\beta}(\cdot, m) \) is positive, simple and isolated.

In the present paper, using the topological degree theory applied to compact operators and operators of \((S+)\) type, we show the existence of a nontrivial solution of problem (1).

2. Preliminaries

In our further considerations we will use the standard spaces \( X = W^{2,p}(\Omega) \cap \mathcal{W}^{1,p} \cap L^{\infty}(\Omega) \), with corresponding norms \( \|u\|_{2,p} = (\|\Delta u\|_{p}^{p} + \|u\|_{p}^{p})^{\frac{1}{p}} \), \( \|u\|_{p} = (\int_{\Omega} |u|^{p} dx)^{\frac{1}{p}} \) and \( \|u\|_{\infty} \) respectively.

Recall that for all \( f \in L^{p}(\Omega) \), the Poisson equation associated with the Dirichlet problem

\[
\begin{cases}
-\Delta u = f & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

is uniquely solvable in \( X \) (cf. [12]). We denote by \( \Lambda \) the inverse operator of \( -\Delta : X \to L^{p}(\Omega) \).

In the following lemma we give some properties of the operator \( \Lambda \) (cf. [11]).

**Lemma 2.1.**

(i) (Continuity) There exists a constant \( C_{p} > 0 \) such that: \( \|\Lambda f\|_{2,p} \leq C_{p} \|f\|_{p} \) holds for all \( p \in ]1, +\infty[ \) and \( f \in L^{p}(\Omega) \).

(ii) (Continuity) Given \( k \in \mathbb{N}^{*} \), for all \( p \in ]1, +\infty[ \) there exists a constant \( C_{p,k} > 0 \) such that for all \( f \in L^{p}(\Omega) \), \( \|\Lambda f\|_{W^{k+2,p}} \leq C_{p,k} \|f\|_{W^{k,p}} \).

(iii) (Symmetry) The following identity: \( \int_{\Omega} \Lambda u.vdx = \int_{\Omega} u.\Lambda vdx \) holds for all \( u \in L^{p}(\Omega) \) and \( v \in L^{p}(\Omega) \) with \( p \in ]1, +\infty[ \).

(iv) (Regularity) Given \( f \in L^{\infty}(\Omega) \), we have \( \Lambda f \in C^{1,\alpha}(\overline{\Omega}) \) for all \( \alpha \in ]0,1[ \). Moreover, there exists \( C_{\alpha} > 0 \) such that \( \|\Lambda f\|_{C^{1,\alpha}} \leq C_{\alpha}\|f\|_{\infty} \).

(v) (Regularity and Hopf-type maximum principle) Let \( f \in C(\overline{\Omega}) \) and \( f \geq 0 \) then \( w = \Lambda f \in C^{1,\alpha}(\overline{\Omega}) \), for all \( \alpha \in ]0,1[ \) and \( w \) satisfies: \( w > 0 \) in \( \Omega \), \( \frac{\partial w}{\partial n} < 0 \) on \( \partial \Omega \).

(vi) (Order preserving property) Given \( f, g \in L^{p}(\Omega) \), if \( f \leq g \) in \( \Omega \) then \( \Lambda f \leq \Lambda g \) in \( \Omega \).

Let \( N_{p} \) be the Nemytskii operator defined by:

\[
\begin{align*}
N_{p}(v)(x) &= |v(x)|^{p-2}v(x) & \text{if } v(x) \neq 0, \\
N_{p}(v)(x) &= 0 & \text{if } v(x) = 0.
\end{align*}
\]

We have \( \forall v \in L^{p}(\Omega) \) \( \forall w \in L^{p}(\Omega) \) \( N_{p}(v) = w \iff v = N_{p}(w) \). The operator \( \Lambda \) enables us to transform the problem (1) to the other problem which we will study in the space \( L^{p}(\Omega) \).

**Lemma 2.2.** [4] The problem (1) is equivalent to problem

\[
\begin{cases}
\text{Find } v \in L^{p}(\Omega) \setminus \{0\} \text{ such that } \\
e^{\beta \cdot x}N_{p}(v) = \Lambda(e^{\beta \cdot x}f(\cdot, \Lambda v, v)) + \Lambda(e^{\beta \cdot x}h) \text{ in } L^{p}(\Omega),
\end{cases}
\]

**Definition 2.1.** We say that \( u \in X \) is a solution of problem (1) if \( v \in L^{p}(\Omega) \), where \( v = -\Delta u \) is a solution of the problem (4).
Let us here recall for the reader’s convenience the theorem of Leray-Schauder [9].
Let $X$ a Banach space, $\mathcal{O} \subset X$ a non-empty set of $X$ and $f : \mathcal{O} \to X$ be a compact mapping. Put

$$\mathcal{E}_1 = \{(id-f, \mathcal{O}, y) : \mathcal{O} \subset X \text{ bounded open}, f : \mathcal{O} \to X \text{ is compact and } y \notin (id-f)(\partial \mathcal{O})\}.$$ 

**Theorem 2.3.** There exists a unique function $d : \mathcal{E}_1 \to \mathbb{Z}$ called the topological degree, satisfying:

(i) $d(id, \mathcal{O}, y) = 1$ for all $y \in \mathcal{O}$.

(ii) (Homotopy invariance): If $h : [0, 1] \times \mathcal{O} \to X$ is a compact mapping and $y : [0, 1] \to X$ a compact mapping such that $y(t) \notin (id-h(t, \cdot))(\partial \mathcal{O})$ for all $t \in [0, 1]$, then $d(id - h(t, \cdot), \mathcal{O}, y(t))$ is independent of $t \in [0, 1]$.

(iii) If $d(id - F, \mathcal{O}, y) \neq 0$, then $(id - F)^{-1}(y) \neq \emptyset$.

(iv) If $f_{|\mathcal{O}} = g_{|\mathcal{O}}$, then $d(id - f, \mathcal{O}, y) = d(id - g, \mathcal{O}, y)$.

(v) (Borsuk’s theorem): If $\mathcal{O}$ is more symmetric with $0 \in \mathcal{O}$ and $f$ is odd on $\mathcal{O}$, then $d(id - f, \mathcal{O}, 0)$ is an odd integer.

3. Fredholm’s alternative

We establish the nonresonance part of the Fredholm’s alternative for the operator $\Theta_{p, \beta}$ which is defined by $\Theta_{p, \beta}u := \Delta p u + 2\beta \nabla (|\Delta u|^{p-2} \Delta u) + |\beta|^2 |\Delta u|^{p-2} \Delta u$ in the case

$$f(x, u, \Delta u) = \Gamma m(x)|u|^{p-2}u$$

where $\Gamma$ is not in the spectrum associated with the operator $\Theta_{p, \beta}$ with weight $m(x)$. Problem (4) remain to

\[ \begin{cases} 
\text{Find } v \in L^p(\Omega) \setminus \{0\} \text{ such that } \\
\quad e^{\beta \cdot x} N_p(v) = \Gamma \Lambda(e^{\beta \cdot x} N_p(\Lambda v)) + \Lambda(e^{\beta \cdot x} h) \text{ in } L^p(\Omega). 
\end{cases} \]

(5)

we have the following result.

**Theorem 3.1.** For all $h \in L^p(\Omega)$, the problem (5) admits at least one nontrivial solution. Moreover, if $h \in L^\infty(\Omega)$, then every solution of (5) is in $C(\Omega)$.

**Proof.** To prove the existence of nontrivial solution of (5), we use the property of Leray-Schauder’s topological degree. Consider the family of operators $(T_t)_{t \in [0, 1]}$ defined from $L^p(\Omega)$ to $L^p(\Omega)$ by

$$\forall v \in L^p(\Omega) \quad \forall t \in [0, 1] \quad T_t(v) = N_p(\Gamma e^{-\beta \cdot x} \Lambda(m e^{\beta \cdot x} N_p(\Lambda v)) + e^{-\beta \cdot x} \Lambda(t e^{\beta \cdot x} h)).$$

Let $(v_n)_n$ be a sequence in $L^p(\Omega)$ such that $v_n \rightharpoonup v$ in $L^p(\Omega)$, then under assertion (i) of lemma 2.1 and by Sobolev’s injection theorem, we have $\Lambda v_n \rightharpoonup \Lambda v$ in $X$ and $\Lambda v_n \to \Lambda v$ in $L^p(\Omega)$. We deduce that for every $t \in [0, 1]$, $T_t$ is a compact operator.

According to the theorem of Leray-Schauder 2.3, it suffices to prove the following a priori estimate

$$\exists \tau > 0 \quad \text{such that } \quad v - T_t(v) \neq 0 \quad \forall v \in \partial B(0, \tau), \quad \forall t \in [0, 1].$$ 

(6)

By contradiction, we assume that

$$\forall n \in \mathbb{N}^* \exists v_n \in \partial B(0, n) \exists t_n \in [0, 1] \text{ such that } \quad T_{t_n}(v_n) = v_n.$$ 

(7)

We set for all $n \in \mathbb{N}^*$, $w_n = \frac{v_n}{\|v_n\|_p}$. The sequence $(w_n)_{n \geq 1}$ is bounded in $L^p(\Omega)$, then there is a subsequence of $(w_n)_{n \geq 1}$, still denoted by $(w_n)_{n \geq 1}$ such that $w_n \rightharpoonup w$.
in $L^p(\Omega)$ and $\Lambda w_n \to \Lambda w$ in $L^p(\Omega)$. Dividing (7) by $\|v_n\|_p$, we obtain

$$w_n = N_p'(\Gamma e^{-\beta \cdot x} \Lambda (me^{\beta \cdot x} N_p(\Lambda w_n))) + t_n e^{-\beta \cdot x} \Lambda (e^{\beta \cdot x} h) / \|v_n\|_p^{p-1}.$$

The fact that $N_p(\Lambda w_n) \to N_p(\Lambda w)$ and $t_n \Lambda(e^{\beta \cdot x} h) / \|v_n\|_p^{p-1} \to 0$ in $L^p'(\Omega)$, we get

$$w_n \to N_p'(\Gamma e^{-\beta \cdot x} \Lambda (me^{\beta \cdot x} N_p(\Lambda w)))$$

in $L^p(\Omega)$. We deduce that $w_n \to w$ in $L^p(\Omega)$ and $w \not\equiv 0$. In conclusion we have

$$\begin{cases} w = N_p'(\Gamma e^{-\beta \cdot x} \Lambda (me^{\beta \cdot x} N_p(\Lambda w))), \\ w \in L^p(\Omega) \setminus \{0\}. \end{cases}$$

Which is contradicts with $\Gamma$ is not in the spectrum of the operator $\Theta_{P,\beta}$. Consequently the estimate (6) holds and one has

$$d(I - T_1, B(0, r), 0) = d(I - T_0, B(0, r), 0),$$

where $d$ is the topologic degree function, $I$ is the identity of $L^p(\Omega)$, $B(0, r)$ is the ball of center 0 and radius $r$ and $\partial B(0, r)$ is its boundary. The Theorem 2.3, (v) assures that

$$d(I - T_0, B(0, r), 0) \neq 0.$$

Thus there exists $v \in B(0, r)$ such that $(I - T_1)(v) = 0$, which will prove the existence of a solution of problem (5). \hfill $\square$

### 4. Non-resonance relative to the first eigensurface of $\Theta_{P,\beta}$

In problem (4), we suppose that the nonlinearity $f$ verifies the following hypothesis

\((H_1)\)

$$\begin{cases} \exists (a, b) \in \mathbb{R}^2 \text{ such that} \\ \forall (s, t) \in \mathbb{R}^2 \quad |f(x, s, t)| \leq a|s|^{p-1} + b|t|^{p-1} + c(x) \quad \text{a.e.} \quad x \in \Omega, \\ \frac{1}{\Gamma_1^p(\beta, 1)} + \frac{1}{\Gamma_1^p(\beta, 1)^{1/p}} < 1, \end{cases}$$

where $c \in L^p(\Omega)$ and $\Gamma_1^p(\beta, 1)$ is the first eigensurface of the operator $\Theta_{P,\beta}$, with $m \equiv 1$. Given in [5] by

$$1/\Gamma_1^p(\beta, 1) = \sup_{v \in L^p(\Omega) \setminus \{0\}} \frac{\int_{\Omega} e^{\beta \cdot x} |\Lambda v(x)|^p dx}{\int_{\Omega} e^{\beta \cdot x} |v|^p dx}$$

(8)

**Theorem 4.1.** If the hypothesis $(H_1)$ holds, then the problem (4) has at least one nontrivial solution for all $h \in L^p(\Omega)$.

**Proof.** To show the existence of a nontrivial solution of (4), we use the properties of monotone type operators (cf. [6]).

We consider the operator

$$T : L^p(\Omega) \to L^p'(\Omega), \quad v \mapsto e^{\beta \cdot x} N_p(v) - \Lambda(e^{\beta \cdot x} f(x, \Lambda v, v)).$$
The operator $N_p$ is of $(S^+)$ type i.e: If $(v_n)_{n \in \mathbb{N}}$ is a sequence in $L^p(\Omega)$ such that
\[
\begin{cases}
    v_n \to v \text{ in } L^p(\Omega) \\
    \limsup_{n \to +\infty} N_p(v_n), v_n - v \leq 0,
\end{cases}
\]
then $v_n \to v$ strongly in $L^p(\Omega)$. Moreover
\[
|f(x, s, t)| \leq a|s|^{p-1} + b|t|^{p-1} + c(x) \quad \text{a.e. } x \in \Omega,
\]
implies
\[
||f(., \Lambda v, v)||_{p'} \leq a||\Lambda v||_p^{p-1} + b||v||_p^{p-1} + ||c||_{p'}.
\]
Then the operator $v \to f(., \Lambda v, v)$ is bounded, hence the operator $v \to \Lambda f(., \Lambda v, v)$ is compact. Thus we deduce that $T$ is of $(S^+)$ type.

Now we show that $T$ is coercive. Using the Hölder inequality and relation (8), we obtain
\[
\frac{\langle Tv, v \rangle}{||v||_p^p} = \int_{\Omega} e^{\beta \cdot x} |v|^p dx - \int_{\Omega} f(x, \Lambda v(x), v(x)) e^{\beta \cdot x} \Lambda v(x) dx \\
\geq m_\beta ||v||_p^{p-1} - m_\beta ||f||_{p'} ||\Lambda v||_p ||v||_p \\
\geq m_\beta \left[ ||v||_p^{p-1} - a \frac{||\Lambda v||_p}{||v||_p} ||v||_p^{p-1} - b \frac{||v||_p^{p-1} - ||c||_{p'}}{||v||_p^{p-1}} ||\Lambda v||_p \right] \\
\geq m_\beta ||v||_p^{p-1} \left( 1 - \frac{a}{\Gamma_p(1, 1)} - \frac{b}{\Gamma_p(1, 1)^{1/p_1}} \right) - m_\beta ||c||_{p'} \\
where m_\beta = \sup_{x \in \Omega} e^{\beta \cdot x}. Since \frac{a}{\Gamma_p(1, 1)} + \frac{b}{\Gamma_p(1, 1)^{1/p_1}} < 1, we have that $T$ is coercive, hence it is surjective, which proves the existence of a solution of problem (4). □

References


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