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Nonresonance conditions for a p-biharmonic operator with weight

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Abstract. This work is devoted to study two nonlinear problems of fourth order governed by the p-biharmonic operators in nonresonance cases. In the first problem we establish the nonresonance part of the Fredholm's alternative, the second is a nonresonance problem relative to the first eigensurface for the spectrum of the operator $\Delta_p^2 u + 2\beta.\nabla(|\Delta u|^{p-2}\Delta u) + |\beta|^2|\Delta u|^{p-2}\Delta u$, where $\beta \in \mathbb{R}^N$ under Navier boundary conditions.

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1. Introduction

We consider the following problem

$$\begin{cases} \Delta_p^2 u + 2\beta.\nabla(|\Delta u|^{p-2}\Delta u) + |\beta|^2|\Delta u|^{p-2}\Delta u = f(x,u,\Delta u) + h(x) & \text{in} \quad \Omega, \\ u = \Delta u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$ $(N \geq 1)$, $\beta \in \mathbb{R}^N$, Δ_p^2 denotes the p-biharmonic operator defined by $\Delta_p^2 u = \Delta(|\Delta u|^{p-2}\Delta u)$, $h \in L^{p'}(\Omega)$, $\left(p' = \frac{p}{p-1}\right)$, and $m \in M = \{m \in L^\infty(\Omega)/meas\{x \in \Omega/m(x) > 0\} \neq 0\}$.

The investigation of existence of solutions for problems at nonresonance has drawn the attention of many authors, see for example [1, 8, 10, 13].

Recently, Ben Haddouch et al. [4, 5, 3], showed that the spectrum of problem

$$\begin{cases} \text{Find} \quad (\beta,\Gamma,u) \in \mathbb{R}^N \times \mathbb{R}_+^* \times X \setminus \{0\} \quad \text{such that} \\ \Delta_p^2 u + 2\beta.\nabla(|\Delta u|^{p-2}\Delta u) + |\beta|^2|\Delta u|^{p-2}\Delta u = \Gamma m(x)|u|^{p-2}u & \text{in} \quad \Omega, \\ u = \Delta u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

contains at least one sequence of positive eigensurfaces $(\Gamma_n^p(.,m))_n$ defined by

$$(\forall \beta \in \mathbb{R}^N) \quad \Gamma_n^p(\beta,m) = \inf_{K\in\mathcal{B}_n} \sup_{u\in K} \int_\Omega e^{\beta.x}|\Delta u|^p dx,$$

and

$$\Gamma_n^p(\beta,m) \to +\infty \quad as \quad n \longrightarrow +\infty,$$

where

$$\mathcal{B}_n = \{K \subset \mathcal{N}_\beta : K \text{ is compact, symmetric and } \gamma(K) \geq n\}$$

and

$$\mathcal{N}_\beta = \{u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega); \int_\Omega m e^{\beta.x}|u|^p dx = 1\}.$$

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Since $m \in C(\overline{\Omega})$ and $m \geq 0$, the authors proved that the first eigensurface $\Gamma_1^p(.,m)$ is positive, simple and isolated.

In the present paper, using the topological degree theory applied to compact operators and operators of $(S+)$ type, we show the existence of a nontrivial solution of problem (1).

2. Preliminaries

In our further considerations we will use the standard spaces $X = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, $L^p(\Omega)$ and $L^\infty(\Omega)$, with corresponding norms $\|u\|_{2,p} = (\|\Delta u\|_p^p + \|u\|_p^p)^{\frac{1}{p}}$, $\|u\|_p = (\int_\Omega |u|^p dx)^{\frac{1}{p}}$ and $\|u\|_\infty$ respectively.

Recall that for all $f \in L^p(\Omega)$, the Poisson equation associated with the Dirichlet problem

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2}$$

is uniquely solvable in X (cf. [12]). We denote by Λ the inverse operator of $-\Delta : X \longrightarrow L^p(\Omega)$.
In the following lemma we give some properties of the operator Λ (cf. [11]).

Lemma 2.1. (i) *(Continuity): There exists a constant $C_p > 0$ such that: $\|\Lambda f\|_{2,p} \leq C_p \|f\|_p$ holds for all $p \in]1, +\infty[$ and $f \in L^p(\Omega)$.*

(ii) *(Continuity) Given $k \in \mathbb{N}^*$, for all $p \in]1, +\infty[$ there exists a constant $C_{p,k} > 0$ such that for all $f \in L^p(\Omega)$, $\|\Lambda f\|_{W^{k+2,p}} \leq C_{p,k} \|f\|_{W^{k,p}}$.*

(iii) *(Symmetry) The following identity: $\int_\Omega \Lambda u.v dx = \int_\Omega u.\Lambda v dx$ holds for all $u \in L^p(\Omega)$ and $v \in L^{p'}(\Omega)$ with $p \in]1, +\infty[$.*

(iv) *(Regularity) Given $f \in L^\infty(\Omega)$, we have $\Lambda f \in C^{1,\alpha}(\overline{\Omega})$ for all $\alpha \in]0, 1[$. Moreover, there exists $C_\alpha > 0$ such that $\|\Lambda f\|_{C^{1,\alpha}} \leq C_\alpha \|f\|_\infty$.*

(v) *(Regularity and Hopf-type maximum principle) Let $f \in C(\overline{\Omega})$ and $f \geq 0$ then $w = \Lambda f \in C^{1,\alpha}(\overline{\Omega})$, for all $\alpha \in]0, 1[$ and w satisfies: $w > 0$ in Ω, $\frac{\partial w}{\partial n} < 0$ on $\partial\Omega$.*

(vi) *(Order preserving property) Given $f, g \in L^p(\Omega)$, if $f \leq g$ in Ω then $\Lambda f < \Lambda g$ in Ω.*

Let N_p be the Nemytskii operator defined by:

$$\begin{cases} N_p(v)(x) = |v(x)|^{p-2} v(x) & \text{if} \quad v(x) \neq 0, \\ N_p(v)(x) = 0 & \text{if} \quad v(x) = 0. \end{cases} \tag{3}$$

We have $(\forall v \in L^p(\Omega))$ $(\forall w \in L^{p'}(\Omega))$ $N_p(v) = w \Longleftrightarrow v = N_{p'}(w)$. The operator Λ enables us to transform the problem (1) to the other problem which we will study in the space $L^p(\Omega)$.

Lemma 2.2. [4] *The problem (1) is equivalent to problem*

$$\begin{cases} Find \quad v \in L^p(\Omega) \setminus \{0\} \quad such\ that \\ e^{\beta.x} N_p(v) = \Lambda(e^{\beta.x} f(., \Lambda v, v)) + \Lambda(e^{\beta.x} h) \quad in \quad L^{p'}(\Omega). \end{cases} \tag{4}$$

Definition 2.1. We say that $u \in X$ is a solution of problem (1) if $v \in L^p(\Omega)$, where $v = -\Delta u$ is a solution of the problem (4).

Let us here recall for the reader's convenience the theorem of Leray-Schauder [9].

Let X a Banach space, $\mathcal{O} \subset X$ a non-empty set of X and $f : \overline{\mathcal{O}} \to X$ be a compact mapping. Put

$$E_1 = \{(id - f, \mathcal{O}, y) : \mathcal{O} \subset X \text{ bounded open, } f : \overline{\mathcal{O}} \to X \text{ is compact and } y \notin (id - f)(\partial\mathcal{O})\}.$$

Theorem 2.3. *There exists a unique function $d : E_1 \to \mathbb{Z}$ called the topological degree, satisfying:*

(i) $d(id, \mathcal{O}, y) = 1$ *for all* $y \in \mathcal{O}$.

(ii) *(Homotopy invariance): If $h : [0,1] \times \overline{\mathcal{O}} \to X$ is a compact mapping and $y : [0,1] \to X$ a compact mapping such that $y(t) \notin (id - h(t,.))(\partial\mathcal{O})$ for all $t \in [0,1]$, then $d(id - h(t,.), \mathcal{O}, y(t))$ is independent of $t \in [0,1]$.*

(iii) *If $d(id - F, \mathcal{O}, y) \neq 0$, then $(id - f)^{-1}\{y\} \neq \emptyset$.*

(iv) *If $f_{/\partial\mathcal{O}} = g_{/\partial\mathcal{O}}$, then $d(id - f, \mathcal{O}, y) = d(id - g, \mathcal{O}, y)$.*

(v) *(Borsuk's theorem): If $\mathcal{O}$ is more symmetric with $0 \in \mathcal{O}$ and f is odd on $\overline{\mathcal{O}}$, then $d(id - f, \mathcal{O}, 0)$ is an odd integer.*

3. Fredholm's alternative

We establish the nonresonance part of the Fredholm's alternative for the operator $\Theta_{p,\beta}$ which is defined by $\Theta_{p,\beta}u := \Delta_p^2 u + 2\beta.\nabla(|\Delta u|^{p-2}\Delta u) + |\beta|^2|\Delta u|^{p-2}\Delta u$ in the case

$$f(x, u, \Delta u) = \Gamma m(x)|u|^{p-2}u$$

where Γ is not in the spectrum associated with the operator $\Theta_{p,\beta}$ with weight $m(x)$. Problem (4) remain to

$$\begin{cases} \text{Find} \quad v \in L^p(\Omega) \setminus \{0\} \quad \text{such that} \\ e^{\beta.x}N_p(v) = \Gamma\Lambda(e^{\beta.x}mN_p(\Lambda v)) + \Lambda(e^{\beta.x}h) \quad \text{in} \quad L^{p'}(\Omega). \end{cases} \tag{5}$$

we have the following result.

Theorem 3.1. *For all $h \in L^{p'}(\Omega)$, the problem (5) admits at least one nontrivial solution. Moreover, if $h \in L^\infty(\Omega)$, then every solution of (5) is in $C(\overline{\Omega})$.*

Proof. To prove the existence of nontrivial solution of (5), we use the property of Leray-Schauder's topological degree. Consider the family of operators $(T_t)_{t\in[0,1]}$ defined from $L^p(\Omega)$ to $L^p(\Omega)$ by

$$\forall v \in L^p(\Omega) \quad \forall t \in [0,1] \quad T_t(v) = N_{p'}(\Gamma e^{-\beta.x}\Lambda(me^{\beta.x}N_p(\Lambda v)) + e^{-\beta.x}\Lambda(te^{\beta.x}h)).$$

Let $(v_n)_n$ be a sequence in $L^p(\Omega)$ such that $v_n \rightharpoonup v$ in $L^p(\Omega)$, then under assertion (i) of lemma 2.1 and by Sobolev's injection theorem, we have $\Lambda v_n \rightharpoonup \Lambda v$ in X and $\Lambda v_n \to \Lambda v$ in $L^p(\Omega)$. We deduce that for every $t \in [0,1]$, T_t is a compact operator.

According to the theorem of Leray-Schauder 2.3, it suffices to prove the following a priori estimate

$$\exists r > 0 \quad \text{such that} \quad v - T_t(v) \neq 0 \quad \forall v \in \partial B(0,r), \ \forall t \in [0,1]. \tag{6}$$

By contradiction, we assume that

$$\forall n \in \mathbb{N}^* \ \exists v_n \in \partial B(0,n) \ \exists t_n \in [0,1] \quad \text{such that} \quad T_{t_n}(v_n) = v_n. \tag{7}$$

We set for all $n \in \mathbb{N}^*$, $w_n = \dfrac{v_n}{||v_n||_p}$. The sequence $(w_n)_{n\geq1}$ is bounded in $L^p(\Omega)$, then there is a subsequence of $(w_n)_{n\geq1}$, still denoted by $(w_n)_{n\geq1}$ such that $w_n \rightharpoonup w$

in $L^p(\Omega)$ and $\Lambda w_n \to \Lambda w$ in $L^p(\Omega)$. Dividing (7) by $||v_n||_p$, we obtain

$$w_n = N_{p'}(\Gamma e^{-\beta.x}\Lambda(me^{\beta.x}N_p(\Lambda w_n)) + t_n e^{-\beta.x}\frac{\Lambda(e^{\beta.x}h)}{||v_n||_p^{p-1}}).$$

The fact that $N_p(\Lambda w_n) \to N_p(\Lambda w)$ and $t_n\dfrac{\Lambda(e^{\beta.x}h)}{||v_n||_p^{p-1}} \to 0$ in $L^{p'}(\Omega)$, we get

$$w_n \to N_{p'}(\Gamma e^{-\beta.x}\Lambda(me^{\beta.x}N_p(\Lambda w))) \text{ in } L^p(\Omega).$$

We deduce that $w_n \to w$ in $L^p(\Omega)$ and $w \not\equiv 0$. In conclusion we have

$$\begin{cases} w &= N_{p'}(\Gamma e^{-\beta.x}\Lambda(me^{\beta.x}N_p(\Lambda w))), \\ w &\in L^p(\Omega) \setminus \{0\}. \end{cases}$$

Which is contradicts with Γ is not in the spectrum of the operator $\Theta_{P,\beta}$. Consequently the estimate (6) holds and one has

$$d(I - T_1, B(0,r), 0) = d(I - T_0, B(0,r), 0),$$

where d is the topologic degree function, I is the identity of $L^p(\Omega)$, $B(0,r)$ is the ball of center 0 and radius r and $\partial B(0,r)$ is its boundary. The Theorem 2.3, (v) assures that

$$d(I - T_0, B(0,r), 0) \neq 0.$$

Thus there exists $v \in B(0,r)$ such that $(I - T_1)(v) = 0$, which will prove the existence of a solution of problem (5). □

4. Non-resonance relative to the first eigensurface of $\Theta_{p,\beta}$

In problem (4), we suppose that the nonlinearity f verifies the following hypothesis

$$(H_1) \qquad \begin{cases} \exists (a,b) \in \mathbb{R}^2 \text{ such that} \\ \forall (s,t) \in \mathbb{R}^2 \ |f(x,s,t)| \leq a|s|^{p-1} + b|t|^{p-1} + c(x) \quad \text{a.e.} \quad x \in \Omega, \\ \dfrac{a}{\Gamma_1^p(\beta,1)} + \dfrac{b}{\Gamma_1^p(\beta,1)^{1/p}} < 1, \end{cases}$$

where $c \in L^{p'}(\Omega)$ and $\Gamma_1^p(\beta,1)$ is the first eigensurface of the operator $\Theta_{p,\beta}$, with $m \equiv 1$. given in [5] by

$$1/\Gamma_1^p(\beta,1) = \sup_{v \in L^p(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega e^{\beta.x}|\Lambda v(x)|^p dx}{\displaystyle\int_\Omega e^{\beta.x}|v|^p dx} \tag{8}$$

.

Theorem 4.1. *If the hypothesis (H_1) holds, then the problem (4) has at least one nontrivial solution for all $h \in L^{p'}(\Omega)$.*

Proof. To show the existence of a nontrivial solution of (4), we use the properties of monotone type operators (cf. [6]).

We consider the operator

$$\begin{aligned} T: \ L^p(\Omega) &\to L^{p'}(\Omega) \\ v &\mapsto e^{\beta.x}N_p(v) - \Lambda(e^{\beta.x}f(x,\Lambda v, v)). \end{aligned}$$

The operator N_p is of (S+) type i.e: If $(v_n)_{n\in\mathbb{N}}$ is a sequence in $L^p(\Omega)$ such that

$$\begin{cases} v_n \rightharpoonup v \text{ in } L^p(\Omega) \\ \limsup\limits_{n\to+\infty} < N_p(v_n), v_n - v >\leq 0, \end{cases}$$

then $v_n \to v$ strongly in $L^p(\Omega)$. Moreover

$$|f(x,s,t)| \leq a|s|^{p-1} + b|t|^{p-1} + c(x) \quad \text{a.e.} \quad x \in \Omega,$$

implies

$$||f(.,\Lambda v, v)||_{p'} \leq a||\Lambda v||_p^{p-1} + b||v||_p^{p-1} + ||c||_{p'}.$$

Then the operator $v \to f(.,\Lambda v, v)$ is bounded, hence the operator $v \to \Lambda f(.,\Lambda v, v)$ is compact. Thus we deduce that T is of $(S+)$ type.

Now we show that T is coercive. Using the Hölder inequality and relation (8), we obtain

$$\begin{aligned} \frac{\langle Tv, v\rangle}{||v||_p} &= \frac{\displaystyle\int_\Omega e^{\beta.x}|v|^p dx}{||v||_p} - \frac{\displaystyle\int_\Omega f(x,\Lambda v(x), v(x))e^{\beta.x}\Lambda v(x)dx}{\|v\|_p} \\ &\geq m_\beta ||v||_p^{p-1} - \frac{m_\beta \|f\|_{p'}\|\Lambda v(x)\|_p}{\|v\|_p} \\ &\geq m_\beta \left[||v||_p^{p-1} - a\frac{||\Lambda v||_p^p}{||v||_p^p}||v||_p^{p-1} - b\frac{||\Lambda v||_p}{||v||_p}||v||_p^{p-1} - ||c||_{p'}\frac{||\Lambda v||_p}{||v||_p}\right] \\ &\geq m_\beta ||v||_p^{p-1}\left(1 - \frac{a}{\Gamma_1^p(\beta,1)} - \frac{b}{\Gamma_1^p(\beta,1)^{1/p}}\right) - \frac{m_\beta ||c||_{p'}}{\Gamma_1^p(\beta,1)^{1/p}}. \end{aligned}$$

where $m_\beta = \sup_{x\in\overline{\Omega}} e^{\beta.x}$. Since $\dfrac{a}{\Gamma_1^p(\beta,1)} + \dfrac{b}{\Gamma_1^p(\beta,1)^{1/p}} < 1$, we have that T is coercive, hence it is surjective, which proves the existence of a solution of problem (4). □

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